Plasma Physics and Applications

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Week – 01

Lecture 05: Debye Potential

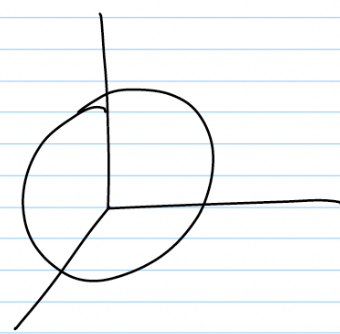
Hello students. We are discussing Debye shielding and we have realized that due to the presence of a test charge in the plasma, charge separation will be created and as a result we will get to see an effective potential which is we called as phi and so del square phi is E times Ne minus Ni divided by epsilon naught. So, it is very important to understand the origin of this potential or the role of this potential in shielding the electric field. So, of course, there is a test charge, there is a potential and there is an electric field. What is the potential? The potential has arised because of the unequal number of electrons and ions in this cloud. So, within that cloud you have more electrons.

$$\nabla^2\phi = \frac{e(n_e - n_i)}{\epsilon_0}$$

$$\phi(r,\theta,\phi)$$

'T'

$$n = n_\infty \exp\left(-E/k_BT\right)$$



$$f(u) = A\exp\left[-\frac{1}{2}mu^2 + e\phi\right]/k_BT$$

'V' $\Rightarrow$ $\underline{eV}$

$\oplus \longrightarrow$ ◯ $\rightarrow (n_e > n_i)$

$\Downarrow$

$\bar{e}$ $\phi \Rightarrow e\phi.$

So, this negative charge more most of the negative charge has given rise to this potential. Now ideally if this potential is visible for the entire plasma then there is no shielding. So, our expression that we are going to derive should be able to give us an inference that this potential will cease to exist after a point of time. After certain distance this will cease to exist.

$$n_i = n_\infty$$

$$n_e = n_\infty e^{-(E+)/k_BT}$$

$$n_e = \int_{-\infty}^{\infty} f(u)\,du.$$

$$\phi = 0 \Rightarrow n_i = n_e = n_\infty$$

$$n_e(r) = n_\infty \exp\left(\frac{-e\phi}{k_BT}\right)$$

$$\boxed{n_e(r) = n_\infty \exp\left(\frac{e\phi}{k_BT}\right)}$$

Only then we can establish the idea of shielding of the electric field by the plasma. So, whatever it is so one thing is as of now even before we discuss the Debye's length or derive the Debye's length. So, plasma is behaving collectively. So, all the plasma is behaving collectively to shield out the electric field. Now let us take it ahead.

$$\nabla^2\phi = \frac{e}{\epsilon_0}\left[n_e - n_i\right]$$

$$\nabla^2\phi = \frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2\frac{\partial\phi}{\partial r}\right] + \frac{1}{r^2\sin\theta}\overset{0}{\cancel{\frac{\partial}{\partial\theta}\left[\sin\theta\frac{\partial\phi}{\partial\theta}\right]}} + \frac{1}{r^2\sin^2\theta}\overset{0}{\cancel{\left[\frac{\partial^2\phi}{\partial\varphi^2}\right]}}$$

$n_i = n_e \Rightarrow \phi = 0$

ϕ is symmetric in θ & φ $\quad \frac{\partial}{\partial\theta}, \frac{\partial}{\partial\varphi} = 0$

$$\frac{1}{r^2}\frac{d}{dr}\left[r^2\frac{d\phi}{dr}\right] = \frac{e}{\epsilon_0}\left[n_\infty \exp\left(\frac{e\phi}{k_B T}\right) - n_\infty\right]$$

$$\frac{1}{r^2}\frac{d}{dr}\left[r^2\frac{d\phi}{dr}\right] = \frac{e n_\infty}{\epsilon_0}\left[\exp\left(\frac{e\phi}{k_B T}\right) - 1\right]$$

So, we have to derive what is the form of phi. So, ideally we take the coordinates r theta phi. So, we need to have r theta phi. So, what does it mean? It means we have to find out how the potential will vary as a distance as a function of distance away from the test charge and how it vary in the angular direction and in the azimuthal direction. So, we have a three dimensional picture and we have something like this.

So, we have to find out with respect to r theta and phi. So, basically the Laplacian can be expanded in r theta and phi then we can use the right hand side. Now at this point of time what is the role of temperature? Because if there is no temperature then the electrons movement will be restricted. They do not have any thermal motions which means that given the temperatures the number density of electrons will be something. It will follow some distribution.

So, we can assume the Maxwellian distribution where the number n is n infinity exponential minus E by k B T. I think all of you are familiar with this distribution. This will just tell you at any given temperature how many number of electrons you can find with respect to the background number of electrons. So, you can use it to write the function f u which is A exponential minus half m u square plus E phi divided by k B T. You see this is the kinetic energy.

Everyone has its kinetic energy by the virtue of its temperature. Now this additional energy E phi when you are accelerating a charged particle with a potential V the energy that it gains is E v. We know this. So, I have used the same E phi. I am calling the potential as phi instead of V.

Now there is a positive charge just a quick recap. Now there is a cloud of electrons around this positive charge. Within this cloud the number of electrons is greater than number of ions which creates a potential and electron sees this potential and gets an energy E phi. So, this is the full picture. These are the ingredients that we have in this particular recipe.

If we want to understand what happens in totality we should account each of these. We cannot neglect any of these and find out an expression. So, we can say that since the number of ions is constant it is not moving. So we will say that number of ions is equal to number of an equilibrium number which is not changing but the number of electrons on the other hand is n infinity e to the power of minus E all of E plus by k B T. This is what we have.

$$e\phi << k_B T$$

$$\frac{1}{r^2}\frac{d}{dr}\left[r^2\frac{d\phi}{dr}\right] = \frac{e n_\infty}{\epsilon_0}\left[\cancel{1} + \frac{e\phi}{k_B T} - \cancel{1}\right]$$

$$\frac{1}{r^2}\frac{d}{dr}\left[r^2\frac{d\phi}{dr}\right] = \boxed{\frac{e^2 n_\infty \phi}{\epsilon_0 k_B T_e}}$$

$$\lambda_D^2 = \frac{\epsilon_0 k_B T_e}{e^2 n_\infty}$$

$$\boxed{\frac{1}{r^2}\frac{d}{dr}\left[r^2\frac{d\phi}{dr}\right] = \phi\,\lambda_D^{-2}}$$

Solution $\underline{\phi}$

So, this E is kinetic energy plus E phi. Now what we have to do is let us say the kinetic energy is there of course when phi is 0 how will you find the number of electrons since we have written the distribution function number of electrons can be minus infinity to infinity F of u du. So, at equilibrium when phi is 0 we have the condition which is number of ions is equal to number of electrons we are going to call this number as the n infinity. So, number of electrons as a function of r will be n infinity exponential minus E phi by k B T minus E by k B T. Here let us say we do not talk about the background kinetic energy we are rather more interested about the role or the influence of the potential phi.

Since charge is negative we will write number of electrons as I am using E the charge of electron as minus E so I will write k B T. So, this is a very important relation the meaning of this relation has a lot of significance in plasma physics. Now let us use these things and try to derive an expression for the device potential. Now we have del square phi is equals to E times n E minus n i. So, far we are good we accounted for the charge density first then we accounted the electric field that can be produced out of that charge density.

$$\frac{1}{r^2}\frac{d}{dr}\left[r^2\frac{d\phi}{dr}\right] = \frac{\phi(r)}{\lambda_D^2}$$

$$\frac{1}{r^2}\left[2r\frac{d\phi}{dr} + r^2\frac{d^2\phi}{dr^2}\right] = \frac{\phi(r)}{\lambda_D^2}$$

$$\frac{2}{r}\frac{d\phi}{dr} + \frac{d^2\phi}{dr^2} = \frac{\phi(r)}{\lambda_D^2}$$

$$\frac{d\phi}{dr} + \underbrace{\frac{d\phi}{dr} + r\frac{d^2\phi}{dr^2}}_{} = \frac{r\phi(r)}{\lambda_D^2}$$

$$\underbrace{\frac{d\phi}{dr} + \frac{d}{dr}}_{}\left[r\frac{d\phi}{dr}\right] = \frac{r\phi(r)}{\lambda_D^2}$$

$$\frac{d}{dr}\left[\frac{d}{dr}(r\phi)\right] = \frac{r\phi(r)}{\lambda_D^2}$$

We wrote the electric field as the negative potential gradient now we have a potential phi. See the potential will tell you the expression for the potential if it is written in all the coordinates will tell you where is it strongest and where is it weakest. So, now we have to see how this del square phi can be expanded in the r theta phi coordinate system. So, del square phi this is standard you can refer to any book on simple vector calculus you can you will find this expression readily available. So, del square phi is 1 by r square dou by dou r of r square dou phi by dou r plus 1 by r square sin theta dou by dou theta of sin theta dou phi by dou theta.

So, this is the expression for del square phi. So, do not think how I got this expression this is pretty standard Laplacian in the spherical polar coordinate system. So, ideally when ni is equal to ne our potential simply becomes 0 this is the initial condition we are deviating away from it. Let us assume the potential phi is symmetric in theta and phi directions theta and phi directions it is symmetric what does it mean? So, it is not varying with respect to theta and phi that is it. So, it is only varying as a function of distance, but along the azimuth it is not varying it is constant.

So, which means that we can neglect dou by dou theta terms and dou by dou phi terms they are 0. Since we are no longer dealing with 3 independent variables and 1 dependent variable we have only 1 independent variable and 1 dependent variable that means we can get rid of all the partial derivatives and we can write the total derivative. So, we will write so now all of this is del square phi. So, we have gotten rid of these terms. So, this is irrelevant and this one.

So, we are only left with this. These are 0's by the way. So, now we can write it as 1 by r square d by dr of r square d phi by dr is equal to e by epsilon naught. We missed an epsilon naught when we got this expression del square phi is I forgot to write an epsilon naught here. So, this is e by epsilon naught ne is n infinity exponential e phi by k B T minus n infinity.

$$\frac{d^2}{dr^2}[r\phi] - \frac{r\,\phi(r)}{\lambda_D^2} = 0$$

$$\text{Let } r\phi = x$$

$$\boxed{\frac{d^2x}{dr^2} - \frac{x}{\lambda_D^2} = 0}$$

$\phi(r)$

You see you have a positive term which appears in the exponential which says that at higher energies you will have that means you have more number of electrons at a given temperature generally the distribution is opposite. But it is indeed true as long as you consider this plasma system within this cloud this is valid that is the beauty of this expression that you see here. Now we can all this is not required 1 by r square d by dr of r square d phi by dr is equal to e by epsilon naught n infinity exponential e phi by k T minus n infinity. So, I will just rearrange the terms for convenience. So, some algebra which is essential for thorough understanding of the physics d phi by dr is e n infinity by epsilon naught times exponential e phi by k B T.

What is k B? k B is the Boltzmann's constant. Now let us say we impose a condition now that e phi is much less than k B T. Why do I need this condition? I want to write the first order approximation of this exponential then I have to have the condition that the power that appears in the exponential is small so I can use it. So, I will write it again 1 by r square d by dr of r square d phi by dr is equal to e n infinity by epsilon naught times 1 plus e phi by k B T minus 1. I have written exponential as 1 plus x because x square and all those terms will become very small as the numerator is very small in comparison to the denominator.

So, only this term will also be very small but we cannot just write it as 1 that is it. So, we have this because we need to have the variation the exponential is representing something so maybe the smallest variation that it represents we have to account for it. So, here is the same thing 1 by r square d by dr of r square d phi by dr is equals to e square n infinity phi by epsilon naught k B T. The phi the potential is a result of charge separation the majority charges that are there inside this sphere or the electron cloud is electrons and the entire approximation is valid for electrons. So, it is appropriate to write the temperature as electron temperature.

So, this is the electrons which are trying to see this potential and get accelerated or get some energy. So, now let us say we will make some substitution. We have a second order differential equation. So, phi is the independent variable here and all of this is a constant which is multiplying it. Let us say we call this constant if you see what will be the dimension of this.

We can derive the dimension of this you will realize that it will be something like 1 by length square. We define we just make some substitution that is it. Lambda d square is epsilon naught k B T e divided by e square n infinity. So, what is lambda d? Lambda d is in the units of length. So, epsilon naught k B T e by e square n infinity is in the units of what you say square root of length.

So, using this I am just substituting this using this we can write the equation as 1 by r square d by dr of r square d phi by dr is equals to phi lambda d raise to the power of minus 2. So, this is a differential equation and the solution of this differential equation.

What is the solution of this differential equation? Solution is phi we have still not obtained the form of the device potential. We only know if there is a radial variation in the device potential what will it represent that is what this equation is telling you nothing more. We still have not derived what is the form for phi.

How this potential has originated? This potential has originated by the introduction of a test charge a positive test charge into the plasma. Let us say we take it ahead and we let us see what we can do about it. So, this is still the same 1 by r square d by dr of r square d phi by dr is I am going to call phi as phi of r. Why? Why because phi only varies as a function of r but with respect to the other two coordinates it is constant. I take this derivative inside I will keep 1 by r square outside you have d by dr of uv taking the derivative inside we have 2 r d phi by dr plus r square d square phi by dr square which is equal to phi of r divided by lambda d square.

Taking the 1 by r square also inside we have 2 by r d phi by dr plus d square phi by dr square is equal to phi of r divided by lambda d square. 2 by r d phi by dr plus d square phi by dr square is so we will conveniently write it as d phi by dr plus d phi by dr plus r d square phi by dr square is equal to r phi of r divided by lambda d square. This is 2 d phi by dr once r is taken to the other side 2 d phi by dr. We have written phi of r plus d phi by dr plus d phi by dr. These two we are going to write it as it will remain d phi by dr 1 d phi by dr will remain plus d by dr of r d phi by dr is equal to r phi of r by lambda d square.

So this is uv again if I take this derivative inside it will be once it will be d square phi by dr square plus you take the it you keep it as v then dr by dr is 1. So this is still the same just rewritten it. We can take d by dr as common and we will write d by dr of r phi is r phi of r by lambda d square. All of this is written like this. You just try to match you will find out that both of them are same.

You first take d by dr of this is uv then you take a derivative on that you will get that this. Maybe you can try it yourself and match both sides. Now essentially all of this now I can rewrite as d square by dr square of r phi minus r phi of r by lambda d square is 0. Go back and see d square by dr square of r phi minus all of this is equal to 0. Let now at this point let r phi is equal to x by changing the variables.

Now we can write our equation as because we have r phi. If you see what I have done actually I have just started from this equation which is 1 by r square d by dr of r square d phi by dr is equal to phi times lambda d raise to the power of minus 2. Instead of all this I rearrange the terms so that this factor r times phi appears on both the sides of the equation that is what I have now. And then I am going to call r phi as x in that case I will write the equation as d square x by dr square. Now here it is very important to note that x is not length.

It is r times the potential. If you take q by 4 pi epsilon 0 r just the basic form the actual form of potential is what actually we are trying to understand or we are trying to derive. But the dimension the point is r phi is not distance d square x by dr square minus x by lambda d square is equal to 0. This equation is for you to accept. It is not something very complicated. We followed all the first principles of derivatives or differentiation and arrived at this second order differential equation.

Now we have to solve it. We our job is still not done by deriving this equation. But let us try to understand the inference of this equation so that in the next lecture we can solve it and we can obtain the analytical form of phi of r. Where is this phi of r residing? It is residing inside this electron cloud. What about it? It is created because of this unequal number of charges inside this or majority of electrons inside this. And it so happens that this potential or the limit of this potential is only up to a particular distance and beyond the distance this positive test charge or its presence cannot be felt.

So, we will conclude this lecture here and in the next lecture we will try to solve this equation and obtain an analytical form for the Debye's potential and also establish how the potential diminishes at the characteristic length which we call as the Debye's length. Thank you.