Foundation of Quantum Theory: Relativistic Approach
Special Relativity 1.2

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Lecture- 09

So let us resume our discussion for the special relativity properties which we were discussing in the previous class and let us try to see how do we go about from having low-inch transformations to get good laws of physics under special relativity transformations.
So remember in the previous class we have learned that in order to keep the speed of light same across all inertial frames we are bound to have transformations between two different inertial frames moving with respect to velocity v with respect to each other through something called the Lorentz transformation. And we obtain the form of the Lorentz transformation through simple algebraic manipulations of constancy of the a line element which is $dt^2 = c^2dt^2 - dx^2\ dy'^2$ that should be the same for speed of light across all frames and we ended up getting the set of low-end transformations Now that was the first premise or the first basic ingredient of the special relativistic properties which we had discussed in the previous class. We wanted to have two properties that speed of light remains same in all inertial frame which we have now achieved through Lorentz transformations. And now the second part of it is the laws of physics should remain invariant form in all inertial frames. We had previously learned that the Newton's law which was the response of a particle or a system to application of a force is not a constant anymore across all frames. It was a constant or invariant for all inertial frames if the frames were connected by Galilean transformations. But if the frames are connected by Lorentz transformations, this is no longer true that the accelerations which is defined as $s = d^2x/dt^2$ is not same under law of transformation across all inertial frames. So, second law of Newton is not holding any more. So, similarly here we want to get a good correct second law of Newton's analog or a correct force law as to say. So let us see why was that that the acceleration as a response to application of a force was invariant under Galilean transformation. And let us see if some of the basic pre-points of that can be again translated into Lorentz transformations or not. So in the Newtonian regime, the force law was actually a vector equation. The force vector was equal to m times the acceleration vector. And therefore one can write a new vector call it a y vector a y vector which can defined as f - ma and that was supposed to be 0 according to newton's law newton's second law and in all inertial frame this was supposed to be 0. That means what is being told that this vector y which is f - ma happens to be a νll vector a vector with zero magnitude or a zero vector and we know that under spatial transformations all with special transformations like rotations and what not a νll vector remains νll therefore. Under Galilean transformation, when we were doing that, what we were doing was just changing spatial coordinate, spatial labeling, new axis, which was just obtainable from spatial transformation. But any spatial transformation cannot make a νll vector a non-νll. So, therefore, in Newton's frame under Galilean transformation, this force law was being respected because it was a vector equation. Some vector was being equal to 0. Therefore it was respected across all spatial transformations. Now we have learnt that Lorentz transformation is not only spatial it has a temporal part as well time also changes. So only spatial transformation respecting vector should not be the principle which we should move along. Better way one way would be to write down the force law in terms of space time vectors, space time 4 vector it is called as a νll equation.

Correct force law

In the Newtonian regime the force law was a vector equation

$$\vec{Y} \equiv \vec{F} - m\vec{a} = 0$$

A null vector remains null under all spatial transformation. $\Rightarrow$ Invariance of the force law

- But Lorentz transformations $\neq$ Spatial
- One needs to write a space-time 4-vector 'null' equation.

Space time 4-vectors

→ Vectors are quantities that transform like co-ordinate differentials

e.g. $dt, dx, dy, dz \equiv dx^{\mu}$

Or, $\dfrac{dt}{d\lambda}, \dfrac{dx}{d\lambda}, \dfrac{dy}{d\lambda}, \dfrac{dz}{d\lambda} \equiv V^{\mu}$

Scalar

or $\dfrac{dV^{\mu}}{d\lambda}$

So, a force law could be $R^{\mu} \equiv F^{\mu} - m\dfrac{d^2x^{\mu}}{d\lambda^2}$

Correct force law

In the Newtonian regime the force law was a vector equation

$$\vec{r} = \vec{F} - m\vec{a} = 0$$

A null vector remains null under all spatial transformation → Invarianceof the force law

→ But Lorentz transformation ≠ Spatial

→ One needs to write a space-time 4 vector null equations.

Space time 4 vectors

→ Vectors are quantities that transform like co-ordinate differentials

e.g. dt, dx, dy, and dx = dx^{μ}

Or, $$\frac{dt}{d\lambda}, \frac{dx}{d\lambda}, \frac{dy}{d\lambda}, \frac{dz}{d\lambda} = V^{\mu}$$

Or,

$$\frac{dV^{\mu}}{d\lambda}$$

So, a force law could be $R^{\mu} = F^{\mu} - md^2x^{\mu}/ d\lambda^2$

Previously, we had written down a νll equation with respect to spatial vectors and we were doing spatial transformations only. So, everything was good.

Now, since we are forced to do a space time transformations, we should rather look for space time 4 vectors and actually space time 4 vectors which are all just like the case of Newton's law. So, that is our guiding philosophy that we will try to find out four vectors and write down force equations in a νll form of those four vectors. In order to do that, we need to know what are the space time four vectors. So, the same question could be casted in spatial three dimensions as well. What are the spatial vectors? So, in all any dimension of space or space time, More generally vectors are defined as vectors are those quantities which transform as the coordinate differentials. So, *dt, dx, dy, dz* these are the four coordinates in space time. In spatial dimensions only *dx, dy, dz* would have been three coordinate differentiable, but in space time there are four coordinate differentiable dt, dx, dy, dz. Collectively, they are given the name dx_{μ} where μ index runs from 0 to 3. It can take value 0, 1, 2, 3 where 0 is reserved for the temporal direction, 1 is reserved for the x direction, 2 is reserved for the y direction and 3 is reserved for the z direction. So, this would be, these are the coordinate differentials. Whatever transforms like a coordinate differentials will be a vector. For instance, if I define the coordinate differential and divide it by some scalar quantity which does not change under frame transformation, then this quantity $dt/d\Lambda$ where Λ is a scalar will also transform like dt because $d\Lambda$ does not transform at all. All it will transform will be through dt. Similarly, this quantity $dx/d\Lambda$ will all it will transform through would be via dx and similarly for dy upon $d\Lambda$. So, if Λs are invariant, then dt upon $d\Lambda$, dx upon $d\Lambda$, dy upon $d\Lambda$ and dz upon $d\Lambda$ are also transforming like coordinate differentials and these can be defined as vectors. So, I can define a four velocity, four dimensional velocity which is not the derivative of coordinate differential with respect to time. Previously again in Galilean transformation we were doing exactly the same thing. We were saying that *dx/dt*, *dy/dt*, *dx/dt* were the velocities because in the Galilean transformation dt was a scalar, it was not changing under transformation. So similarly in Lorentz transformation so that dt is starting to change, so I cannot use dt in the denominator of the division. I should rather find out some scalar which does not change under coordinate transformation through Lorentz transformation and that scalars division or the rate of change of coordinates *t, x, y* and *z* with respect to that invariant coordinate, invariant parameter should be defined as a velocity. When I say invariant, it does not mean constant. I am not saying that Λ should not change from time to time or something. I am not saying that Λ should not change from time to time or something. Just like time was flowing in the Galilean transformation, it was not constant. Only that in different frames, they all agree that if one frame declares it value 5, all other frames declare it to value 5 as well. If one frame declares it to be value 6, all other frames will declare it to value 6. So similarly, in Lorentz transformation, we should find out a parameter whose value is universally agreed upon by all the inertial observers. So those things will be scalars. If I find such a scalar, which we will see an example soon, that scalars further derivative, for example, I can take this v μ and take one more derivative with respect to Λ. So that would be something like $d^2x_\mu/d\Lambda^2$. Again the same thing since v mute transforms like a vector and d upon $d\Lambda$ is invariant then this quantity is also a vector so basic building block is whatever transform like coordinate differentials is a vector and coordinate differentials derivatives with respect to invariant quantities are also transforming like coordinate differentials themselves so therefore all those things are vectors. Similarly, the second derivative of the coordinate differential, which is the first derivative of the velocity is the acceleration, acceleration 4 vector. So, in that sense, if I can define a acceleration 4 vector, which is $d^2x_\mu/d\Lambda^2$, then I can write down a force law which is a generalization of the previous force law like this that 4 vector f μ some 4 force a space time force previously only special force which we are dealing about should be equal to m times the space time acceleration previously we were writing only the spatial acceleration if we could write such a law then this law this equal to 0 again will respect the conditions like before in one frame if the 4 vector r μ is 0 in all other frames which are inertially connected to it and zero vector will remain zero. So, that is the idea that one needs to find out a vector laws and in order to get vector laws or invariant laws that those things would be physically meaningful. So, let us try to see what kind of invariant quantities which we can come about.

So, one invariant quantity we already know in this game is speed of light. That does not change from frame to frame but too bad we cannot use it as a Λ because Λ s needs to be varying as well not only c speed of light is invariant across all frame it is constant as well that it does not change its value ever we do not want that we want some parameter which changes value but when it changes value. The new value is also universally agreed upon by all inertial frame, the previous value was also agreed upon all inertial frames and so on. So, we do not want constants, we want invariants. So, let us try to find out what kind of invariants which we can get from Lorentz transformations.

Under Lorentz transformation

$$\begin{pmatrix} c\,dt \\ dx \end{pmatrix} \longrightarrow \begin{pmatrix} c\,dt' \\ dx' \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{1-v^2/c^2}} & -\frac{v/c}{\sqrt{1-v^2/c^2}} \\ \frac{-v/c}{\sqrt{1-v^2/c^2}} & \frac{1}{\sqrt{1-v^2/c^2}} \end{pmatrix} \begin{pmatrix} c\,dt \\ dx \end{pmatrix}$$

$$\begin{pmatrix} c\,dt' \\ dx' \\ dy' \\ dz' \end{pmatrix} = \underbrace{\begin{pmatrix} \frac{1}{\sqrt{1-v^2/c^2}} & \frac{-v/c}{\sqrt{1-v^2/c^2}} & 0 & 0 \\ \frac{-v/c}{\sqrt{1-v^2/c^2}} & \frac{1}{\sqrt{1-v^2/c^2}} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}}_{\Lambda} \begin{pmatrix} c\,dt \\ dx \\ dy \\ dz \end{pmatrix}$$

In matrix rep.

$$dx^\mu \longrightarrow dx'^\mu = \sum_\nu \Lambda^\mu{}_\nu \, dx^\nu$$

If co-ordinates transform through Λ, the

$$\boxed{\frac{\partial}{\partial x^\mu} \longrightarrow \frac{\partial}{\partial x'^\mu} = \sum_\nu (\Lambda^{-1})^\nu{}_\mu \frac{\partial}{\partial x^\nu}}$$

$$\begin{pmatrix} cdt \\ dx \end{pmatrix} = \begin{pmatrix} cdt' \\ dx' \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{1-v^2/c}} & \frac{-v/c}{\sqrt{1-v^2/c}} \\ \frac{-v/c}{\sqrt{1-v^2/c}} & \frac{1}{\sqrt{1-v^2/c}} \end{pmatrix} \begin{pmatrix} cdt \\ dx \end{pmatrix}$$

$$\begin{pmatrix} cdt' \\ dx' \\ dy' \\ dz' \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{1-v^2/c}} & \frac{-v/c}{\sqrt{1-v^2/c}} & 0 & 0 \\ \frac{-v/c}{\sqrt{1-v^2/c}} & \frac{1}{\sqrt{1-v^2/c}} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} cdt \\ dx \\ dy \\ dz \end{pmatrix}$$

In the matrix representation

$$\boxed{dx^{\mu} \rightarrow dx'^{\mu} \rightarrow \sum_{\nu} (\Lambda^{-1})^{\mu}_{\nu} \frac{\partial}{\partial x^{\nu}}}$$

So, for that I will just write down the Lorentz transformation one more time which we had previously written that cdt' is some combination of previous cdt and previous dx and dx' is also some combination of previous cdt and previous dx. So, together the two equations here which here are given here in the box form can be rewritten in a matrix notation. So I can write down a two cross two matrix, which is connecting the old time and old spatial x to new time and new spatial x and their differentials.
Similarly, if I expand the domain to four dimensions, three + one dimension, if I'm moving along only X direction, nothing is happening along Y and Z direction. So dy' should remain dy and dz' should remain dz. That means they are connected by some identity matrix. While t and x are getting mixed. So, they are connected by some non-trivial matrix which is here. So, in 3 + 1 dimension, this block diagonal matrix is appearing as a matrix of transformation and that we give a name Λ. So, coordinate differentials or vectors transform like Λ. This is the rule that coordinate differential will transform through the matrix Λ. However, we can see if this is true, the following statement is also true due to the chain rule of derivatives that the partial derivative with respect to the coordinate μ or Jacobian transformation rule, the partial derivative with respect to some coordinate x_{μ} goes to del upon delx'_{μ} and that can be shown transforms like the previous partial derivatives but this time with a new matrix Λ^{-1} so coordinate differentials themselves where could have been written in terms of their older versions through matrix Λ coordinate derivatives can be written in terms of their previous version through a matrix Λ^{-1}. So, that is what one can trivially find out from these that you can write down from Jacobian transformation such a thing is possible. So, coordinate differentials themselves transform with Λ coordinate derivatives transform with Λ^{-1}. And you can see that in coordinate differentials, the indexes appear in the upper side while the coordinate derivatives index appear in the lower side. So, this is the hand rule, a thumb rule which we can use that whatever has a upper index will transform with matrix Λ, whatever has a lower index will transform with a matrix Λ^{-1}, where Λ^{-1} will be the inverse matrix of this. So, let us see one example, suppose there is a upper index object P_{μ}, what should be its new version or new inertial frame.

★ Each upper index object transforms with one Λ matrix.

$$p^{\mu} \to p'^{\mu} = \sum_{\nu} \Lambda^{\mu}{}_{\nu}\, p^{\nu}$$

⇒ As many Λ matrices as the number of indices

$$p^{\mu\nu} \to p'^{\mu\nu} = \sum_{\alpha} \Lambda^{\mu}{}_{\alpha} \sum_{\beta} \Lambda^{\nu}{}_{\beta}\, p^{\alpha\beta}$$

★ Each lower index object transforms with one Λ^{-1} matrix

$$U_{\mu} \to U'_{\mu} = \sum_{\nu} (\Lambda^{-1})^{\nu}{}_{\mu}\, U_{\nu}$$

⇒ As many (Λ^{-1}) matrices as the number of lower indices

$$U_{\mu\nu} \to U'_{\mu\nu} = \sum_{\alpha} (\Lambda^{-1})^{\alpha}{}_{\mu} \sum_{\beta} (\Lambda^{-1})^{\beta}{}_{\nu}\, U_{\alpha\beta}$$

★Each upper index object transforms with one Λ matrix.

$$P^{\mu} \to P'^{\mu} = \sum_{\nu} \Lambda^{\mu}_{\nu} P^{\nu}$$

⟶ *As many Λ' matrices as the number of indices*

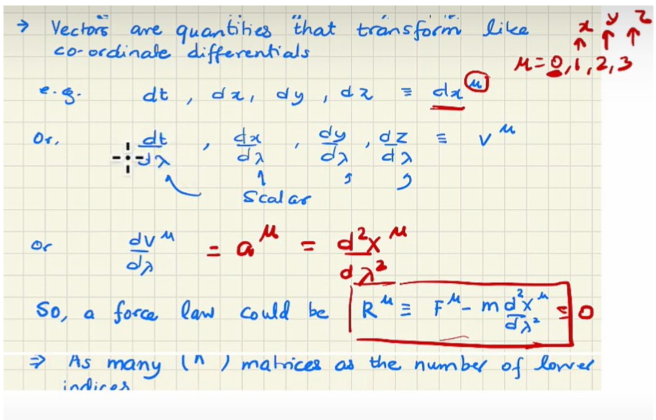
$$P^{\mu\nu} \to P'^{\mu\nu} = \sum_{\alpha} \Lambda^{\mu}_{\alpha} \sum_{\beta} \Lambda^{\nu}_{\beta} P^{\alpha\beta}$$

★ Each lower index object transform with one Λ^{-1} matrix

$$U_{\mu} \to U_{\mu}' = \sum_{\nu} (\Lambda^{-1})^{\nu}_{\mu} U_{\nu}$$

→ *As many* (Λ^{-1}) matrices as the numbre of lower indices

$$U_{\mu\nu} \to U_{\mu\nu}' = \sum_{\alpha} (\Lambda^{-1})^{\alpha}_{\mu} \sum_{\beta} (\Lambda^{-1})^{\beta}_{\nu} U_{\alpha\beta}$$



→ Vectors are quantities that transform like co-ordinate differentials.

e.g. dt, dx, dy ie $dx = dx^{\mu}$

or

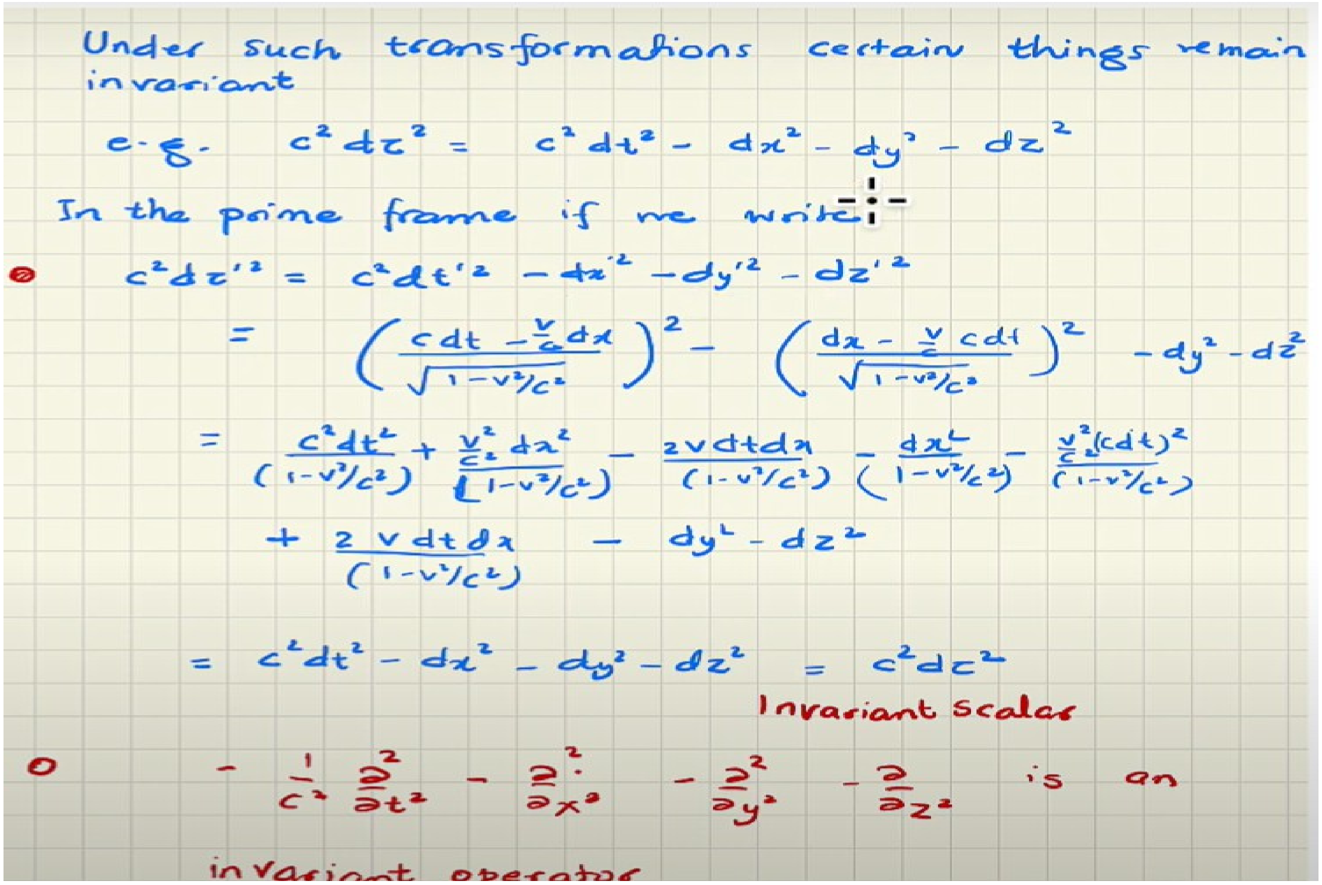
$$\frac{dt}{d\lambda}, \frac{dx}{d\lambda}, \frac{dy}{d\lambda}, \frac{dz}{d\lambda} = V^{\mu}$$

all are scalars

So, a force law could be $\boxed{R^{\mu} = F^{\mu} - m\frac{d^2X^{\mu}}{d\lambda^2} = 0}$

→ As many (n) matrices as the number of lower indices.

It should be P^{μ} and since it has an upper index, it should transform with matrix Λ multiplication. Similarly, if there is a two index object which has two upper index, call it p mu ν each upper index will bring in one Λ. So, in a new inertial frame, $P'^{\mu\nu}$ will be obtainable from the previous P through two Λ multiplications. So, see in this one index object, the index became the first index of the matrix, the second index got summed over,μ was getting summed over. Here also in the two index object, the upper two indices become the upper two indices of the two matrices when the lower indices of the two matrices get summed over. So, this is how a general rule will follow as many upper indices are there those many Λ s will appear in the game in the transformation and all the lower indices of the Λ will get summed over. The similar kind of thing is true for lower index objects as well. We have seen that coordinate derivatives transform with inverses of Λ. So all lower index objects transform with a Λ^{-1} matrix. And this time the upper index gets summed over. Previously the lower index was getting summed over and upper index remained intact. This time the lower index will remain intact. And upper indices will be summed over and the matrix will be replaced by Λ^{-1}. Similarly, for more than one lower index object, those many Λ^{-1}s will appear and upper indices of all of them will get summed over and the lower indices will remain intact. So, this is the general prescription of transforming upper index objects or lower index objects.Upper index object transform with matrix Λ, lower index objects transform with matrix Λ^{-1}s. So, now we have this at our hand, then we can try to find out if there are invariant quantities around or not, which was our goal to start with. So, let us look at these first demands which we had. This quantity I am going to define as c^2 times some invariant $d\tau^2$. So, this quantity I am just right now proposing, which is just c^2dt^2 - dx^2 - dy^2 - dz^2. So, this is the quantity I am looking at.



Under such transformations certain things emain invariantly e.g.

$cd\tau^2 = c^2dt^{2\,=} = c^2dt^2 - dx^2 - dy^2 - dz^2$

In the prime frmae if we write

$cd\tau'^2 = c^2dt'^2 - dx'^2 - dy'^2 - dz'^2$

$$= \left(\frac{cdt - v/cdx}{\sqrt{1-v^2/c^2}}\right) - \left(\frac{dx - v/cdt}{\sqrt{1-v^2/c^2}}\right) - dy^{2-} dz^2$$

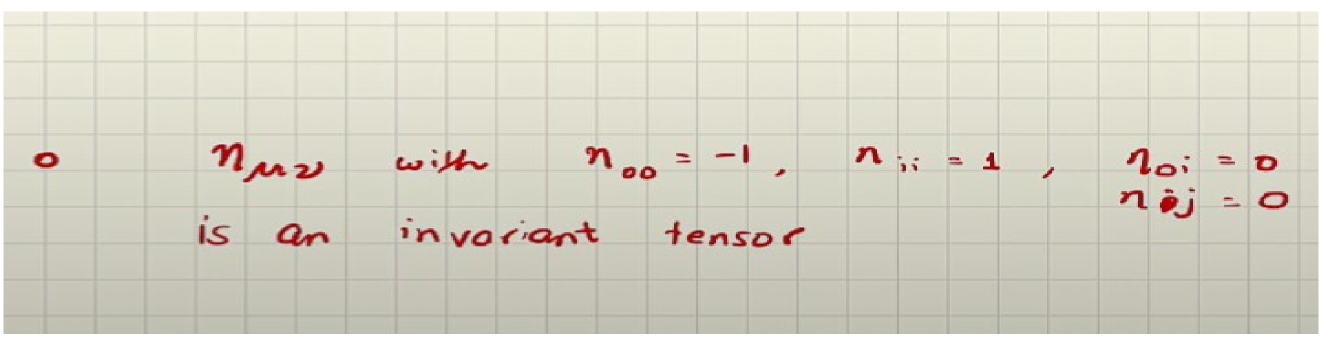
$$= \left(\frac{cdt - v/cdx}{\sqrt{1-v^2/c^2}}\right)^2 - \left(\frac{dx - v/cdt}{\sqrt{1-v^2/c^2}}\right)^2$$

$$= \frac{c^2dt^2}{1-v^2/c^2} + v^2/c^2\frac{dx^2}{1-v^2/c^2}\frac{2vdtdx}{1-v^2/c^2} - \frac{dx^2}{1-v^2/c^2} - v^2/c^2\frac{(cdt)^2}{1-v^2/c^2} + \frac{2vtdx}{1-v^2/c^2} - dy^2 - dz^2$$

$= c^2dt^2 - dx^2 - dy^{2-} dz^2 = cd\tau^2$ which is an invariant scalar.

$$\circ - \frac{i}{c^2}\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial z^2}$$

Let us find out what happens in the primed frame. In the primed frame, it will become c^2 remain c because it is a constant. d tau should become d tau prime which is this and its appearance should be $c^2 dt'^2 - dx'^2 – dy' -dz'^2$. And I know under lowest transformation how do these quantities transform. First of all dy' and dz' do not transform at all. So, they will just become plain and simple dy^2 and dz^2 . dx' is *equal to this* $\frac{1}{\sqrt{1-v^2/c^2}}(dx - v^2/cdt)$. This is the rule for dx which can be obtainable either from this row or the boxed equation which we had written previously. From here, you can take the derivative from both sides, you will get dx' is equal to $dx - vc \times c\, dt$. So, that is what we have written over here, that we have written over here, that dx' is this quantity. Similarly, dt' is this quantity, cdt' is this quantity. Now, we in the new frame this quantity $d\tau'^2$ is written in this form. We can open up those brackets and try to collect things. So, you will see that opening this bracket will give rise to $a^2 + b^2 - 2ab$. Opening this bracket will give again rise to three terms $a^2 + b^2 - 2ab$ and then dy^2 and dz^2. Now it so happens you can combine various first of all you can see that these cross terms between dt and dx exactly cancel each other and when you combine terms like dt^2 from here and here and terms like dx^2 these two you will get exact cancellations of factors and you will get just c^2dt^2 and dx^2. So you see if I started with a prime frame I got the thing in the unprimed frame, which is the same. That means d tau primes is also equal to d tau. Not only c is same across all frame, *dτ's* are also same across all frame. d tau is not a constant, but it is invariant. It's value may change, but not across frames. So sometimes for some *dx, dt, dy* and *dz*, that d tau would be, let us say, 7 units, then all observers, all inertial frames will say that it is 7 units. They will not agree on what is dt. They will not agree on what is *dx, dy* or *dz*. They will say that they will have their own *dt', dx'* . But they all agree that this combination of *dt, dx, dy* and *dz*, that is invariant. So, this is an invariant scalar. So, we can use d tau as a $d\Lambda$. Previously we were looking for a Λ parameter which was invariant right in order to define velocities and what not. Now we have this quantity $d\Lambda$ which can play the role of we have this quantity d tau which can play the role of $d\Lambda$. So we can define the velocities and accelerations as derivative with respect to this parameter d tau which is an invariant parameter it is not a constant parameter but it is an invariant parameter and this is given a name proper time so now we have a velocity which is defined with respect to the derivative of with respect to proper time so that is given a name proper velocity similarly acceleration can be also obtained from the double derivative of the positions, the space time positions with respect to proper time. So, that would be proper acceleration.



∘ $\eta_{\mu\nu}$ with $\eta_{00} = -1$, $\eta_{ii} = 1$, $\eta_{oi} = 0$ $\eta_{oj} = 0$ is an invariant tensor.

Similarly, we can search for other invariant quantities. You can prove using the transformation relations of dt, dx, dy and dz. Just like previously this combination was an invariant combination and it gave you an invariant scalar this time there is a holy day tomorrow tomorrow is holy days So, just like previously here should be a positive sign. So, just like this combination c^2dt^2 - dx^2 - dy^2 dz^2 was invariant, then we can prove from the similar logic of transformations that this particular combination of coordinate differential derivatives is also invariant. This is an exercise for you.

You can prove along the same lines that this operator does not, this combination does not change. It is invariant, but it is not an invariant number or invariant scalar. This is invariant operator. It has to act on some function to give you a number. So therefore, I cannot use this as a $d\Lambda$ because it is not a number yet. This is the previous one d tau was a number so that is why I could use that with respect to the as a mimic for Λ I cannot use this because Λ is not obtainable yet unless it acts on something but still it is some invariant operator that means acting on some function it will give a number acting on a scalar it will give a number which will be agreed upon by all inertial frames more importantly. If I take a lower index object $\mu\nu$, I know that it has to transform through two Λ^{-1}s because the rule we have set up is that anything has a lower index it will transform with as many Λ^{-1}s. And the transformation laws which we had written was like this. So similarly$\eta_{\mu\nu}$ if I define should be $\eta_{\mu\nu'}$ which will in which α will come above α would be the sum index which is upper μ here then there is a β an $d\Lambda^{-1}$ here μ here and β and β α β. This is how any lower index tensor will transform. Now, if I give you a specific tensor whose properties are these that it is 0 0 component is η'_{ν} . 1, all other diagonal components are 1, 0 on all off diagonal components are 0. That means if I want to write it in terms of matrix form. Is it is - 1 1 1 1 and all other quantities are 0 then using this transformation properties like here one can prove that this quantity $\eta_{\mu\nu}$ also does not change under Lorentz transformation so this is one of the special two index object which does not change. So, $\eta'_{\mu\nu}$ is also the same set that it is - 1, 1, 1, 1 and all other entries are 0. This happens only for this special 2 rank object. Any other 2 rank object might not do this. So, this is a special invariant tensor. So, this is given a name metric tensor. So, we have learnt about three invariant objects. One was an invariant scalar which gives you a number which is universally agreed upon. Then there is an invariant operator which acts on a scalar and then gives you a number which is universally agreed upon. And then there is a tensor whose components are universally agreed upon. Okay. Fine.

The Λ matrix can be written as

$$\Lambda \equiv \begin{pmatrix} \cosh\alpha & \sinh\alpha & 0 & 0 \\ \sinh\alpha & \cosh\alpha & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\Lambda^{-1} = \begin{pmatrix} \cosh\alpha & -\sinh\alpha & 0 & 0 \\ -\sinh\alpha & \cosh\alpha & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\Lambda(\alpha)\,\Lambda(\alpha') = \Lambda(\alpha+\alpha')$$

$$\Lambda(-\alpha) = \Lambda^{-1}(\alpha)$$

$$\Lambda(0) = \mathbb{1}$$

So, out of these invariant quantities and apart from that speed of light is also one of the invariant quantities. But it is a constant quantity as well so we can construct various we have seen that examples of various invariant quantities now we can look at what is the prescription of getting invariant quantities so for that it is suggestible to look at the structure of the matrix transformation the transformation which we wrote down Λ. Remember the Λ matrix was what? Λ matrix was a block diagonal matrix.Here was something, here was something, then nothing in this block, nothing in this block and identity matrix in this block.

The Λ matrix can be writtern as

$$\Lambda = \begin{pmatrix} \cosh\alpha & \sinh\alpha & 0 & 0 \\ \sinh\alpha & \cosh\alpha & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

and

$$\Lambda^{-1} = \begin{pmatrix} \cosh\alpha & -\sinh\alpha & 0 & 0 \\ -\sinh\alpha & \cosh\alpha & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$\Lambda(\alpha)\,\Lambda(\alpha') = \Lambda(\alpha + \alpha')$

$\Lambda(-\alpha) = \Lambda^{-1}(-\alpha)$

$\Lambda(0) = 1$

Under Lorentz transformation

$$\begin{pmatrix} c\,dt \\ dx \end{pmatrix} \longrightarrow \begin{pmatrix} c\,dt' \\ dx' \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{1-v^2/c^2}} & -\frac{v/c}{\sqrt{1-v^2/c^2}} \\ -\frac{v/c}{\sqrt{1-v^2/c^2}} & \frac{1}{\sqrt{1-v^2/c^2}} \end{pmatrix} \begin{pmatrix} c\,dt \\ dx \end{pmatrix}$$

$$\begin{pmatrix} c\,dt' \\ dx' \\ dy' \\ dz' \end{pmatrix} = \underbrace{\begin{pmatrix} \frac{1}{\sqrt{1-v^2/c^2}} & \frac{-v/c}{\sqrt{1-v^2/c^2}} & 0 & 0 \\ \frac{-v/c}{\sqrt{1-v^2/c^2}} & \frac{1}{\sqrt{1-v^2/c^2}} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}}_{\Lambda} \begin{pmatrix} c\,dt \\ dx \\ dy \\ dz \end{pmatrix}$$

Under Lorentz transformations

$$\begin{pmatrix} cdt \\ dx \end{pmatrix} = \begin{pmatrix} cdt' \\ dx' \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{1-v^2/c}} & \frac{-v/c}{\sqrt{1-v^2/c}} \\ \frac{-v/c}{\sqrt{1-v^2/c}} & \frac{1}{\sqrt{1-v^2/c}} \end{pmatrix} \begin{pmatrix} cdt \\ dx \end{pmatrix}$$

$$\begin{pmatrix} cdt' \\ dx' \\ dy' \\ dz' \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{1-v^2/c}} & \frac{-v/c}{\sqrt{1-v^2/c}} & 0 & 0 \\ \frac{-v/c}{\sqrt{1-v^2/c}} & \frac{1}{\sqrt{1-v^2/c}} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} cdt \\ dx \\ dy \\ dz \end{pmatrix}$$

Now, you pay the close attention to this, the upper 2 cross 2 dimensional block. This is a very peculiar matrix. The same quantity appear in the diagonal and same quantity appear in the off diagonal. And more so diagonals multiplied - off diagonal multiplied is 1. You can verify that this square this times that - that times that is 1.

That means you have a structure like cos hyperbolic αs for some α parameter in the diagonal and sin hyperbolic αs for some α parameter in the off diagonal such that diagonal multiplied - off diagonal multiplied are 1. So, this is a Λ matrix, the inverse of that matrix would be just α going to - α. So, this becomes the inverse matrix, - sine hyperbolic α and - sine hyperbolic α. So, therefore, we have a structure which is a something called a group theoretic structure that we have association between two different parameters like if I take one Λ with a parameter α multiply with another Λ with a parameter α prime I will get a resultant matrix which is also a Λ like this kind of matrix with parameter α replaced by $\alpha + \alpha'$. So, this is the association law. Then we have an inverses I know that for a given α I have a - α which is inverse of the element and then α is equal to 0 is identity matrix. So, this makes it a group people familiar with a bit of group theory would find it useful that this low rate transformations constitute a group a contivous group contivous in the sense α can take any value just like velocity v could have taken any contivous value α can take any contivous value okay so this is how this group elements are marked by different choices of α. Now using these one can further look at as we have already seen that the eta ν ν under the action of this group which is just another name of transforming through Λ^{-1} matrices does not change. And we have already summarized that any object which has two lower indices would be given a name rank 0 to tensor. Anything with upper index will be given rank 2 0 or 1 0 tensor. So, upper index objects are in this side, lower index objects are in that side. So, for example $\eta_{\mu\nu}$ does not have any upper index. So, it will have upper index count 0 and has two lower indices. So, it has a lower index count 0. So, therefore, it is a rank 0 to tensor.

The quantity $\eta_{\mu\nu}$ is a rank $(0,2)$ tensor which happens to invariant.

$$\eta_{\mu\nu} \longrightarrow \eta'_{\mu\nu} = \sum_{\alpha} (\Lambda^{-1})_{\mu}{}^{\alpha} \sum_{\beta} (\Lambda^{-1})_{\nu}{}^{\beta} \eta_{\alpha\beta}$$

$$= \eta_{\mu\nu}$$

* Rank $(1,2)$ tensor after contracting with rank $(1,0)$ tensor gives a rank $(0,1)$ tensor.

V^{μ} $(1,0)$ tensor

$\eta_{\mu\nu} V^{\alpha} \equiv P^{\alpha}_{\mu\nu}$ $(1,2)$ tensor

$\sum_{\nu} \eta_{\mu\nu} V^{\nu} \equiv V_{\mu}$ $(0,1)$ tensor

$\sum_{\mu} V_{\mu} X^{\mu}$ is $(0,0)$ tensor

$$\rightarrow \sum_{\mu} V'_{\mu} V'^{\mu} = \sum_{\alpha}\sum_{\beta}\sum_{\mu} (\Lambda^{-1})_{\mu}{}^{\alpha} (\Lambda)^{\mu}{}_{\beta} V^{\beta} V_{\alpha}$$

$$= \sum_{\alpha}\sum_{\beta} \delta^{\alpha}{}_{\beta} V^{\beta} V_{\alpha} = \sum_{\alpha} V_{\alpha} V^{\alpha}$$

The quantity $\eta_{\mu\nu}$ *is a ran (0, 2) tensor which happens to be invariant.*

$$\eta_{\mu\nu} \rightarrow \eta'_{\mu\nu} = \sum_{\alpha} (\Lambda^{-1})^{\alpha}_{\mu} \sum_{\beta} (\Lambda^{-1})^{\beta}_{\nu} \quad \eta_{\alpha\beta} = \eta_{\mu\nu}$$

★ *Rank (0,2) tensor after contracting with rank (1,0) tensor gives a rank (0,1) tensor.*

V^{μ}*(1,0) tensor after contracting with rank (1,0) tensor gives a rank (0,1) tensor.*

$H_{\mu\nu} V^{\alpha} = P^{\alpha}{}_{\mu\nu}$ *(1,2) tensor.*

$$\sum_{\nu} \eta_{\mu\nu} \quad V^{\nu} = V_{\mu} \text{ (0,1) tensor.}$$

$$\sum_{\mu} V_{\mu} X\mu \quad \text{is (0,0) tensor}$$

$$\rightarrow \sum_{\mu} V_{\mu} X\mu V'^{\mu} = \sum_{\alpha} \sum_{\beta} \sum_{\mu} (\Lambda^{-1})^{\alpha}_{\beta} (\Lambda)^{\mu}_{\beta} V^{\beta} V_{\alpha} = \sum_{\alpha} \sum_{\beta} \delta^{\alpha}_{\beta} V^{\beta} V_{\alpha} = \sum_{\alpha} V_{\alpha} V^{\alpha}$$

Similarly, dx_{μ} has only one upper index, so its upper index count will be 1 and no lower index, so its lower index count will be 0. And I know that any upper index object transform with a Λ and lower index objects transform with as many Λ^{-1}s as the lower index count is. So, therefore, suppose I have a quantity which is one object with two lower indices and one object with one upper index so then I have a three index object one upper index and two lower indices so then it will be called rank one two tensor because it has one upper index so upper index count will be one two lower index then its lower indices count is two however there is a special case in which I do not just put eta, μ, ν and V_{α} side by side. I take this α, put it to value ν and then sum it over. This operation is called contraction. So, previously here I had put α to be any value and ν could have taken any value, μ could have taken any value. So, these are independent. So, then it became a three index object. However, if I do not allow α and ν to be take any values independently but I force that α will take only those values which ν takes and then I sum over the ν then this operation is called contraction and you can prove from here that if this transform with two Λ^{-1}s And this transform with one Λ. And I have forced that the lower index of this and upper index of that have to be the same and summed over. That effectively multiplies these two matrices. And therefore, this will become identity and resulting quantity will just transform with one Λ matrix. Therefore, if I just put these two objects $\eta_{\mu\nu}$ and V_{α} side by side, I get a rank one two tensor. But instead of putting side by side, I contract their indices. Then I lose one pair of Λ^{-1} an $d\Lambda$ and resultant thing will transform with only one Λ. So therefore it would be the resultant thing will transform with only one Λ^{-1}. So it has two Λ^{-1}s here and one Λ. So these two got cancelled out under contraction. Resultant thingis transforming only with one Λ^{-1}. So, therefore the resultant quantity will be a rank 0, 1 tensor. Similarly, if I take one lower rank index tensor and one upper rank index tensor and contract them, do not just put them side by side. The lower index of this and upper index of this has to be the same and they are summed over, then they are contracted. Then you can prove that this transform with 1 Λ^{-1}, this transform with 1 Λ lower index of this and upper index of that are being summed over. That means it is matrix multiplication and this becomes identity. That means nothing is left over. This becomes a rank 0, 0 tensor. No Λ is left out after this operation so it is a rank 0 0 tensor that means it does not change under with any Λ under low end transformation so a rank 0 0 tensor is also an invariant or a scalar that is what we were looking for right so those quantities which do not change under low range transformations are rank 0, 0 tensors. And rank 0, 0 tensors can be obtained from contracting a lower index object with an upper index object or any rank 2 upper index object with a rank 2 lower index object.

If you contract both the indices, you will get a scalar. So, this is one example one again the same thing that one lower index object v prime lower μ and v prime upper μ. This is the let us say velocity $dx'/d\tau$ is $dx'^{\mu}/d\tau$ is the velocity 4 space time velocity as seen in the prime frame. This v lower μ is obtainable from contracting $dx'_{\mu}/d\tau$ with, so this should be μ, with $\eta_{\mu\nu}$. So, as we saw that if I contract the lower index of $\eta_{\mu\nu}$ with the upper index of v, I will get a lower index object. So, here therefore this v primed μ is just the contraction of $V^{\mu'}$ with $\eta_{\mu\nu}$. So, f V_{μ}' is contraction $\eta_{\mu\nu}$ and V^{μ}, $V^{\mu'}$. So, if I take a velocity If I take a velocity 4 vector in any frame and contract it with its lowered version, which is this is the lowered version, then the resultant quantity will be an invariant rank 0, 0 tensor. And you can prove that its value is the same as if it is in the unprimed coordinates. So, this is a rank 0, 0 tensor or a vector contracted with itself is an invariant quantity that is like a magnitude of a vector. This time I am not doing just plane sum over all the components $dt^2 + dx^2 + dy^2 + dz^2$. I am doing a contraction. A contraction brings in $\eta_{\mu\nu}$ and that $\eta_{\mu\nu}$ if you remember had a - 1 along one of the diagonals. So, therefore it is not just plane summing over one of the element is picking up a negative sign. So, this is a contraction. So, this is how through contraction of various vectors we can generate invariant quantities rank 0, 0, 10. So, you take any vector make its lowered version by contracting it with $\eta_{\mu\nu}$ and then take the lower index object and upper index object contract them together that would be an invariant. So, this is how we would be able to constitute various invariant quantities.

So, this is how we would be able to constitute various invariant quantities. And once we are comfortable with that, we will try to write down the similar kind of things for quantum theories. What laws of quantum theories are invariant under Lewis transformations? So, I stop over here for this lecture. In the next lecture, we will build upon from here. So, now that we know what are the invariant quantities, we will try to how to get the invariant quantities. Then we will try to use some invariant quantities to write down good laws and good relations. So, I stop over here.