Foundation of Quantum Theory: Relativistic Approach
Special Relativity 1.1
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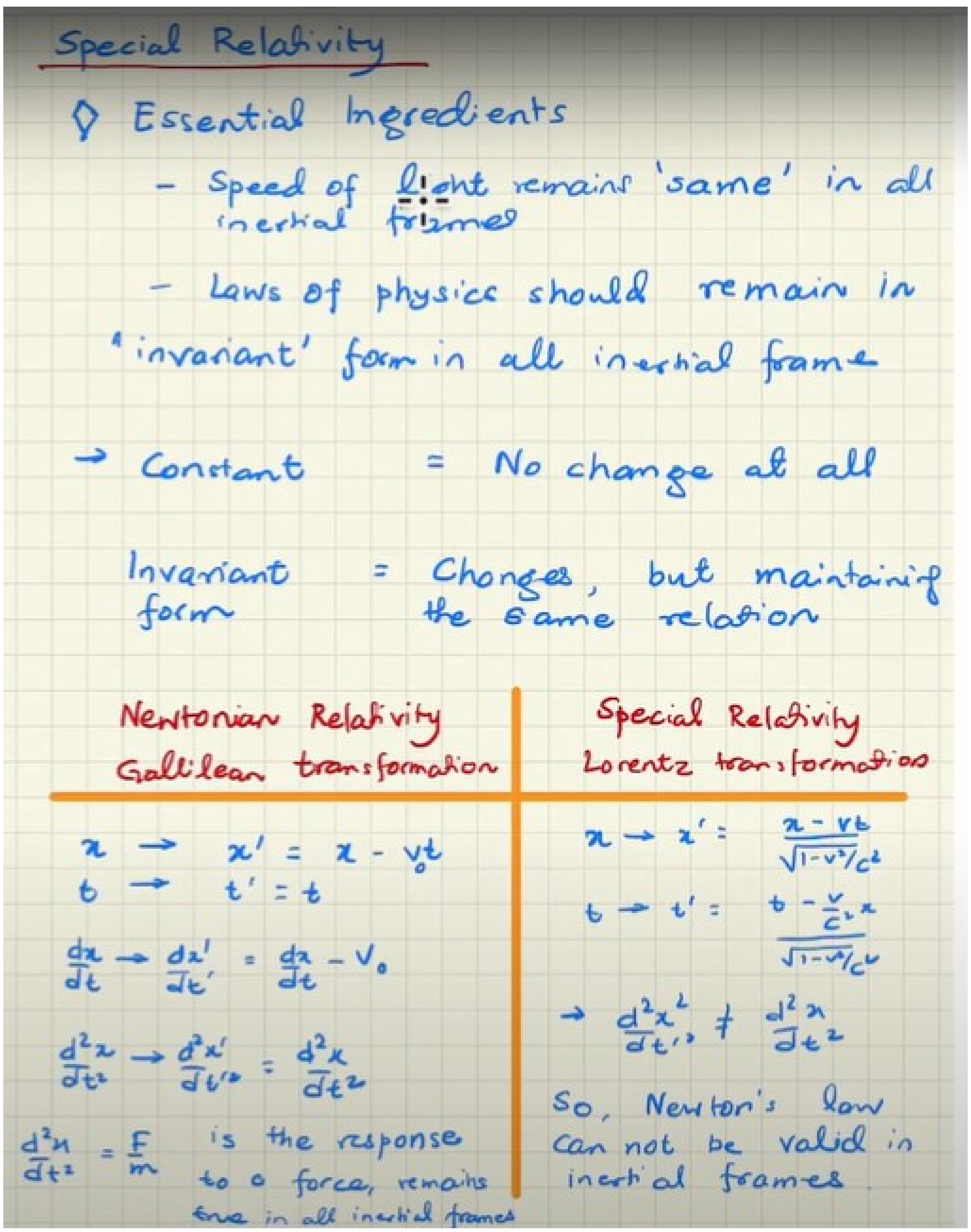
Lecture- 08

So in today's discussion, we will start discussing things regarding special relativity.



Newtonian Relativity
Galilean Transformation

$x \to x' = x - v_0 t$

$t \to t' = t$

$$\frac{dx}{dt} = \frac{dx'}{dt'} = \frac{dx}{dt} - V_0$$

$$\frac{d^2x}{dt^2} \to \frac{d^2x'}{dt'^2} = \frac{d^2x}{dt^2}$$

$$\frac{d^2x}{dt^2} = \frac{F}{m}$$ is the response to a force remains true in all inertial frames.

Special Relativity
Lorenetz Transformations

$$x \to x' = \frac{x - vt}{\sqrt{1 - \frac{v^2}{c^2}}}$$

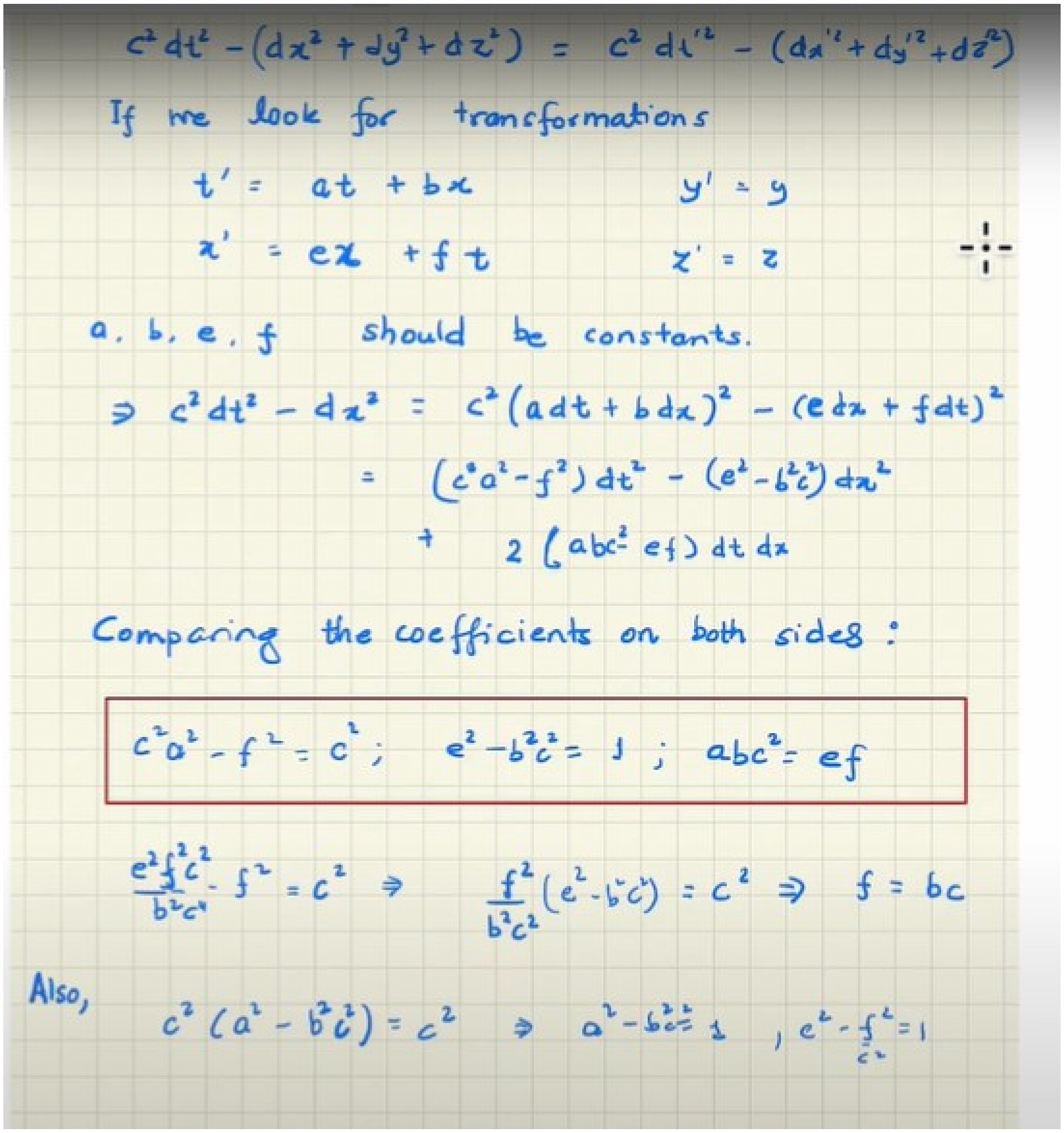
$$t \to t' = \frac{t - \frac{vx}{c^2}}{\sqrt{1 - \frac{v^2}{c^2}}}$$

$$\to \frac{d^2x'}{dt'^2} \neq \frac{d^2x}{dt^2}$$

So, Newton's law can not be valid in an inertial frames.

We will touch upon the concepts of special relativity and we will see the manifestations of various invariants which special relativity comes up with. And then we will try to see in the coming lectures what happens when the concepts of special relativity is applied to quantum mechanics and what their marriage gives rise to the so-called relativistic quantum mechanics and quantum field theory as well.
So let us get going. In this discussion, I am going to touch upon the basic ingredients of special relativity and the two central concepts which are very core to the discussion of relativity or special relativity is that when we talk about physics in inertial frames, two things should be very sacred to all inertial frames. One is that the speed of light that should remain same in all inertial frames. So that is one proposal special relativity comes up with. And the secondly, whatever we describe as a laws of physics should also remain somewhat something same in all inertial frame. That is to say a technical word for that is they should be of invariant form. in all inertial frames so there are two different things which these two concepts give rise to the first thing is constant the speed of light remains same that is the same constant value in all inertial plane so the word constant will mean no change at all that speed of light in one frame is whatever value is there in other inertial frames the same value is recovered however In the second concept, which is the laws of physics, we are using the word called invariant form. The invariant form tells us about that the parameters or the variables which are there in the laws of nature or laws of physics, which are expressed in terms of some equations, those variables or different parameters in the equations, they are allowed to change. However, the relation maintaining different variables should remain the same relation across different frames. So, in that invariant form, we are saying that the structures of the equation should remain same, not their values. I am not demanding that if $f = ma$, f should remain the same and a should remain the same. Despite we are demanding that f should always be equal to ma, whatever value you find in the different inertial frames. So, these are the two core concepts with which most of the discussions of spatial relativity relevant for our course of quantum theory would be useful, can be dealt with. So, just to remind you about various things, the various subtle points which are different in spatial relativity when compared to Newtonian gravity. Is that Newtonian gravity is essentially dictated by transformations which are known to be Galilean transformations. So, different inertial frames where we are trying to write laws of physics are

identified or related to each other through Galilean transformations in Newtonian relativity. What are Galilean transformations? Galilean transformation is that if you are inertial frame and there is another inertial frame which is moving with respect to you with respect to some velocity called v_0. Then the coordinates of your inertial frame and the other inertial frame which is moving with respect to you are related by these transformations, that your coordinates are x and t, the other inertial frames coordinates will be x' and t'. And the Galilean transformation suggests that your coordinate and the new, the different inertial frames coordinates are related by a velocity times the time difference. Meaning, if another frame is moving away from you with some velocity v_0, it would find that its location is given by this expression x - v_0t. Similarly, for time in Galilean transformation, it is suggested that time would not change across two different inertial frames. So, time will remain the same. So, if we have one inertial frame which is S and another inertial frame which is moving with respect to *S,* we call it *S'*. Then suppose we are talking about a point *X* over here. The same, so call it point P rather, the point P has a distance x in the frame S while it has a distance x primed in the frame *S'*. Now as the frame *S'* is moving towards the point, its distance gets shorter and shorter over time. So whatever was x, the distance of the origin from the projection of that point would get shorter and shorter. So, that is how this x' is equal to x - v_0 t is written. So, let us say this between this s and *S'* which is having coordinate x prime this is t and this is t'. So, this is moving with respect to the first inertial plane with velocity v_0. Now, the time remains the same. The time flows in the same fashion across these two inertial planes, which is what Galilean transformation assumes. Under this, it so happens that the velocity of a particle as seen by frame S, suppose there is a particle, suppose there is a particle here rather. I should not draw it on the temporal axis, but just on the spatial axis so suppose there is a particle which is moving with certain velocity call it some velocity which is v as seen by observer in the frame s so that observer says that the particle what we are seeing is moving with respect to it with the velocity v so dx/dt is v Now, the same particle will appear moving slightly slower when compared to an observer in *S'* frame because *S'* frame itself is moving with velocity v_0. So, from these transformation laws, you can find out what is the velocity as seen by the observer in *S'*. So, that would be dx'/dt' which is the velocity as seen by the observer in the *S'* frame and it so happens that x' is x - v_0t L while t' is just t. You can convert this equation into that. So, you can see that the velocity as seen by observer in the second frame *S'* is velocity as seen from the first frame $dx/dt - v_0$. So, it is somewhat lower the velocity. So, velocity is due change under inertial transformations detected by this Galilean wave. However, under this transformation what remains preserved is the acceleration. So, you can similarly compute what is the acceleration of the particle as seen by the two frames. So, from the first frame the acceleration will be d^2x/dt^2 while in the second frame it would be d^2x'/dt'^2 . Again, use the information that t' is just t and x' is $x - v_0t$. You can see that under second derivative, the deviation term, this v_0t terms is killed out. Under double derivative, it does not contribute. So, therefore, the accelerations across two different inertial frames remain the same. So, that is why the Newton's law which was given as acceleration is equal to f by m, the second Newton's law. In an inertial frame the acceleration generated by the action of any force is given in terms of its force divided by the inertial mass remains the same across all inertial frames so given a force how much of acceleration will be generated is a frame independent answer in the newtonian mechanics so this was the setup under which newton's laws operate However, as we have seen from the second point itself, the velocities do change across different frames. So, therefore, it is in direct conflict with the first assumption the special relativity tries to propose that the speed of light should remain the same in all inertial frames. In order to accommodate this demand, people try to come up with another set of transformations which respects this demand that the speed of light remains the same across all inertial frames and ultimately we ended up landing on a set of transformations which are called Lorentz transformations which are different from Galilean transformation in the sense that in this transformation not only x and x' s change and the changing relation is different compared to what it was in the Galilean transformation. The times also do not flow with the same rate as proposed in the Galilean transformations.Time also change across two different frames. So, in one frame if the particles location is given by x, in the second frame it is not $x - vt$ as it was proposed in the Galilean transformation. It is $(x-vt)\sqrt{1-v^2/c^2}$. Similarly, time is in the second coordinate is not equal to the previous time, t' is not t, but it is t' is equal to t - some v upon c^2t imes x, actually all the v there should be v_0s. So, t' is equal to $t - v_0x/c^2$ divided by this strange function square root of $1 - v_0^2/c^2$. So if you try to accommodate the demand of constancy of speed of light, actually one can work out that when the particle moves at the speed of light, when it so happens from this transformation, if dx/dt happens to be c, you can prove from this relation that dx'/dt' also happens to be the c. This is what is special about this transformation that it protects c. This doesn't happen for any other number. If dx/dt is not c, then its counterpart in S' frame is not the same value. It will change. Only for the value c it works out. So therefore, these low-range transformations respect the first demand of special relativity that speed of light should remain the same in all inertial frames. And therefore, in order to accommodate that we are forced to adopt these transformations which are different from Galilean transformations. One can see that whenever v_0 upon c^2 or v_0 upon c is a very small quantity, very ignorably small quantity, then these transformations become the Galilean transformation. So, the strange looking functions in the Lorentz transformation have this property that v_0/c tending to zero limit is the Galilean transformation. So, therefore, if we are moving with a speed much smaller compared to the speed of light, I can work with Galilean transformations equally well because this is the limiting case of low-range transformations. But as soon as I start dealing with particles which are moving with higher and higher velocities, then the transformations should not be used as a Galilean transformation because they do not respect this demand that the speed of light should remain the same. This is a whole set of discussions why speed of light should remain the same in all inertial frames, which I encourage you to look on any special literature, Michelson-Modley experiments and what not. But given this premise that speed of light is supposed to be same in all inertial frame, we are forced to use low-range transformations rather than Galilean transformations. The in order to accommodate the demand of constancy of speed of light we landed upon these transformations and the post we pay is this that unlike the galilean transformations where the accelerations was the same in all inertial frames then it so happens that s d^2x/dt^2 does not remain same in all inertial frames under loading transformations So therefore, if this was supposed to be law of nature that the response of a particle in one application of any force, then this response does not remain same across all inertial frames. Under Galilean transformation, it was true that the response which was generated which is d^2x/dt^2 had the same value across all inertial frames. So, that was this, it was an acceptable law of physics. However, if I adopt low-range transformation, this statement is no more true. The accelerations do not remain same. Therefore, the Newton's law which tells me that s d^2x/dt^2 is the response of a force does not remain same across all inertial frames. So, therefore, this is cannot be acceptable as a law of physics. The second law of Newton's mechanics is not the law to be applicable on particles which are moving with faster velocities and do see the Lorentz transformation as a correct set of transformations across different inertial frames. Just to summarize the discussion, where do these functions, strangely looking functions come up with? I assume you have seen it before as well, but I will just quickly wrap up the discussion with this small set of exercises from where one can see the demand of constancy of a speed of light, how does it uniquely give rise to the Lorentz transformation. This is set upon by the demand that if speed of light has to be same across all inertial frame, then the distance traveled in one inertial frame by the light $dx^2 + dy^2 + dz^2$ should be the same as c^2dt.



$$c^2 dt^2 - (dx^2 + dy^2 + dz^2) = c^2 dt'^2 - (dx'^2 + dy'^2 + dz'^2)$$

If we look for transformations

$t' = at + bx \qquad y' = y$

$x' = ex + ft \qquad z' = z$

a, b, e, f should be constants.

$$\Rightarrow c^2 dt^2 - dx^2 = c^2(adt + bdx)^2 - (edx + fdt)^2$$

$$= (c^2a^2 - f^2)dt^2 - (e^2 - b^2c^2)dx^2 + 2(abc^2 - ef)\,dt\,dx$$

Comparing the coefficients on both sides:

$$c^2a^2 - f^2 = c^2; \quad e^2 - b^2c^2 = 1; \quad abc^2 = ef$$

$$\frac{e^2f^2c^2}{b^2c^4} - f^2 = c^2 \Rightarrow \frac{f^2}{b^2c^2}(e^2 - b^2c^2) = c^2 \Rightarrow f = bc$$

Also, $c^2(a^2 - b^2c^2) = c^2 \Rightarrow a^2 - b^2c^2 = 1, \quad e^2 - \frac{f^2}{c^2} = 1$

$c^2dt^2 – (dx^2 + dy^2 + dz^2) = c^2dt'^2 – (dx'^2 + dy'^2 + dz'^2)$

If we look for transformation

$t' = at + bx, \quad y' = y$

$x' = et + fx, \quad z' = z$

a, b, e, f should be constants.

$\rightarrow c^2dt^2 – dx^2 = c^2(adt + bdx)^2 – (edx + fdt)^2$

$=(c^2a^2 - f^2)dt^2 – (a^2 - b^2c^2)dx^2 + 2(abc^2 – ef)dtdx$

Comparing the coefficients on both sides we get

$c^2a^2 - f^2 = c^2, \; e^2 - b^2c^2 = 1; \; abc^2 = ef$

$e^2f^2c^2/b^2c^4 – f^2 = c^2 \rightarrow f^2/b^2c^2(c^2 - b^2c^2) = c^2$

$\rightarrow f = bc$

Also,

$c^2(a^2\text{-}b^2c^2) = c^2 \rightarrow (a^2\text{-}b^2c^2) = 1,\ e^{2\,-}\ f^2/c^2 = 1$

$$a = \frac{1}{\sqrt{1 - v^2/c^2}} \quad ; \quad \text{from} \quad a = \frac{ef}{bc^2}$$

$$\Rightarrow \boxed{t' = \frac{1}{\sqrt{1 - v^2/c^2}}\left(t - \frac{vx}{c^2}\right)}$$

$$a = \sqrt{1 - \frac{v^2}{c^2}} \quad ; \text{from} \quad a = \frac{ef}{bc^2}$$

that the speed of light remains the same they might have traveled for different distances in different inertia they might have traveled for different temporal durations in different inertial frames. But the ratio of the distance travelled by the time elapsed remains same across all inertial frames. So, that is what our demand is that this quantity should remain same in all inertial frame. That is why we are looking for a transformation which keeps this invariance protected.How should we relate t and *t'*, x and *x'* , *y* and *y'* and *z* and *z'?* So, since in a simple case we can assume that particle is moving only along the x axis or two different inertial frames are just moving away from each other along one particular axis, call it x axis. Then nothing changes in y and z direction. So, *y'* should be *y* and *z'* should be *z*. And I suppose that there should be a linear relation combining a linear relation relating the *t'* and *x'* s to their counterparts in another inertial frame so linear transformation a*t + bx; x + ft.* These *a, b, e* and *f* are supposed to be constants I am using symbols *a b* not *c* and *d* because *c* I am using for speed of light and I'm not using b because *dx dy dz* d appears very at various places therefore in order to not to confuse people I'm using a b e and f now this a b e and f should be constants of time and space both they cannot depend on time and they cannot depend on x Otherwise, the relation will change from time to time. If it is a function of time, then that relation what we are proposing will change from time to time. Today, if this is this relation, tomorrow it will become some other relation. And if it is a function of x, it will become, the relation will change from place to place. If here the relation is some quantity, some expression, then at some other places, let us say in other part of the earth, it will become some other relation. If I want the same physics to be operating at all places of inertial, all places of universe let us say, inertial universe and at all time it should be protected. That means there is no special day. Today should not be a special day compared to tomorrow or day after. And this point should not be special to some other point in space. Then a, b, *c* and f should be constants. They should not be a functions of x and t. With this assumption that these are functions, these are not functions of x and t and constants, we try to see what kind of constants will protect this relation. So, since we have assumed *y'* is equal to y and *z'* is equal to *z*, therefore *dy'* will be equal to *dy* here and *dz'* would be equal to *dz* there. So, therefore these two can be cancelled out on both sides. Now, in order for this relation to work out, left hand side remains like this, right hand side I am going to write for *t'* the first relation and for *x'* the second relation. So, $(adt + bdx)^2$ because dt'^2 is supposed to come here and d*x'* square is supposed to sit here. So, d*x'* would be *edx* + *fdt* from this expression and then square it. Now you see from the square of the first quantity something times dt^2 will be generated, something times dx^2 will be generated and a cross term which is dt dx will be generated and same thing will happen for the second terms of square as well. Something times dx^2 will be generated, something times dt^2 will be generated and a cross term twice of e times f times dx dt will be generated. So in the next step what I do, I collect all the terms which are proportional to dt^2. So all the terms which come with dt^2 is collected one place. All the terms which come with dx^2 is collected at another place. And all the terms which are cross multiplication between dt and dx is collected at one place. So I just opened it up. The coefficient of dt^2 will be c^2 times a^2 from the first term and - f^2 from the second term. So, dt^2 will come with the coefficient $c^2a^2 - f^2$. dx^2 will come with the coefficient c^2b^2, c^2b^2 with a + sign, which is - and - here. I have written in a funny way in order to compare it on the left hand side. And from the second term, the coefficient of dx^2 will be just e^2 with a - sign. So, this is fine.

Now the last term which is the cross term which is getting generated from the first term I will get a term twice of c^2ab which is here and from the second term I will get twice of ef which is here. So ultimately the right hand side is a collection of something times dt^2 something times dx^2 and something times $dtdx$ and that should better match the expression on the left hand side. So therefore component wise The coefficient of dt^2 should match the coefficient of dt^2 here. Coefficient of dx^2 should match the coefficient of dx^2 here. And coefficient *of dt* and *dx* should vanish because left hand side there is no *dt* and *dx*. So, that gives rise to various identities across these constants, the four constants. $c^2a^2 - f^2$ which is the coefficient of dt^2 on the right hand side should match the coefficient on the left hand side which is $c^2.e^2 - b^2c^2$ which is the coefficient of the dx^2 should match the coefficient of dx^2 on the left hand side which is 1. And twice of abc^2 - ef should be 0 which gives rise to identity abc^2 is ef. So, these are the three identities relations amongst the constants *ab ef* are generated by demand of constancy of speed of light. Now, there is plain simple algebra to follow. SI nce here I can see that a is nothing but ef upo*n* bc^2 a is ef divided by this bc^2 and that a I can feed up here so I will get this relation $e2f2/b^2c^4$ which is coming from the ratio ef upon bc^2 which was a so this relation can be written like that Alright, once I have written like that you can see that c^2 is cancelling the c^4 here so that it will become 1 over c^2 and then I can write this expression as I can pull out f square upon $b^2 c^2$ here and inside I will get e^2 - $b^2 c^2$.

So, this left hand side of this can be written neatly as this. Now use the second identity that e^2 - $b^2 c^2$ is one so you will see that $e^2 - b^2$ is one and this side I have a c^2 already so I would get a relation $f^2/b^2 c^2$ is c^2 this is one therefore f^2 will be b^2c^4 and therefore $\sqrt{e}$ will be f is equal to b c^2 okay f should be b c^2 all right this is fine then I feed this information in the first equation again first equation was $c^2a^2 - f^2$, f^2 will become b^2c^4 therefore I can pull out a c^2 common to both the terms and I will have this identity which again c^2 will get cancelled and I will have a simple relation a^2 - $b^2 c^2$ is one a^2 - $b^2 c^2$ is one and secondly from here since *bc* will be f upon *c* from here $\frac{f}{c}$ is *bc*. I can write this $(e^2 - bc)^2$ as $e^2 - f/c^2$. So these two identities I am going to, I have converted these identities here into these two identities which are useful. This is plain and simple algebra, just mathematical jugglery we have done.

Now let us apply the mathematical, the physical input that we want to look at.

Again as we discussed there are two frames S and *S'* and *S'* frame is moving away from S with velocity v_0. So, at time t, if this is my origin of s here, the origin of *S'* will be at location x is equal to v_0 t. So, x should be the location of the origin of *S'*. This would be v_0 t according to observer in s. According to observer in S', the location of the origin will still be *X*' is equal to 0. Because for this observer in this frame, the other frame is moving backwards. Its own frame is stationary as seen by this guy. So therefore, this guy will say that okay, my location is *X*' is equal to 0. So therefore, after time t as seen from frame s, the location of the origin of the second frame is v_0t. And while the second observer says that location of the origin of its own frame is at *x'* is equal to 0. So *x'* is equal to 0 should be obtainable from the expression we just wrote down. *x'* should be ex + ft. So, *x'* is equal to 0 should be e x, e x was v_0 t. I am writing v_0 t in a funny way, v_0 divided by *c* and multiplying by *c* . This is just $v_0\ t + f\ t$, f was b c^2. We had written f was b c^2 and *x'* was $e\ x + ft$. So, I have a relation that 0 should be equal to e times v_0 upon *c* times ct + b times c^2 t. So, you can see that *c* is common in various places. So, you can use this relation and write down that e v_0 upon c^2 is - b. This expression, different side being 0, gives you v_0/c^2 *is* - *b*, which is fine. And now we have everything set up. The second identity, which we just wrote

down here, now let us put that to business $e^2 - f^2/c^2$ is supposed to be 1, which is here.
Now we use the identities, which we have just derived. e^2 I will leave like that. f was b c^2. So, which is square and it will be b^2c^4/c^2 coming from here. So, this will be this relation and b itself is e times v_0^2/c^2. So, use all of this information and this information will nicely convert itself into e^2 - e^2 times v_0^2/c^2 And therefore, E should be root of $1/(1 - v_0/c^2)$. So, the first parameter E, which appears over here, has been identified to be root of $1/(1 - v_0/c^2)$. The second parameter b can be obtained from this expression. Once I know e what I have to just do multiply *a* - of v_0^2/c^2. That is what I do. So, *b* is also obtained from this relation. Similarly, f can be obtained from the relation f is equal to bc^2. And I will get just this f. And lastly, a is *ef*/bc^2, which was obtained from the last relation over here. So, once I knew e, all other parameters can be fixed. Then I know what is a, a is just $1/(1 - v_0/c^2)$, b is just - v_0/c^2 times this factor, e is just square root of this quantity again and *f* is this - v_0 upon this. So, therefore, I get a transformation which initially I wrote down as *at* + *vx* and then v*x* + *ft*, this expression ultimately gives rise to the low-range transformation. So, the low-range transformation which we wrote down earlier as a proposal is now derived as a result of the demand of constancy of speed of light. Okay so you can just do the algebra once again at your convenience to settle this idea that all we have to use is to demand that speed of light remains the same across two inertial frames that is this quantity the same *c* should appear on both sides and there and after that you just demand that it is a linear transformation because of homogeneity and isotropy of space and time And after that the algebra is straight forward to lead you to the fixing of the coefficient *a, b, e* and *f* and you do land upon Lorentz transformations or given algebra. So, I stop here for this discussion session.In next discussion sessions we will see what kind of new physics this Lorentz transformations give rise to and what kind of new laws of physics we land upon. So, I stop here.