Foundation of Quantum Theory: Relativistic Approach
Time dependent perturbation theory 1.3
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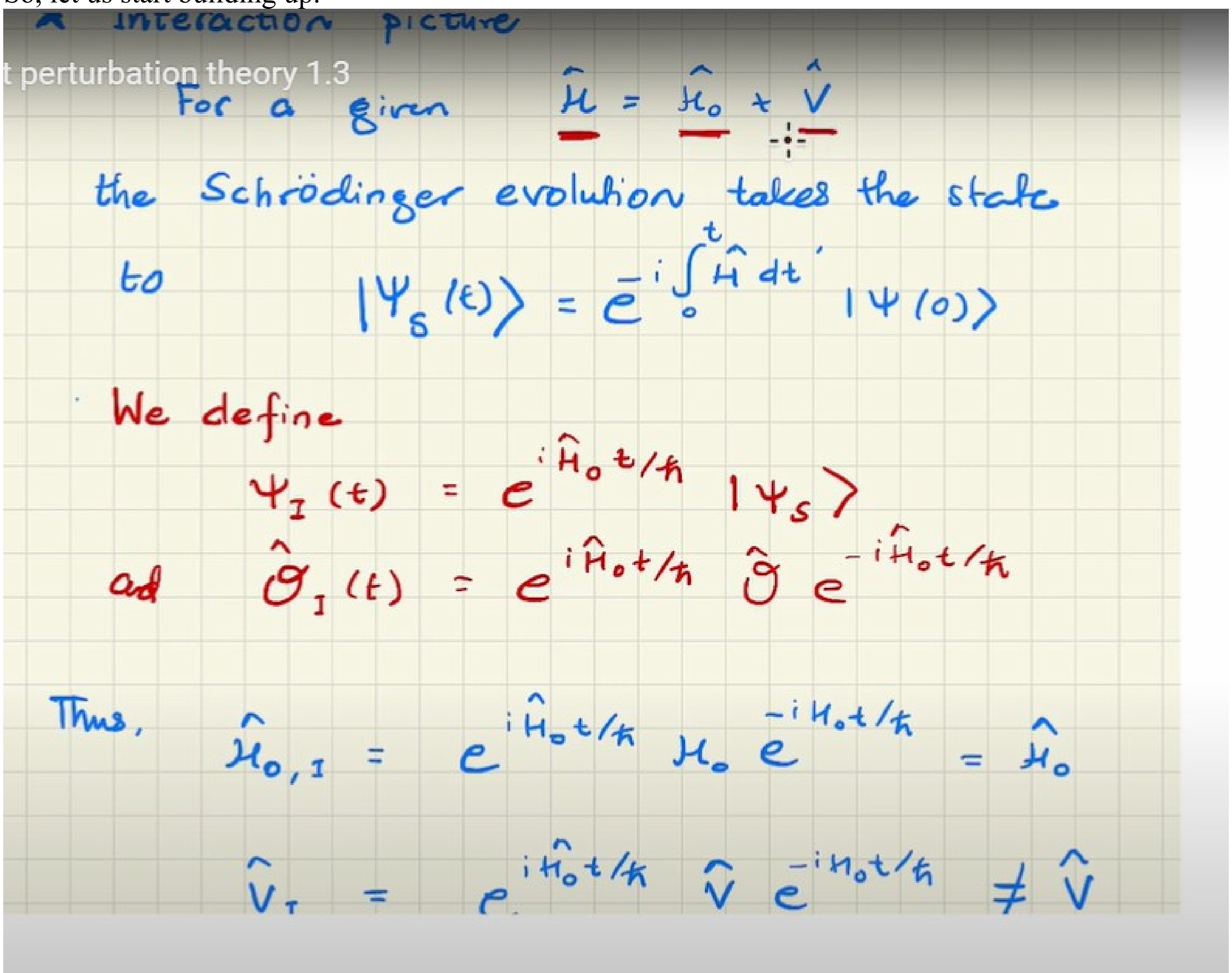
Lecture- 07

So, continuing the discussion in time-dependent perturbation theory, today we will discuss about the concepts of interaction picture which will be very handy when we discuss matter field interaction in the later segment of this course and Dyson series expansion which is also one way of neatly writing the time-dependent perturbation theory and it is more useful in dealing with systems like quantum fields.
So, let us start building up.



For a given

$$\hat{H} = \hat{H}_0 + \hat{V}$$

the Schrodinger evolution takes the state

$$|\psi_s(t)\rangle = e^{-i\int_0^t \hat{H}\frac{t}{\hbar}}\ \mathrm{I}\ |\psi(0)\rangle$$

$|\psi_I(t)\rangle = I \quad e^{i\hat{H}_0 t/\hbar} \quad |\psi_s\rangle$

Thus $I \quad \hat{H}_{0,I} = e^{i\hat{H}_0 t/\hbar} H_0 e^{-i\hat{H}_0 t/\hbar} = \hat{H}_0$

$\hat{V}_I = e^{i\hat{H}_0 t/\hbar} \hat{V} e^{-i\hat{H}_0 t/\hbar} \quad \neq \quad \hat{V}$

So till now we know that if suppose there is a Hamiltonian which drives a state which evolves time evolves a state and it is made up from two pieces one is the free part whose solution we exactly know and then there is a perturbation part or the potential part which has been added to it in presence of which we do not exactly know how to solve the equations exactly and we try to compute its time evolution or eigenfunction questions approximately under different perturbation theory expansions. So, we know under Schrodinger evolution if I ask this question, then the total Hamiltonian if it is time dependent also, then this particular time evolution operator e to the power - I integration of the Hamiltonian over dt, $e^{-i\int_0^t H_0 \frac{t}{\hbar}}$ will act upon the state initial state $\psi(0)$ and give me the final state at time t. So, this is the time evolution operator, you must be more familiar with time independent version of that which is just I $e^{i\hat{H}_0 t/\hbar}$. So, I am missing an $\hbar$ in this, but in the more general case where Hamiltonian is expected to be time dependent as well, then the exponential has a integration of the Hamiltonian. So, this is the standard Schrodinger evolution. Now what we are going to do is to define a new kind of interaction picture in which I define a new state ψ_i, ψ interaction. So, actually this I can write as a ket.

Remember the h0 was just one piece in the total Hamiltonian. So, that piece which I exactly know how to solve, exponential of that hits the wave function ψ_s and whatever I get I am going to define it as a ψ interaction ψ_i. Similarly, another piece of information which we are going to define is called the interaction version of operator. So, what I do, I take the operator O and squeeze it between the unitary operators of the free Hamiltonian. This is the time evolution of a system when there were no perturbation. So, I $e^{iH_{0t}/\hbar}$ from the left and $e^{-iH_{0t}/\hbar}$ from the right. So, effectively it is hat U_0 in the right hand side and $\hat{U}_0^\dagger$ on the left hand side. So, where $\hat{U}_0$ is just $e^{i\hat{H}_0 t/\hbar}$. So, this is the way I am going to define the operators evolution. Remember in Schrodinger evolution, the full Hamiltonian evolves the state. I am not evolving with respect to the full Hamiltonian, only the free part.

Similarly, in the Heisenberg picture, the operator is evolved with respect to the full Hamiltonian.

Therefore,

$$\frac{d}{dt}|\psi_I(t)\rangle = \frac{i}{\hbar}\hat{H}_0 e^{+i\hat{H}_0 t/\hbar}|\psi_s\rangle + e^{+i\hat{H}_0 t/\hbar}\frac{d|\psi_s\rangle}{dt}$$

$$= \frac{i}{\hbar}\hat{H}_0|\psi_I(t)\rangle - \frac{i}{\hbar} e^{i\hat{H}_0 t/\hbar}\hat{H} e^{-i\hat{H}_0 t/\hbar}|\psi_I\rangle$$

$$= \frac{i}{\hbar} e^{i\hat{H}_0 t/\hbar}(\hat{H}_0 - \hat{H}) e^{-i\hat{H}_0 t/\hbar}|\psi_I\rangle$$

$$\Rightarrow \boxed{i\hbar\frac{d}{dt}|\psi_I(t)\rangle = \frac{i}{\hbar}\hat{V}_I|\psi_I\rangle}$$

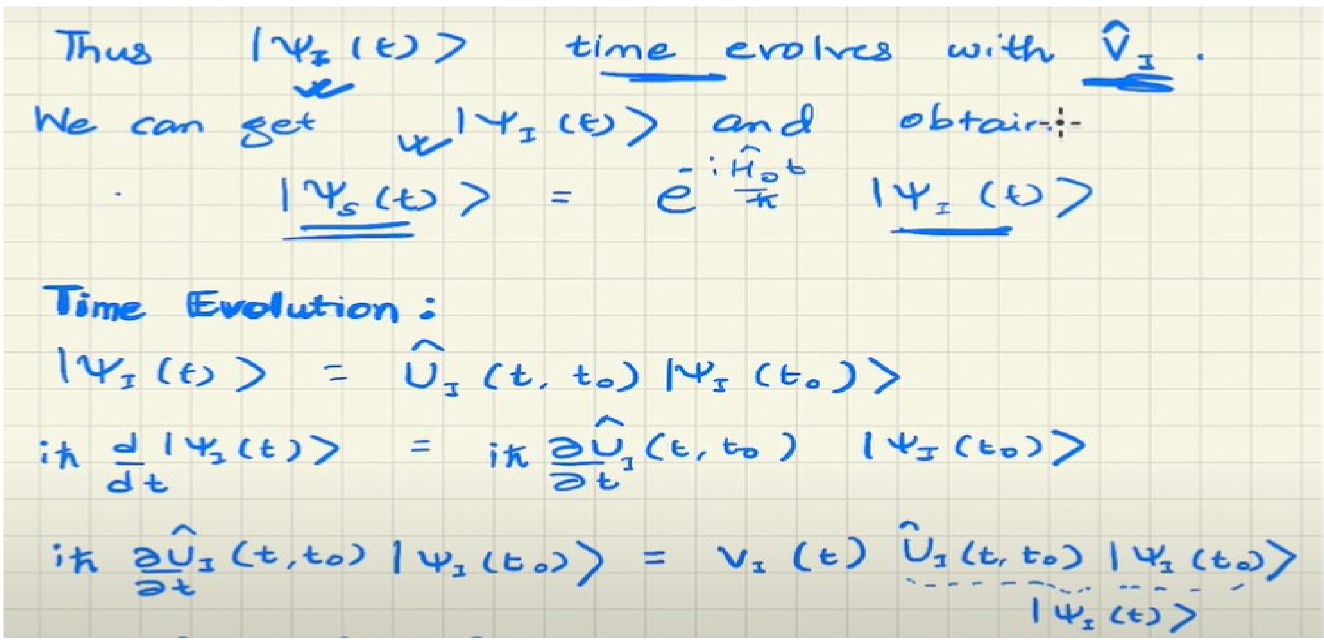
Therefore

$$\frac{d}{dt}\ |\psi_I(t)> = \ \frac{i}{h}\bar{\hat{H}}_0 e^{i\hat{H}_0 t/\bar{h}}\ |\psi_s> \ + e^{i\hat{H}_0 t/\bar{h}}\frac{d}{dt}\ |\psi_s>$$

$$= \ \frac{i}{h}\bar{\hat{H}}_0 e^{i\hat{H}_0 t/\bar{h}}\ |\psi_I(t)> - \ \frac{i}{h} e^{i\bar{\hat{H}}_0 t/\bar{h}}\hat{H}_0 e^{-i\hat{H}_0 t/\bar{h}}\ |\psi_I(t)>$$

$$\boxed{\rightarrow i\hbar\ \frac{d}{dt}\ |\psi_I(t)> = \ \frac{i}{\hbar}\hat{V}_I\ |\psi_I> I}$$

Again, I am not doing evolution with respect to the full Hamiltonian, only the free part. So, therefore, I am in between. The state has been evolved and the operator has also been evolved and we are in between. It is not being done with respect to the full Hamiltonian, but something, one piece of the Hamiltonian which is exactly solvable. How this is useful, it will be very clear in a couple of minutes. Let us go along with this. If any operator can sit here between $\hat{U}_0$ and $\hat{U}_0^\dagger$ from right and left, I can put the free Hamiltonian also in itself. So, suppose I take O as $\mathscr{H}_0$ itself. So, then $\mathscr{H}_0$ will sit across these two unitary operators. $e^{iH_{0t}/\bar{h}}$ from the left and $e^{-iH_{0t}/\bar{h}}$ from the right. However, it so happens that this operator and this operator commute. Some Hamiltonian and its exponential commute among themselves.



Thus $|\psi_I(t)>$ evolved with $\hat{V}_I$

We can get $|\psi_I(t)>$ and obtain

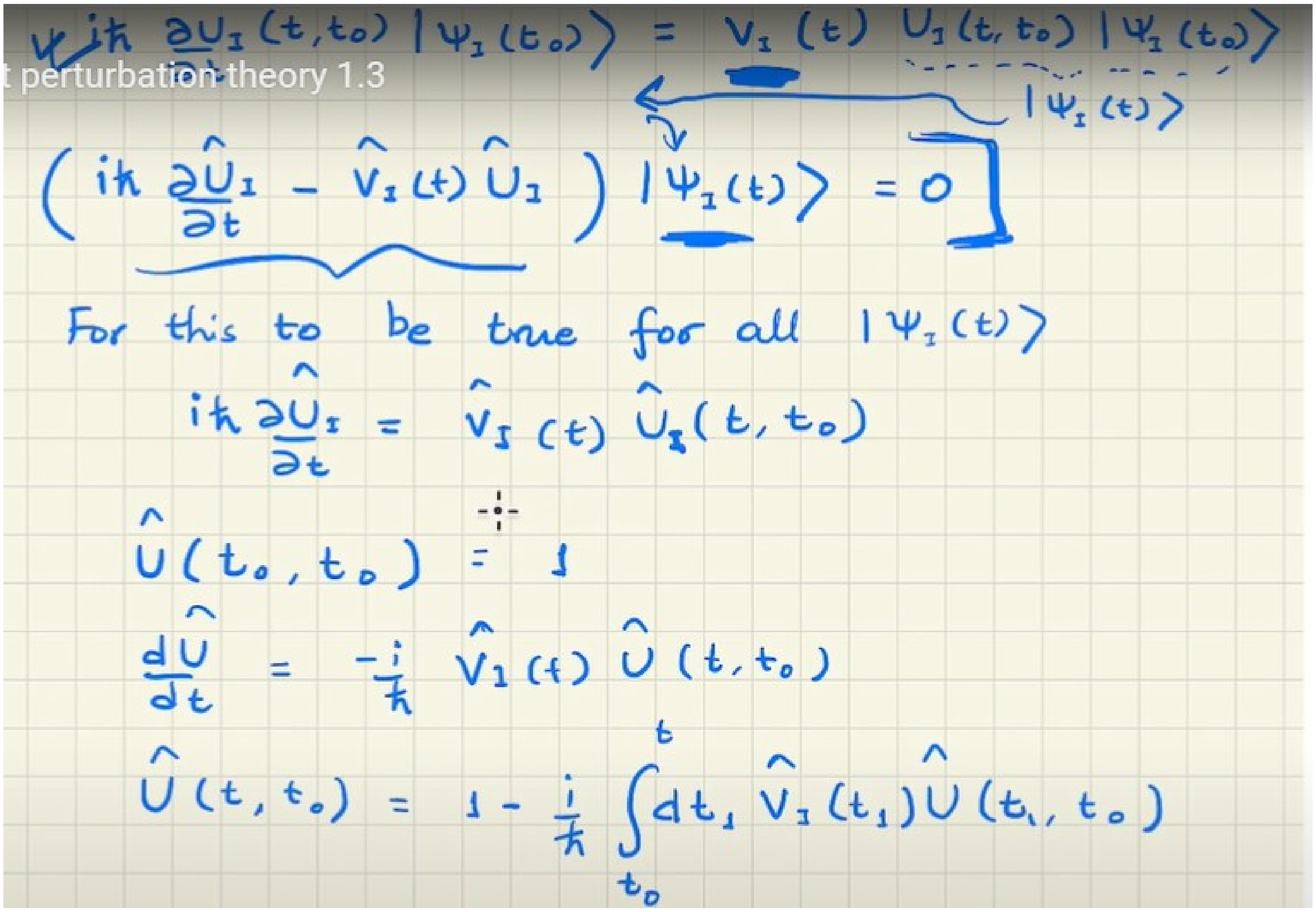
$|\psi_s(t)> = \ e^{i\hat{H}_0 t/\bar{h}}\ |\psi_I(t)>$

Time Evolution:

$|\psi_I(t)> = \ \hat{U}_I(t, t_0)\ \ |\psi_I(t_0)>$

$$i\hbar\frac{d}{dt}\ |\psi_I(t)> = \ i\hbar\frac{\partial \hat{U}_I(t,t_0)}{\partial t}\ \ |\psi_I(t_0)>$$

$$i\hbar\frac{\partial \hat{U}_I(t,t_0)}{\partial t}\ \ |\psi_I(t_0)> = V_I(t)\ \ \hat{U}_I(t,t_0)\ \ |\psi_I(t_0)>|\psi_I(t)>$$



$|\psi_I(t_0)\rangle =$ $\hat{V}_I(t)\hat{U}_I(t,t_0)$ $\psi_I(t_0)\rangle|\psi_I(t)\rangle$

$i\hbar\frac{\partial \hat{U}_I}{\partial t}$ - $\hat{V}_I(t)\hat{U}_I(t,t_0)$ $|\psi_I(t)\rangle = 0$

For this to be true for all $|\psi_I(t)\rangle$

$$i\hbar\frac{\partial \hat{U}_I}{\partial t} = \hat{V}_I(t)\hat{U}_I(t,t_0)$$

$$\hat{U}_I(t,t_0) = 1$$

$$\frac{d\hat{U}}{dt} = \frac{-i}{\hbar}\hat{V}_I(t)\hat{U}_I(t,t_0)$$

$$\hat{U}_I(t,t_0) = 1 - \frac{i}{h}\int_{t_0}^{t} dt_1 \hat{V}_I(t)\hat{U}_I(t,t_0)$$

So, therefore, I can flip this. So $\mathcal{H}_0$ can come on the left and this can go to the right because these two operators commute. If they were not commuting, I would not be able to flip. They are not numbers. So once I flip, this $e^{iH_{0t}/\hbar}$ comes here and it unitary acts upon the second part of the unitary operator. Effectively, I get $\hat{U}_0^{\dagger}$ on the right-hand side of $\mathcal{H}_0$ and ultimately I get $\mathcal{H}_0$ bad. I

$\hat{U}_0^\dagger$

$\psi_i(t)$ $e^{iH_{0t}/\bar{h}}$ $\psi_s(t)$.

Remember this is a plus sign here, unlike the usual Schrodinger equation where a - $i\mathscr{H}_0\ t/\ \hbar$ appears. Okay but if I go along do the time derivative so the time derivative will hit this part as well as this part both of them are time dependent so from the first part if I do then the exponential will throw out a iota $\mathscr{H}_0$ operator and divided by $\hbar$ and the exponential back that exponential combined with the Schrodinger picture ψ_s which is appearing here gives back me the interaction picture wave state so I get this extra factor I upon $\hbar$ times $\mathscr{H}_0$ times the interaction picture wave function back. And the second part when the time operator hits the Schrodinger state ψ_s, I know the Schrodinger evolution happens with the Schrodinger evolution happens with the full Hamiltonian. So, that is what we have to do. But let us keep waiting and let us see where do we get up with this. So, the first term is this as we have seen. In the second term, I know that this will be equal to, so here let us elaborate upon this piece and let us see what do we get out of it. So, $e^{iH_{0t}/\bar{h}}$ I acting upon $d\psi_s(t)/dt$ and I know that Schrodinger picture del ih upon dt of ψ_s is equal to the full Hamiltonian times $\psi_s(t)$ is equal to 0.So, it is not time t is equal to 0 at the same time ψ_s. So, therefore, I can put for this I This expression can be written as $e^{iH_{0t}/\bar{h}}$ and for the time derivative of the Schrodinger picture I can write it as *-i/ħ* which is *i/ħ* going to the right hand side will become -i/ $\hbar$ then I will get a h times ψ_s which can further be written as $e^{-i\hat{H}_0 t/\bar{h}}$ then what I do I have this $\hbar$ here $\mathscr{H}_0$ operator here I insert a $e^{-i\hat{H}_0 t/\bar{h}}$ and $e^{iH_{0t}/\bar{h}}$ plus then there was a ψ_s I have just inserted an identity in between what does it do this part if I ook at it is just again the ψ_i back so what I am supposed to get for this. I am going to get $e^{iH_{0t}/\bar{h}}$ will be common outside then $e^{iH_{0t}/\bar{h}}$ then there will be an h and then there will be I $e^{iH_{0t}/\bar{h}}$ which is here and then the ψ interaction which is this. So, you see ultimately I do get after the time derivative of things I do end up getting these two terms. *iħ I* upon $\mathscr{H}_0$ acting on sin direction and then the h, the full Hamiltonian squeeze between the free unitaries $\hat{U}$ and $\hat{U}_0^\dagger$ and sin interaction back. So, together if I combine them, I will get, so since this $\mathscr{H}_0$ can also be written I This $\mathscr{H}_0$ can also be written as $e^{iH_{0t}/\bar{h}}$ and $e^{-iH_{0t}/\bar{h}}$ So, I will use this. So, ultimately you can see both sides will have similar kind of structure, only thing that here it will be $\mathscr{H}_0$ and here it will be h.

So, I will get $e^{iH_{0t}/\bar{h}}$ from the left and $e^{-iH_{0t}/\bar{h}}$ on the right and in between I will have a $\mathscr{H}_0$ - h. So, that would be the, this would be the, so that will be the full description of the time derivative of the interaction. So, probably I am missing a sign in the next step, but anyway. This $\mathscr{H}_{0\,g}$ - H is - of the potential. So, there should be a - over here. A - of the potential because remember the full H was $\mathscr{H}_0$ plus the potential. And then the potential gets squeezed between $\mathscr{H}_0$(t) and $e^{-iH_{0t}/\bar{h}}$ to give rise to the V interaction. I will get a V interaction over here and left hand side therefore this should not have been there. So, left hand side therefore will be d upon dt of ψ it which now can be neatly written as *iħ I* d dt of $\psi_{i\,I}$ that is equal to vi and ψ_i. The boxed equation and this equation are the same thing. So, now you see the interaction picture I is like a Schrodinger equation evolution with the role of Hamiltonian played by the interaction potential, not just the potential but its interaction version. Interaction version meaning the potential is squeezed between $\hat{U}_0^\dagger$, $\hat{U}_0$, $\hat{U}_0$ and $\hat{U}_0^\dagger$. The way we had defined $\hat{U}_0$ and $\hat{U}_0$ I was the other way. So, let me write it once more. It is $\hat{U}_0^\dagger \hat{V} \hat{U}_0$. So, this V_i plays the role of the Hamiltonian for interaction gate. The interaction wave function evolves as it is undergoing a Schrodinger evolution with only the perturbation or the potential terms interaction version playing the role of the Hamiltonian. Okay all right so the information which we just obtained by going to the interaction picture that the interaction gets ψ_i time evolves with V_i , $V_{i\,I}$ *is I* the role $V_{i\,I}$ is the potential or the I perturbation Hamiltonian and its interaction time evolved version and that plays like a role and of Hamiltonian for the ψ_i and you can evolve the system with respect to this new Hamiltonian and obtain at a future time what would be $\psi_{i\,t}$. And since I know the relation between ψ_i t and the Schrodinger $\psi_s(t)$, $\psi_i(t)$ and schrodinger $\psi_s(t)$ are related by this operation if I know the left hand side I will obtain the right hand side by just taking $e^{-iH_{0t}/\hbar}$ or better way of saying that multiplying $e^{-iH_{0t}/\hbar}$ to both the sides so therefore I will also know this what is the schrodinger time evolved wave function so interaction time evolved wave function will be obtained from the perturbation Hamiltonian its interaction version and multiplying the three parts I unitary, I will get the Schrodinger picture wave function, its time evolved version. So, this is a useful tool as we will see that if I just focus about on the time evolution of the interaction picture, I then as we saw that the time evolution is being governed by a unitary operator which is made from the perturbation potential or perturbation Hamiltonian completely.I am going to write it like U_i. So, that is the $\hat{U}_I$ between time t_0 and t. So, you start in the interaction picture some wave function or the ket at time t_0 and some unitary operator takes $\hat{U}_I$ to some future I get ψ_{it}. So, therefore, if I take the derivative of this on the left hand side with the $i\hbar$ *I* times del del t. So, right hand side I will get $i\hbar$ *I* upon del upon del t since there are two variables and I am taking the derivative with respect to t and while t_0 is a fixed quantity I can convert it into a partial derivative. And while $\psi_i(t_o)$ does not depend on t. So, only derivative hits the unitary operator $\hat{U}_I$. So, therefore, I get the structure that the time evolved version $\psi_i(t)$, $\psi_i(t)$ is just this much, its time derivative is this much which I put on the left hand side and right hand side is just the interaction Hamiltonian as I know V_i and $\psi_i(t)$ which is this. So, this is I the derivative of $\psi_i(t)$ and this is just $\psi_i(t)$. So, I bring then this equation right hand side into the left in order to write neatly as $i\hbar$ *I* time derivative of the unitary - the interaction Hamiltonian $V_{i\,I}$ times the unitary they should together act on the interaction ψ and that annihilated. I So, that means this operator acting on ψ_i should be 0. This is just coming from this expression and writing ψ_i as the unitary times ψ_i at initial time. And now this equation should be true for all wave function not for just wave function. I am not looking for a solution of this equation for what wave function it happens. I am demanding that it should happen for all wave function because given any wave function I can go to its interaction version and that interaction version will evolve time evolve like this and I will get this equation. So, if this equation has to be true for all wave functions of the universe that can only mean that this operator itself has to be 0. That means the time derivative of the unitary should be related to the potential, interaction potential times the unitary itself. So, should be unitary over here. So, then I am just dropping the I notation in the $\hat{U}_I$ because I will not write $\hat{U}_I$ every time. I just write $\hat{U}_I$ and it should be clear that we are talking about $\hat{U}_I$. So, there this equation is satisfied by the $\hat{U}_I$ that I time evolution operator. In case of free Hamiltonian, I know that time evolution operator is the exponential of the free Hamiltonian. In the interaction picture, in a presence of perturbation, in presence of interaction picture, I am trying to find out what is the unitary. So, what properties of unitaries we are looking for? We want the unitary to satisfy a trivial identity that if I do not move in time, nothing should happen to the wave function. That means the unitary at of no time evolution, t to t_0 should be identity operator. So, this should be 1 or identity operator. Further, as we have just seen from the previous expression, this should be respected. That means time derivative of unitary should be equal to this, right hand side which is this.

Dyson Series

Feeding the $\hat{U}(t_1, t_0)$ back into its integral expression

$$\hat{U}(t,t_0) = 1 - \frac{i}{\hbar}\int_{t_0}^{t} dt_1\, \hat{V}_I(t_1) \overbrace{\left(1 - \frac{i}{\hbar}\int_{t_0}^{t_1} dt_2\, \hat{V}_I(t_2)\, \hat{U}(t_2,t_0)\right)}^{\hat{U}(t_1,t_0)}$$

$$= 1 - \frac{i}{\hbar}\int_{t_0}^{t} dt_1\, \hat{V}_I(t_1) + \left(-\frac{i}{\hbar}\right)^2 \int_{t_0}^{t} dt_1\, \hat{V}_I(t_1) \int_{t_0}^{t_1} dt_2\, \hat{V}_I(t_2)\, \hat{U}(t_2,t_0)$$

Again feeding $\hat{U}(t_2,t_0)$ back into this expression again and again

$$\hat{U}(t,t_0) = 1 - \frac{i}{\hbar}\int_{t_0}^{t} dt_1\, \hat{V}_I(t_1) + \left(-\frac{i}{\hbar}\right)^2 \int_{t_0}^{t} dt_1\, \hat{V}_I(t_1) \int_{t_0}^{t_1} dt_2\, \hat{V}_I(t_2)$$

$$+ \left(-\frac{i}{\hbar}\right)^3 \int_{t_0}^{t} dt_1\, \hat{V}_I(t_1) \int_{t_0}^{t_1} dt_2\, \hat{V}_I(t_2) \int_{0}^{t_2} dt_3\, \hat{V}_I(t_3) + \cdots$$

$$= 1 + \sum_{n=1}^{\infty} \left(-\frac{i}{\hbar}\right)^n \int_{t_0}^{t}\int_{t_0}^{t_1} \cdots \int_{0}^{t_{n-1}} dt_1 \cdots dt_n\, \hat{V}_I(t_1) \cdots \hat{V}_I(t_n)$$

Dyson Series

Feeding the $\hat{U}_I(t,t_0)$ back into it's integral expression

$$\hat{U}_I(t,t_0) = 1 - \frac{i}{\hbar}\int_{t_0}^{t} dt_1\, \hat{V}_I(t_1) 1 + \left(-\frac{i}{\hbar}\right)^2 \int_{t_0}^{t} dt_1\, \hat{V}_I(t_1) \int_{t_0}^{t} dt_2\, \hat{V}_I(t_2)\, \hat{U}(t_{2,}t_0)$$

Again feeding hat $U_I(t,t_0)$ back into the expression again and again.

$$\hat{U}_I(t,t_0)=1-\frac{i}{\hbar}\int_{t_0}^{t} dt_1\,\hat{V}_I(t_1)1+\left(-\frac{i}{\hbar}\right)^2\int_{t_0}^{t} dt_1\,\hat{V}_I(t_1)\int_{t_0}^{t} dt_2\,\hat{V}_I(t_2)+\left(\frac{-i}{\hbar}\right)^3\int_{t_0}^{t} dt_1\,\hat{V}_I(t_1)\int_{t_0}^{t} dt_2\,\hat{V}_I(t_2)\int_{0}^{t_2} dt_3\,\hat{V}_I(t_3)$$

$$=$$

$$1+\sum_{n\neq 1}^{\infty}\left(\frac{-i}{\hbar}\right)^n\int_{t_0}^{t}\int_{t_0}^{t_1}\ldots\int_{0}^{t_{n-1}} dt_1 dt_2\ldots dt_n\,\hat{V}_I(t_1)\ldots\hat{V}_I(t_n)$$

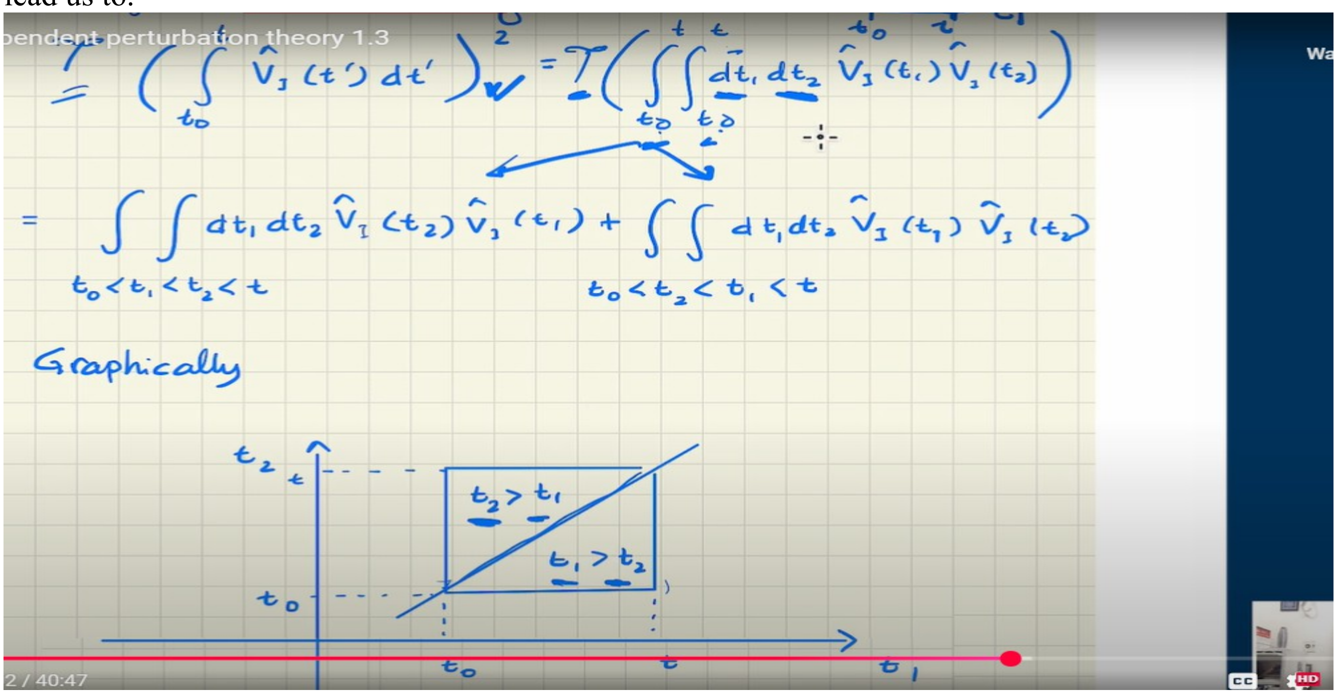
That means if I time, if I do this integration over time on this side and integration of time on that side between t_0 to t, left hand side I will get I the unitary at time t - unitary at time t_0. Unitary at time t_0 is identity t_0, therefore, I can write as unitary as identity or *1 -iħ* and this integration. So, this is the structure of the unitary. It is an integral equation that unitary at any stage is known from the integration of the convolution of I the potential, interaction potential and the unitary itself. That means it is an iterative equation that I have this information. So, you see this is the t_1, this is the t_1, this is the t_1 and this t_1 is undergoing integration from t_0 to P. So, in between whatever was the unitary, its total value determines what will be the unitary at a future time between t_0 to t, dt_1 takes the value and unitary at all points during t_0 to t, unitary along all points during t_0 to t together combines with the respective interaction potential at all times and gives me a unitary at time t. So, it is a convolution, it knows about its past, it knows about the history because the convolution goes from t_0 to t with a variable t_1.

So, therefore again I can use the same expression which we have just written here. The same expression I can feed it back for u of t_1. $U(t)$ *is* the integration of identity - *iħ I* over this integration. This I can feed up for $U(t_1)$. That means I can write down I 1 - *iħ I* times V_i was there. Here it was sitting as $\underline{U(}t_1, t_0)$. So, therefore I can write u of t_1, t_0 as again identity - *iħ* . This time integration from running from t_0 to t_1. Previously the integration was running from t_0 to t. Since I want to know what is U at t_1 here. So, therefore I have to run the integration from t_0 to t_1. So, then a new dummy integration variable t_2 will appear and this will contain a unitary up to time t_2 and t_2 runs from t_0 to t_1. So, you see again I have obtained a $U(t_2)$ which I can again feed in this equation expression and iteratively it becomes a bigger and bigger and bigger kind of integrations. So, ultimately if I start collecting things, this one will be there, then this *iħ I* times this quantity is multiplying the whole bracket. In the whole bracket, there is an identity which first multiplies this outside integral and I get the single integral dt_1 of interaction potential V_i . Then this term here multiplies the outside term, then I would get a -$(i/\hbar)^2$ whole square, *1-i/ħ f*rom here and *1-i/ħ* from here. So, together they get the squared quantity.

Then double integral, one integral here and one integral here. So, dt_1 of the interaction potential and dt_2 of the interaction potential times the unitary up to second integration limit. And again you can feed I the same expression back into $\hat{U}\,e^{t_0}$ and this loop will start growing more and more. So, you will get first 0 integral, single integral, double integral, again you write as a 1 - integration of a third quantity, you will get a triple integral, then the fourth integral, fifth integral and ultimately it will keep building up. So, you will get this kind of a structure, I *1-i/ħ* single integral, - $(i/\hbar)^2$ whole square double integral, *-i/ħ* cube triple integral, similarly fourth integral, fifth integral and so on. And ultimately you can write it as a neat sum as identity or 1 plus a power series in - $(i/\hbar)^n$ and n integrals of $V_{i.}$ s. This is the structure you will ultimately end up getting and the last time you will do this would be the you see the integration limit is shrinking initially it was the full range from t_0 *to* t_1 *then* t_1 was t_0 *, T to t* then inside that the second integral the second integral came up with somewhere in between t_1, this was just t. So, in between there was some t, t_1 and the integral ran from t_0 to t_1. In the third part even the t_2 came between t_0 and t_1. So, the integral ran from t_0 to t_2. So, similarly the third integral will further go smaller, it will run from t_0 to t3. And fourth integral will be t_0 to t_4 where the size of the integral will keep shrinking down and down and down and ultimately it will reduce to a point t_0 itself where the unitary is supposed to be 1. So, ultimately you will get this series and the final u at t_0, t_0 which was 1. So, therefore I have a power series expansion.

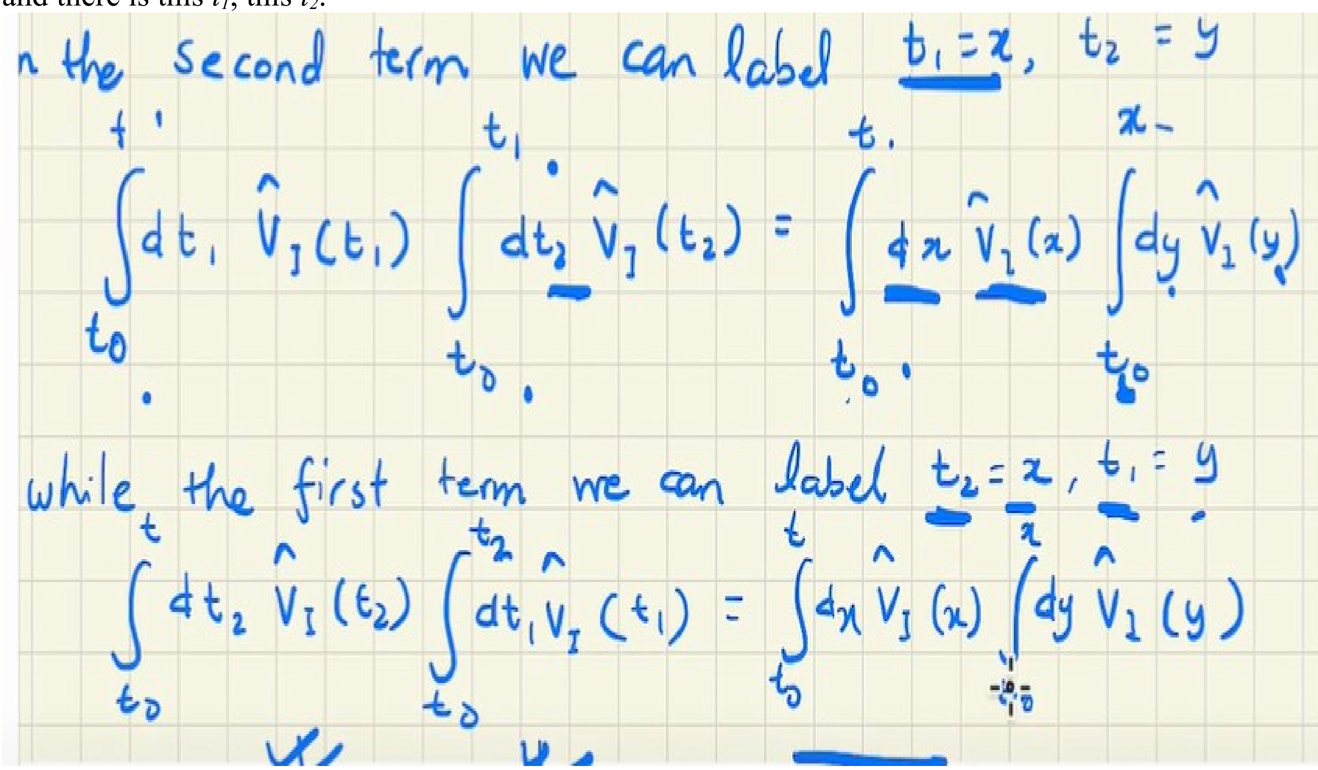
So, I know the unitary from t_0 to t I as this simple power series expansion $-(i/\hbar)^n$, the n integrals dt_1 dt n and then the V_i is in between t_1 tn. I So, this is the structure of the unitary. Remember in the ordinary free theory unitary operator was just the exponential of the free Hamiltonian. That means in free theory $\hat{U}_0$ was $1 - (i/\hbar)^n$ and $(\mathscr{H}_0)^n/n!$. So, we are trying to obtain the version of the unitary I time evolution operator which used to be the plain simple exponential in the free theory. So, in the interaction picture some u which is actually u of interaction which for which we have obtained expression which is this integral power series expansion. Let us try to see if we can bring it into more neater form or not. And in order to do that we are going to define something called a time ordering that means I which way we order time in the integrals. For example, let us take these two double integral that is two integrals square. $V_i dt'$, t_0 to t and its square would be just square.

So, I will do integration twice, one with respect to t_1 running from t_0 to t and another time again with respect to another variable dt_2 with that is also running for the same limit. That means that is meant by the square. Remember in our expression we do not have this square, we have for example in double integral the first limit runs from 0 to t I while the second limit runs from t_0 to t_1 or you can interpret in other way that dt_2 integral runs from t_0 to t_1 and while dt_1 integral runs from t_0 to t. So, they are not running over the same limit, the second integral is running up to first variables up to t_1 which becomes the first variable into the second equation, second integration and that integration runs from t_0 to t. We do not have a square at hand, but still I am just writing a square over here and try to see where does that lead us to.



$$\Gamma' = \left(\int_{t_0}^{t} \hat{V}_I(t')\right) = \Gamma\left(\int_{t_0}^{t}\int_{t_0}^{t} dt_1 dt_2 \hat{V}_I(t_1)\hat{V}_I(t_2)\right) = \int\int_{t_0<t_2<t_1<t} dt_1 dt_2 \hat{V}_I(t_1) V_I(t_2)$$

So, that square can be written as just a plain simple double integrals of the same limits. Then I define something called a time ordering on both these quantities. This symbol curly t is a time ordering. That means I will order the integral along the time direction that is I will do the dt I up to the limit till which it is greater than the dt_1 integral and then vice versa. It will become in a minute what does it mean according to the time ordering. So, the same integral which have the same limit t_0 to t and t_0 to t. That means I am thinking of two variables. One is the variable t_1 and another is variable t_2. Both start from t_0. and this goes up to t and the t_2 also goes up to t. So, I am thinking of a area of this square where both the sides are t_0 to t, the first integral and the t_0 to t along then t_2 which is the second integral. So, I have to obtain the integration inside this square box. This square box can be broken into two triangles, a triangle in the upper half I and a triangle in the lower half. In the upper half, following is true that t_2 is greater than t_1.In the lower half, the other is true that t_1 is greater than t_2. I So, therefore, this double integral of the same limit can be broken down into double integral over the two triangles. First triangle where t_2 is greater than t_1 and the second triangle where t_1 is greater than t_2. So, this is the upper triangle and this is the lower triangle. So, here it was integral over the whole square. So, now you see the picture like structure is coming about. We had similar kind of integration where t_2 was running from t_0 to t_1 and t_1 was running from t_0 to t. So, you see that in all this thing t_2 was always smaller than t_1, its upper limit was t_1. So, this integral here is like the integral where t_2 is smaller than t_1, the lower limit, the lower triangle integral. So, we had this integral at hand, while I know the double integral which I have written as a square over here is made up of two pieces, out of that one piece the lower triangle looks like the integral which we want. So, what do we do with the upper triangle? So, the upper triangle can also be converted into a lower triangle integral because you see the integrands are the same t_2, t_1 and there is this t_1, this t_2.



The second term we can label $b_1 = x$, $t_2 = y$

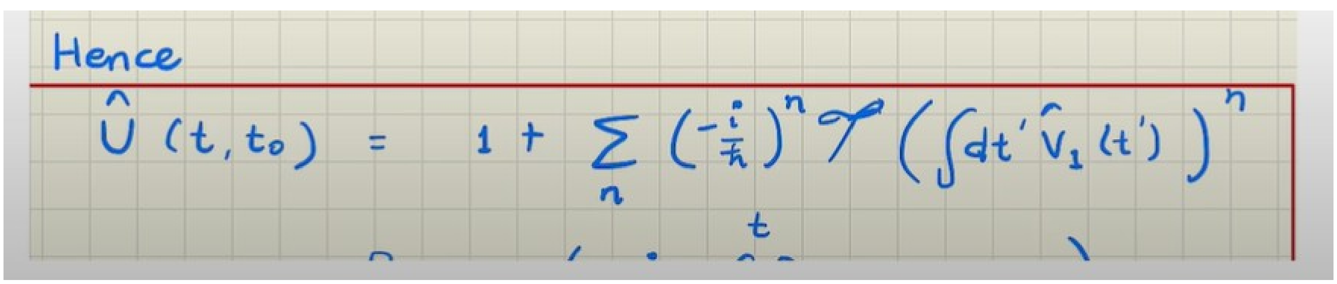
$$\int_{t_0}^{t} dt_1 \hat{V}_I(t_1) \int_{t_0}^{t_1} dt_2 \hat{V}_I(t_2) = \int_{t_0}^{t} dx \hat{V}_I(x) \int_{t_0}^{x} dy \hat{V}_I(y)$$

While the first term we can label $t_2 = z,\ b_1 = y$

$$\int_{t_0}^{t} dt_2 \hat{V}_I(t_2) \int_{t_0}^{t_2} dt_1 \hat{V}_I(t_1) = \int_{t_0}^{t} dx \hat{V}_I(x) \int_{t_0}^{x} dy \hat{V}_I(y)$$

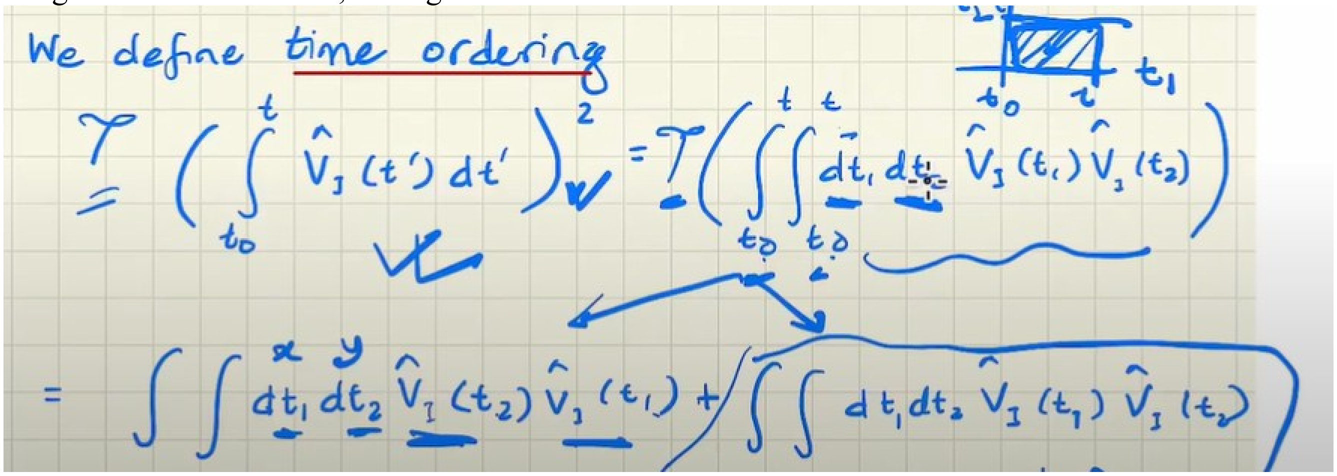
What I can do? I can relabel call this thing as x and this thing as y. Then therefore, this integral in the, so let us say first look at this lower triangle integral. Then the integral which we are running from t_2 is running from 0 to t_1 and then t_1 is running from t_0 to t. I am doing a variable transformation in which I am going to call t_1 as x and t_2 as y. Then this integral in the lower triangle is equivalent to integration of dx with vi of x and x runs from t_0 to t while the second integral is dy. vi of y where y runs from t_0 to x. So, this is the integral form. I am just calling t_1 as x and t_2 as y. While for the upper triangle, I can again do a relabeling. In the upper triangle, I am going to call t_2 as x and t_1 as y. If I do that, you can realize that even the upper integral becomes the same thing. There is a dy which runs from t_0 to x like here and this x runs from t_0 to t like here. I So, you see the upper triangle and the lower triangle are not different, they are exactly one and the same thing. So, therefore, the full square is just double of one of the triangle, either upper one or the lower one. So, I am going to write the full square as the time ordered full square, the time ordered full square as twice of the lower integral. And therefore the twice of the just the lower integral will be one half of the time ordered squared.

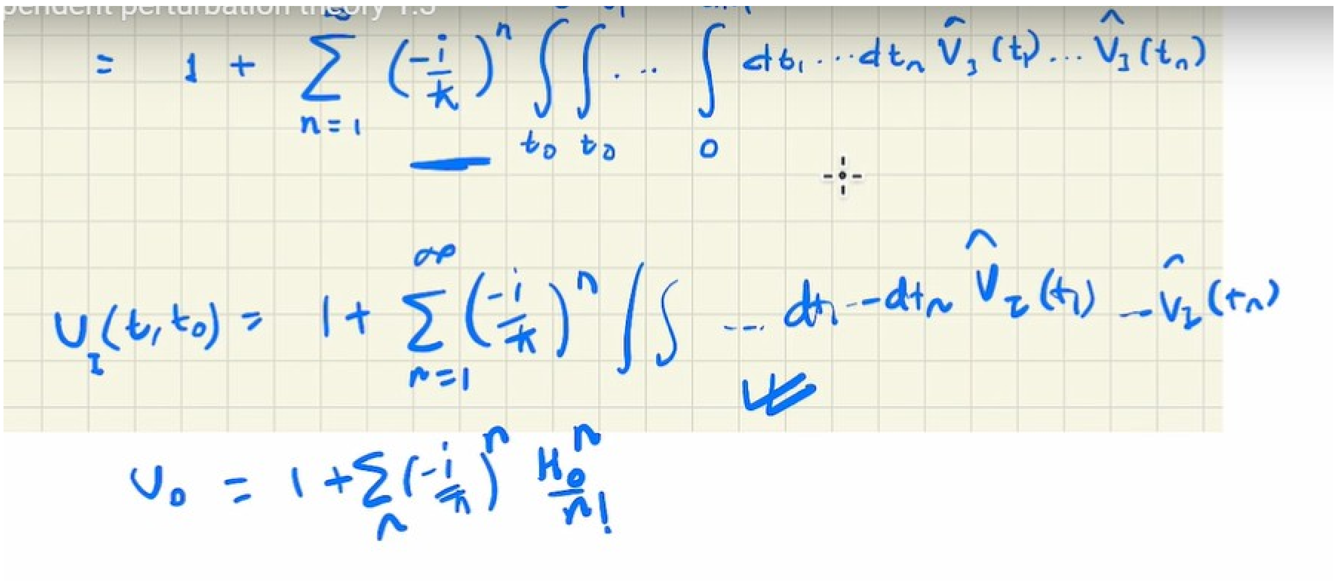
So, the integral which was appearing in my unitary which was here this integral, this double integral is just the lower triangle double integral we just obtained which is one half of the full square, both the limits being same and the full square. So, therefore I I can write the first term after identity as 1 by 2 of the whole square of the integration Hamiltonian, the interaction picture Hamiltonian. Remember interaction picture Hamiltonian is just vi.



$$\hat{U}_I(t, t_0) = 1 + \sum_{n \neq 1}^{\infty} \left(\frac{-i}{\hbar}\right)^n \Gamma\left(\int dt' \hat{V}_I(t')\right)^n$$

So, this is what we get. Similarly, for 3 triple product integral which is appearing over here, I triple product integral which is appearing in the next term over here, then that can be shown in by the same logic. It is 1 by 3 factorial times of the time ordered of cubic product. So, the triple integral which was appearing over here would be 1 by 3 factorial times cubic product of the same thing. Similarly, 4 integrations will be 1 by 4 factorial I quartic product of the same thing and similarly the nth integral that nth integral which appears in my expression it will be just time ordered 1 by n factorial times time ordered version of n product of the same integral. So, therefore the power series expansion which we just wrote down which was here I I saw it is 1 plus - iota upon $\hbar$ to the power n and then n product. And now I know n product is nothing but 1 by n factorial times the time ordered of n in the product of n integrals of the same limit, n integrals of the same limit.





We define time ordering

$$\Gamma' = (\int_{t_0}^{t} dt' \hat{V}_I(t'))^2 \quad = \quad \Gamma(\int_{t_0}^{t}\int_{t_0}^{t} dt_1 dt_2 \hat{V}_I(t_1)\hat{V}_I(t_2)) \quad =$$

$$\int\int dt_1 dt_2 \hat{V}_I(t_2)\hat{V}_I(t_1) + \int\int dt_1 dt_2 \hat{V}_I(t_1)\hat{V}_I(t_2)$$

$$= \boldsymbol{I} \quad 1 + \sum_{n \neq 1}^{\infty} (\frac{-i}{\hbar})^n \int_{t_0}^{t}\int_{t_0}^{t_1} \ldots \int_{0}^{t_{n-1}} dt_1 dt_2 \ldots dt_n \hat{V}_I(t_1)\ldots\hat{V}_I(t_n)$$

$$1 + \sum_{n \neq 1}^{\infty} (\frac{-i}{\hbar})^n \int_{t_0}^{t}\int_{t_0}^{t_1} \ldots \int_{0}^{t_{n-1}} dt_1 dt_2 \ldots dt_n \hat{V}_I(t_1)\ldots\hat{V}_I(t_n)$$

$$U_0 = 1 + \sum_{n} (\frac{-i}{\hbar})^n \frac{H_0^n}{n!}$$

Therefore, the unitary which we are looking for, the interaction picture unitary which we are looking for $\sum_n (\frac{-i}{\hbar})^n$ I and the time ordered version of the n expansion, the n product of the integral. The time ordering operator can be taken out then it just becomes 1 plus power series expansion of I had missed 1 by n factorial 1 – $(i/\hbar)^n$ I and then this integration of the interaction $\mathscr{H}_0{}^n/n!$ factorial which ultimately gives me exponential of the integration of the Hamiltonian in the interaction picture. Remember again the potentials interaction version plays the role of the potential in interaction picture. So, therefore the unitary operator in the interaction picture I is not just the exponential but time ordered version of the exponential of the effective Hamiltonian in the interaction picture. Remember again I am to compare in the free theory it was just $e^{-i\hbar/p}$. This would have been the operator of time evolution. In interaction picture role of $\mathscr{H}_0$ is being played by the V and role of exponential is going to be played by the time ordered exponential and while the role of t is being played by this integration. So, ultimately I have a unitary operator which resembles something like the unitary of the free theory but not quite because time ordering has come up and the integration in the exponential has come up. And therefore, I can using this unitary, I can take this unitary act on initial state in the interaction picture and get the final state in the interaction picture. I And once I have the final state in the interaction picture, I can get the final state in the Schrodinger picture by just plain simple multiplication of the free unitary $\hat{U}_0$.

Hence

$$\hat{U}_I(t,t_0) = 1 + \sum_n \left(-\frac{i}{\hbar}\right)^n \mathcal{T}\left(\int dt' \frac{\hat{V}_I(t')}{n!}\right)^n$$

$$= \mathcal{T} \exp\left(-\frac{i}{\hbar}\int_{t_0}^{t} \hat{V}_I(t')\,dt'\right)$$

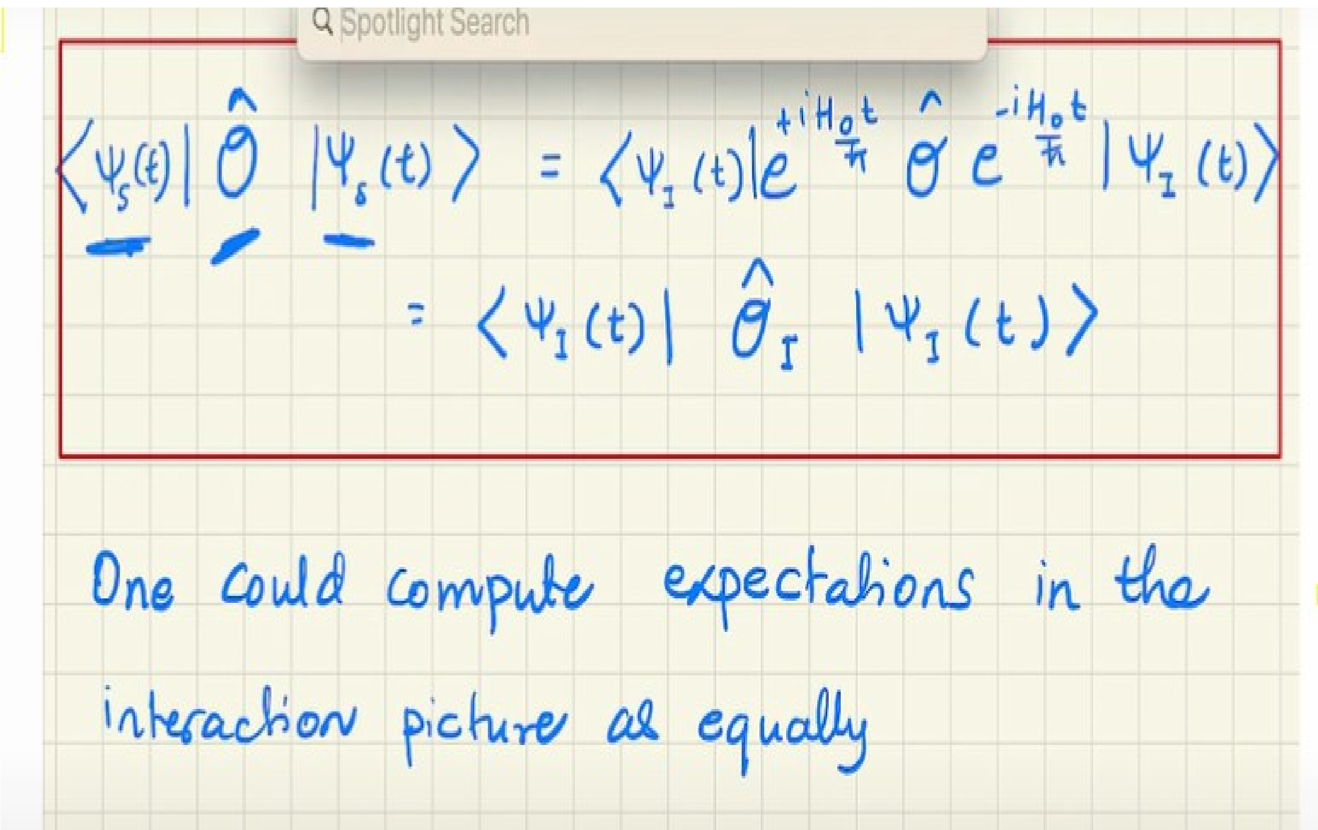
$$|\psi_I(t)\rangle = \hat{U}(t,t_0)\,|\psi_I(t_0)\rangle$$

$$\hat{U}_I(t,t_0) = 1 + \sum_n \left(\frac{i}{\hbar}\right)^n \Gamma\left(dt'\frac{\hat{V}_I(t')}{n!}\right)^n = \Gamma\, e^{\frac{-i}{\hbar}\int_{t_0}^{t}\hat{V}_I(t')dt'}$$

$|\psi_I(t)\rangle$ I $\hat{U}_I(t,t_0)$ $|\psi_I(t_0)\rangle$

So, this is a game in which I can just focus on the potential or the perturbation Hamiltonian do its I time evolution with the free Hamiltonian, free unitary $\hat{U}_0$ and then I get a vi which behaves as if it is the Hamiltonian of interaction ket ψ_i and that ψ_i is related to the ψ_s through a unitary multiplication. Why we have done all this exercise?

Because you can prove that if I am wanting to know expectation value of any operator, whether I can do in Schrodinger picture I I can take this operator squeeze it between time evolved Schrodinger states, but you can prove that by multiplication I know that $\psi_s(t)$ is nothing but $e^{-iH_{0t}/\hbar}$ $\psi_i(t)$. Remember $\psi_i(t)$ was defined as $e^{-iH_{0t}/\hbar}$ acting on $\psi(t)s$.. So, therefore, if I reverse this game, I will get $\psi(t)s$ is $e^{-iH_{0t}/\hbar}$ $\psi_i(t)$. I $\psi_i(t)$. And this $\psi_i(t)s$, I feed here and I feed here. Then I get $\psi_i(t)$, $\psi_i(t)$and then the free Hamiltonian unitary on left and right and this becomes the interaction picture operator. So, if I want to know expectation value, I can entirely work in the Schrodinger picture or I can entirely work in the interaction picture.



Moreover

$$\boxed{\langle \phi_1(t) | \Psi_I(t) \rangle = \langle \phi_s(t) | e^{\frac{iH_0 t}{\hbar}} e^{\frac{-iH_0 t}{\hbar}} | \Psi_s(t) \rangle = \langle \phi_s(t) | \Psi_s(t) \rangle}$$

All inner products can also be computed in the interaction picture

$$< \psi_s(t)\, \hat{O}\, |\psi_s(t)> = < \psi_I(t)|\, I\, e^{+iH_0\frac{t}{\hbar}} \hat{O}\, e^{-iH_0\frac{t}{\hbar}} |\psi_I(t)> \;= < \psi_I(t)|\, \hat{O}_I\, |\psi_I(t)>$$

One could compute expectations in the interaction picture as equally.
Moreover

$$<\phi_I(t)|\psi_I(t)> =<\phi_s(t)|\; e^{+iH_0\frac{t}{\hbar}} e^{-iH_0\frac{t}{\hbar}} \;|\psi_s(t)> =<\phi_s(t)|\psi_s(t)>$$

All inner products can also be computed in the interaction picture.
I just compute what is the interaction wave functions and what is the interaction operator. Their expectation value will correctly give me the expectation value of the operator in the Schrodinger picture. and it so happens in some of the interactions some of the discussions which we will have doing computation in the interaction picture is more convenient than doing computation in the schrodinger picture none of them are different from each other but mathematical convenience will be a guiding philosophy so we will realize that for many of the computation which we will discuss later on interaction picture becomes a smaller computation so in using this philosophy and using this knowledge that the expectations value do not change from one picture to another picture I can compute things in schrodinger picture or I can do computation of things in interaction picture answers will not change similarly one can prove that even the inner product between the wave functions do not change under schrodinger picture or interaction picture so therefore the probability is the projections, the probability, the coefficients, all these things in interaction picture or Schrodinger picture will remain the same. So, therefore, I am not going to lose any information by going to the interaction picture where we will see in later classes I that many of the computations become much more convenient and small.

So, better to work in interaction picture rather than a lengthy Schrodinger picture because ultimately all information in both two pictures are the same. I can work in any of this and I will choose to work in interaction picture in later discussions in this course and there when you see do not worry about why we are working in interaction picture because all the information in interaction picture gets exactly mapped to the information we are looking in the Schrodinger picture. So, let us stop over here for this analysis of interaction picture and the Dyson series. The Dyson series was this integral expansion of time ordered potential exponential. So, we stop over here for this class of time dependent perturbation series.