Foundation of Quantum Theory: Relativistic Approach

Shift in eigenenergies via field interaction

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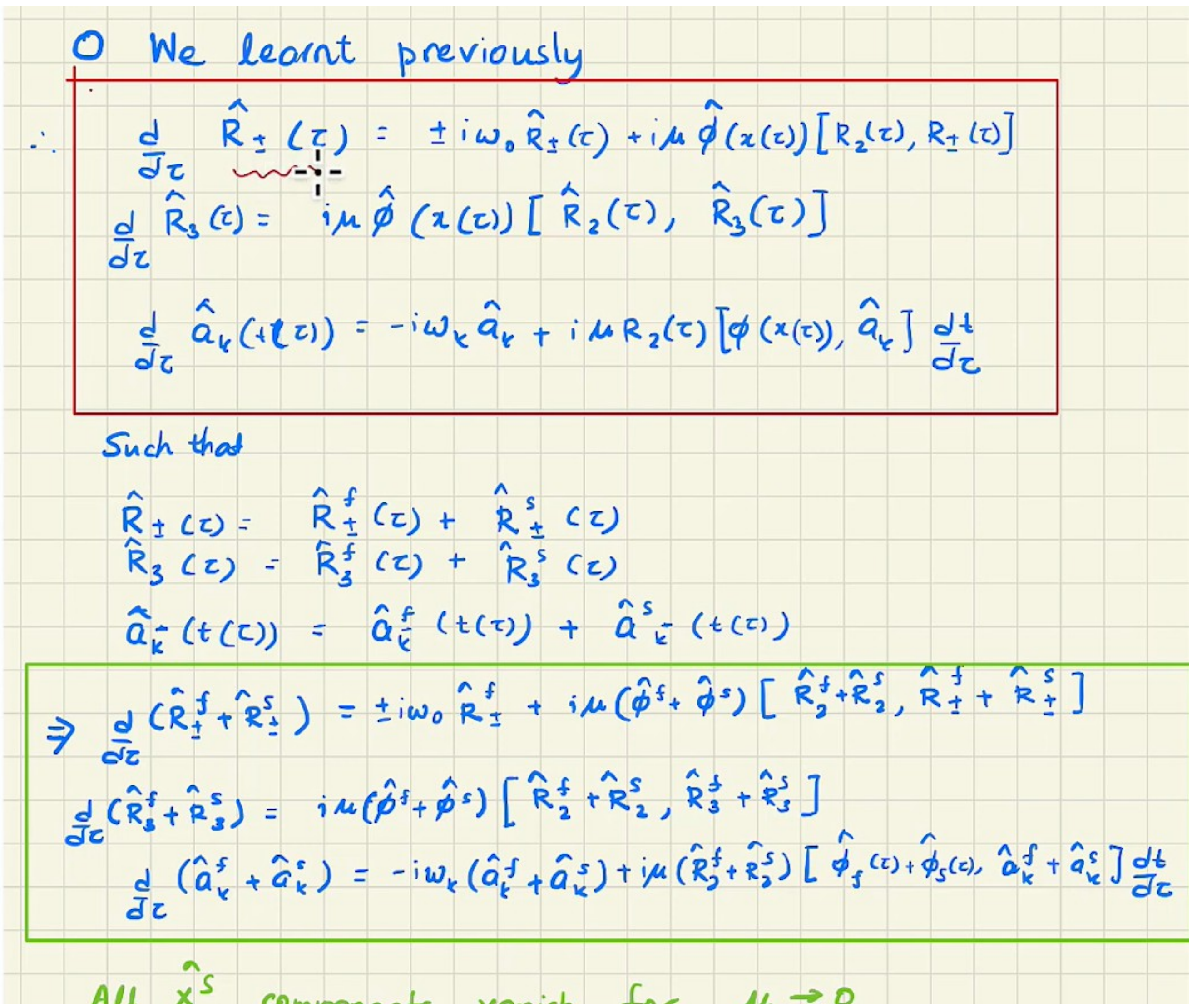
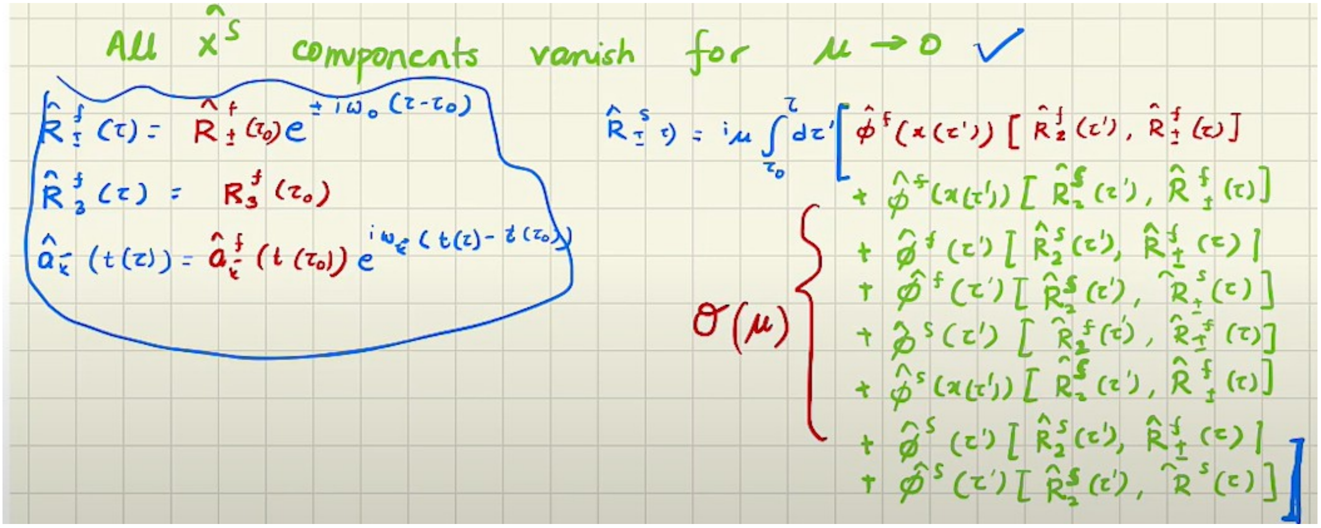
DDC formalism

Lecture- 42

So, in this last discussion today, we are going to touch upon the couple of salient features of atom's interaction with background quantum fields and inherent quantum fluctuations therein



See Lecture 41



Eqn

All $\hat{x}^s$ components vanish for $\mu \to 0$

$$\hat{R}^f_\pm(\tau) = \hat{R}^f_\pm(\tau_0) = \hat{R}^f_\pm(\tau_0)\, e^{\pm i\omega_0(\tau-\tau_0)}$$

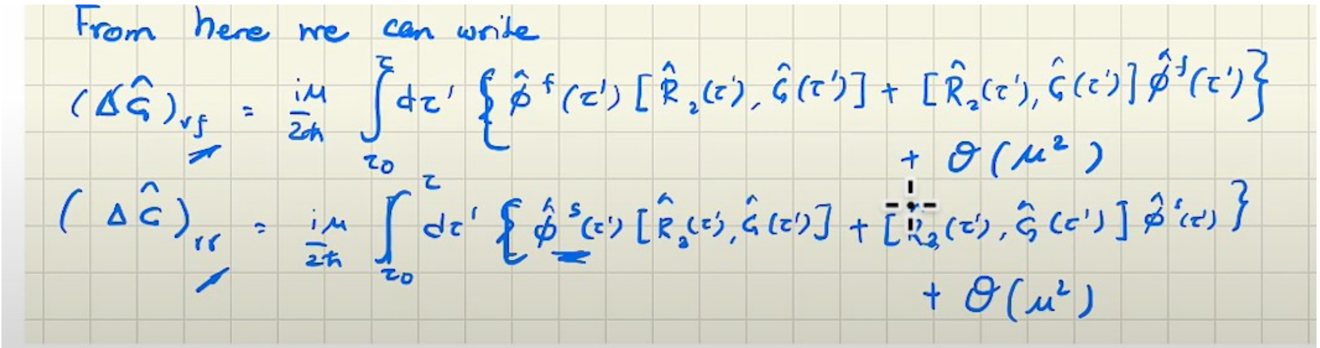
$$R^{\hat{f}}_{3(\tau)} = R^{\hat{f}}_{3(\tau_0)}$$

$$\hat{a}_{\vec{k}}(t(\tau)) = \hat{a}^f_k(t(\tau_0))\, e^{+\omega_{\vec{k}}(t(\tau)-t(\tau_0))} \quad]$$

$$R^{\hat{s}}_{3(\tau)} = i\mu \int_{\tau_0}^{\tau} \left[\hat{\phi}^f(x(\tau'))[\hat{R}^s_\pm(\tau'), \hat{R}^s_\pm(\tau)] \right]$$

$$\theta(\mu) \begin{cases} +\hat{\phi}^f(x(\tau))[\hat{R}^s_\pm(\tau'), \hat{R}^s_\pm(\tau)] \\ +\hat{\phi}^f(\tau')[\hat{R}^s_\pm(\tau'), \hat{R}^s_\pm(\tau)] \\ +\hat{\phi}^f(\tau')[\hat{R}^s_\pm(\tau'), \hat{R}^s_\pm(\tau)] \\ +\hat{\phi}^s(\tau')[\hat{R}^s_\pm(\tau'), \hat{R}^s_\pm(\tau)] \\ +\hat{\phi}^s(x(\tau'))[\hat{R}^s_\pm(\tau'), \hat{R}^s_\pm(\tau)] \\ +\hat{\phi}^s(\tau')[\hat{R}^s_\pm(\tau'), \hat{R}^s_\pm(\tau)] \\ +\hat{\phi}^s(\tau')[\hat{R}^s_\pm(\tau'), \hat{R}^s_\pm(\tau)] \end{cases}$$

So, in the previous set of discussions, we have learned that if matter and the field are talking to each other through say some monopole coupling or any other coupling, what happens for any observable both from the matter sector as well as from the atomic sector that they get changed



Eqn

From here we can write

$$(\Delta \hat{G})_{ij} = \frac{i\mu}{2\hbar}\int_{\tau_0}^{\tau} d\tau' \{\hat{\phi}^f(\tau')[\hat{R}_2(\tau), \hat{G}(\tau')] + [\hat{R}_2(\tau), \hat{G}(\tau)], \hat{\phi}^f(\tau')\} + \theta(\mu^2)$$

$$(\Delta \hat{G})_{\gamma\gamma} = \frac{i\mu}{2\hbar}\int_{\tau_0}^{\tau} d\tau' \{\hat{\phi}^s(\tau')[\hat{R}_2(\tau), \hat{G}(\tau')] + [\hat{R}_2(\tau), \hat{G}(\tau)], \hat{\phi}^s(\tau')\} + \theta(\mu^2)$$

So, the rate of change of both the sectors becomes different and we have operators belonging to the atomic sector, raising or lowering operator for example, they change in a particular way The eigenstate of the operators which are corresponding to the free Hamiltonian of atom that also changed in a particular way While from the field sector, the ladder operator also keeps changing in a particular way and all of them are sensitive to the coupling The atom is having with the background field So, you see if there is no coupling, there are three important elementary results which quickly become clear to us that the raising and lowering operator in the atomic side will just be related to their $i\omega$ energy gap difference The operator which is the free Hamiltonian of the atom would not change at all.
Why it should? Because it will commute with itself if there is no coupling There is total Hamiltonian will be that of a atom and that of a field and the Hamiltonian of the atom commutes with both of them So, it will not change at all And the atomic sector it will the ladder operators will change with this term only which is the feature of a pre-field theory If the theory of the fields is compromised with a correction term in the Lagrangian which is talking about the interaction with the atom, it will give you right to another correction term which is the sourced correction. So, all the three operators which are rather four operator $\hat{R}_+ \hat{R}_- \hat{R}_3$ in the atomic sector and d ladder operators $\hat{a}_k$ and $a_k^\dagger$ which can be derived from $\hat{a}_k$ can be written in terms of their free parts and in terms of additionally there are source part So,

you see free parts are without term Remember in this setting we are talking about times m times $\mu\hat{\phi}$ is some smallness parameter which we are keeping it handy because most of the time we are interested about linear order behavior And in also to see what terms are sourced by interaction So, is just a switching function of interaction Whenever is 0, there is no coupling between atom and the field So, you see most of the source, all the source terms which is appearing $\hat{R}_s$ in – operator or $\hat{R}_3$ operator or the ladder operator of the field has this source term All of them are necessarily a function of a $\mu\mu$ is 0, all these source terms are 0 So, therefore, we can put it these definitions into the equations just above and we will get a new set of equations connecting both the free part and the source first time evolution with whatever appears on the right hand side So, that is what we should start writing This also should be, by the way, there should be $\hat{R}_f$'s term also, which I have mentioned So, this – comes with the source term as well here, which I have mentioned Anyway, so every term, for example, here $\hat{R}_f$ was appearing, I have written $\hat{R}_{f+}$ $\hat{R}_f$, was appearing, I have written ϕ^f ϕ^f and $\hat{R}_2$, similarly $\hat{R}_\pm$ here as well. So, everywhere you will see whatever appears has been broken down into the free part and the source part So, everywhere you will see whatever appears has been broken down into the free part and the source part And we have assumed, we know that all the things which start with order or above, they are just the source term because they will vanish when goes to 0 That means if coupling turns off, whatever piece turns off, that is the source part So, that means free part is made only from independent terms While dependent terms are all the source part For example, if I am trying to write down for the time evolution of R's, the source part in $\hat{R}_\pm$, I will collect everything which depends on So, you see the first term, the first term over here is not, the first term over here is not function of a mu, only R's term is appearing over here Now, over here if I try to write down, you will get this thing that there is a over here and there is a ϕ^f ϕ^s, $\hat{R}^f$ $\hat{R}^s$, $\hat{R}^f{}_2\, s$ and $\hat{R}_\pm{}^f$ and $\hat{R}_\pm$ So, what I would try to start collecting things from, so I will start collecting things from d the dependent part only So, of course, when I had initially missed out this $\hat{R}_\pm{}^s$, that means in the free part there is no source term So, it should not exist as a independent term Therefore, it is not written over here Only thing it would appear in the dependent part which is here such that if tends to 0, there should not be R's If I had put it from the beginning, then the time derivative of $\hat{R}_s$ will depend on this $\hat{R}_s$ appearing over here It should not get appear because if coupling goes down, all the terms of $\hat{R}_s$ should go down That means $\hat{R}_s$ time devolution, the source terms time evolution is only controlled by this second piece over here So, if I open it up, there are six, actually eight terms So, two choices, whether I choose ϕ^f or ϕ^s here, $\hat{R}^f{}_2$ or $\hat{R}^s{}_2$ here and $\hat{R}_+$ S, F or $\hat{R}_+$ S over here So, you see eight choices can be written So, I have written all the eight choices.

So, now you see i is appearing outside and we have known that whenever $\hat{R}_s$ appears, $\hat{R}_s$ can only appear through an appearance of $\hat{R}_s$ itself starts at order So in all these eight terms, there are various sort of terms In this ϕ^f, here I have put pi $\hat{R}_f$ and here also I have put $\hat{R}_f$ Then I will get the first term In the second term, I have kept the first pi as f, the second r has become the source term and then the third r has remained the free part So you see one $\hat{R}_s$ has appeared Similarly, another case would be there when I can put the second $\hat{R}_s$ over here The second $\hat{R}_s$ over here actually There should be 8 terms, so first one is s, second one is f or it could be first one is f, second one is s or it could be both of them are s like this Then again if I start with a ϕ^s over here, so first one is *f*, second one is *f*, one source function is here, then second one, first one is s, second one is f Then the first one is f here and second one is s here And lastly both of them are s So, you see in this term there is no source term, only ϕ^f, $\hat{R}_f$, $\hat{R}_f$ appears In this one $\hat{R}_s$ term appears here In this also one $\hat{R}_s$ term appears which is here In this also, in this second term two $\hat{R}_s$ appears Here also, only one source function appear which is here In the second term, two source functions appear, one here and one inside the commutator Similarly, go below, two source functions appear, one here and one over here Lastly, three source functions appear here, here and here And each source function starts with order term So, you see the first term which is in red is order 0 and outside it gets multiplied with i So, overall first term is order All other terms are higher order terms

Either order square or order cube or order 4 That means when this coupling is linear order which is V I can just forget about all the green terms and only red term can be retained That means once I have put an $i\mu$ in the computation, all other fields can be put to their free forms such that that would be the leading order correction That would be the leading order source correction that would be generated So, that is what we do We write down all the interior commutators and the field appearing as their three forms because I do not want to put a source form inside that will make the correction term second order So, I am truncating my correction up to first order interaction Similarly, we have learned that if we go to the correction to any operator in the phase space of atom, d This would be now talking to the Hamiltonian of the atom as well as the coupling Hamiltonian because this also contains operators from the atomic side So, effectively these two commutators will decide the discourse of the time evolution of any operator in the phase space of atom As before, this can also be broken down into two pieces One is the free part and one is the coupling part And even in the coupling part, there are two terms which we see it can be broken into One is coming through the vacuums, so one is called the vacuum fluctuation term and another is called the self-reaction term or radiation reaction Vacuum fluctuation term is made from the free part of the background field and the forced correction induced changes in the operator G They are called the radiation reaction because the field itself gets changed slightly because of interaction and this give rise to a pi of s and that pi of s in turn decides the change in the operators sector at operators in the atomic sector So, therefore, this is radiation reaction that change in the background photonic field or any background field causes the change in the operators rate of d tau of any operator that is sourced by d the correction term, this is radiative correction in the field, while the free part of the field also determines the change in the operator, but that is named as a vacuum fluctuation generated This is intrinsic to the background field while this is due to the coupling between the atom and the field because you see d ϕs is itself coming because of a coupling of atom to the field And therefore, the second term, this is radiation reaction term is a second order thing Because ϕs itself is a first order effect of change This is due to coupling If there is no coupling, ϕs will be 0 So, that means it starts at order μ So, overall the second, this term is order μ^2 While in the first term, everything is free part And therefore, this is order μ term So, vacuum fluctuations are slightly dominating correction But there is a second order correction overall which is hiding which is radiation reaction correction to any operator So, therefore, from this expression we can see that we can integrate this equation out and we can find out what is the change in any operator due to vacuum fluctuation part and radiation reaction part which is just the integration of these two equations And higher order effects are ignored in this Only the source part is written here and inside I do not write any other higher order thing For example, there could have been source term here as well, source term here as well and so on That I am not writing Inside one source term is written, other things are their free version Okay, so there is a truncation at the leading order It works out in such a way that this term is order μ and this term is order μ^2 already and inside that this order μ^2 term inside I throw away that means overall order μ cube term I am throwing away in this second term and order μ^2 term I am throwing away in the first term So, one particular example for that could be the d phase space operator which is the atomic Hamiltonian itself,, H of a So,, H of a previously we have seen, it was $\hbar\omega_0$ times $\hat{R}_3$ and it can again be written in terms of its free part and its sourced part Free part is free as it should have evolved, which it does not evolve at all, but there is a sourced part which is over here Now, in the same way if I write down the time evolution of the operator $H_{a.}$, then the operator , H_a would be changing like the phase, the general structure which we learnt about over here from these two equations So, I will put instead of G everywhere, H_a So, if I do that, I will gett he $\hbar\omega$ outside and $\hat{R}_3$ over here So, this is how I would proceed for the computation of this operator So, let me write, there will be an extra $\hbar$, which I have missed because there is $\hbar$ already sitting in the denominator, you can see So, outside there will be $\hbar$ coming from this operators adoption So, you can forget the $\hbar$ in the denominator for the computations However, the structure would be like $\hat{R}_f$, $\hat{R}_2$ of f was coming and here the space for here was to be filled by the operator G If you remember from the slide above, the first term is $\hat{R}_2$ and whatever operator you are going to feed in So, that should be the g That is our case is $\hbar\omega\hat{R}_3$ So, $\hat{R}_3$ will sit here and $\hat{R}_3$ will sit here as well at these two commutative structure So, you will get this structure $\hat{R}_2$, $\hat{R}_3$ and here also $\hat{R}_2$, $\hat{R}_3$ with ϕ^f multiplying outside and ϕ^f multiplying left side outside And then there is a second order correction term which comes about d which can be written in terms of this object, which is coming from the source part that I will write down the $\hat{R}_s$ So, $\hat{R}_s$ will go and sit at the first commutator and $\hat{R}_s$ will go and sit at the second commutator If I have written every, H over here as $\hat{R}^f_3$ $\hat{R}^s_3$ So, first term is this, $\hat{R}^s_3$ should have been sitting inside, but recall $\hat{R}^s_3$ which is appearing over here, we should have appeared over here That itself we had previously solved for equations and $\hat{R}^s_3$ is given in terms of a commutator of free parts of the field multiplied with the commutator between $\hat{R}^f_2$ of one time and $\hat{R}^f_3$ of another time So, this $\hat{R}_3$ which has appeared over here, $\hat{R}^s_3$ which has appeared over here has been written in the integral form So, therefore, one integration and μ^2 comes about x and therefore, you get this expression

$$\left(\frac{d\hat{H}_A}{d\tau}\right)_{vf} = \frac{1}{2}\frac{i\omega_0\mu}{\hbar}\left(\hat{\phi}^f[\hat{R}^f_2, \hat{R}^f_3] + [\hat{R}^f_2, \hat{R}^f_3]\hat{\phi}^f\right) - \frac{1}{2}\frac{\omega_0}{\hbar}\mu^2\int_{\tau_0}^{\tau} d\tau' \{\hat{\phi}^f(\tau), \hat{\phi}^f(\tau')\}\,[\hat{R}^f_2(\tau'), [\hat{R}^f_2(\tau), \hat{R}^f_3(\tau)]]$$

$$\left(\frac{d\hat{H}_A}{d\tau}\right)_{rr} = -\frac{1}{2}\frac{\omega_0}{\hbar}\mu^2\int_{\tau_0}^{\tau} d\tau' \,[\hat{\phi}^f(\tau), \hat{\phi}^f(\tau')] \times \left(\hat{R}^f_2(\tau')[\hat{R}^f_2(\tau), \hat{R}^f_3(\tau)] + [\hat{R}^f_2(\tau), \hat{R}^f_3(\tau)]\hat{R}^f_2(\tau')\right)$$

If the field is in its vacuum state

$$\langle 0|\left(\frac{d\hat{H}_A}{d\tau}\right)_{vf}|0\rangle = -\frac{\omega_0\mu^2}{\hbar}\int_{\tau_0}^{\tau} d\tau'\, C^F(x(\tau), x(\tau')) \times [\hat{R}^f_2(\tau'), [\hat{R}^f_2(\tau), \hat{R}^f_3(\tau)]]$$

$$\langle 0|\left(\frac{d\hat{H}_A}{d\tau}\right)_{rr}|0\rangle = \frac{\omega_0\mu^2}{\hbar}\int d\tau'\, \chi^F(\tau, \tau')\{\hat{R}^f_2(\tau'), [\hat{R}^f_2(\tau), \hat{R}^f_3(\tau)]\}$$

where

$$C^F(\tau, \tau') = \frac{1}{2}\langle 0|\{\hat{\phi}^f(\tau), \hat{\phi}^f(\tau')\}|0\rangle$$

$$\chi^F(\tau, \tau') = \frac{1}{2}\langle 0|[\hat{\phi}^f(\tau), \hat{\phi}^f(\tau')]|0\rangle$$

The resulting operator is operator on atomic states

$$(\delta E_-)_{vf} = \langle -,0|\,\hat{H}_A\,|-\,0\rangle$$

$$= -\frac{1}{2\hbar}\,\omega_0\mu^2 \int_{\tau_0}^{\tau}\int_{\tau_0}^{\tau} d\tau' d\tau''\; C^F(\tau',\tau'')\,\chi^A_{--}(\tau',\tau'')$$

$$\chi^A_{--} = \langle -|[\hat{R}^f_2(\tau''),[\hat{R}^f_2(\tau'),\hat{R}^f_3(\tau')]]|-\rangle$$

$$(\delta E_-)_{rr} = -\frac{1}{2\hbar}\,\omega_0\mu^2 \int_{\tau_0}^{\tau}\int_{\tau_0}^{\tau} d\tau' d\tau''\; \chi^F(\tau',\tau'')\; C^A_{--}(\tau',\tau'')$$

$$C^A_{--}(\tau',\tau'') = \langle -|\{\hat{R}_2(\tau''),[\hat{R}^f_2(\tau'),\hat{R}^f_3(\tau')]\}|-\rangle$$

Eqn

$$\left(\frac{d\hat{H}_A}{d\tau}\right)=\frac{1}{2}\frac{i\omega_0\mu}{\hbar}\left(\hat{\phi}^f[\hat{R}^f_{2,}\hat{R}^f_3]+[\hat{R}^f_{2,}\hat{R}^f_3]\hat{\phi}^f\right)-\frac{1}{2}\omega_0\mu^2\int_{\tau_0}^{\tau} d\tau'\hat{\phi}(\tau),\hat{\phi}(\tau')[\hat{R}^f_2(\tau'),[\hat{R}^f_2(\tau),\hat{R}^f_3(\tau')]]$$

$$\left(\frac{d\hat{H}_A}{d\tau}\right)=\frac{1}{2}\omega_0\mu^2\int_{\tau_0}^{\tau} d\tau'\hat{\phi}(\tau),\hat{\phi}(\tau')\times[\hat{R}^f_2(\tau'),[\hat{R}^f_2(\tau),\hat{R}^f_3(\tau')]]$$

If the field is in the vacuum state

$$\langle 0|\left(\frac{d\hat{H}_A}{d\tau}\right)_{vf}|0\rangle=\frac{-\omega_0\mu^2}{\hbar}\int_{\tau}^{\tau_0} d\tau' C^F(x(\tau),x(\tau'))\times[\hat{R}^f_2(\tau'),[\hat{R}^f_2(\tau),\hat{R}^f_3(\tau')]]$$

$$\langle 0|\left(\frac{d\hat{H}_A}{d\tau}\right)_{\gamma\gamma}|0\rangle=\frac{\omega_0\mu^2}{\hbar}\int_{\tau}^{\tau_0} d\tau'\chi^F(\tau,\tau')\times[\hat{R}^f_2(\tau'),[\hat{R}^f_2(\tau),\hat{R}^f_3(\tau')]]$$

where $C^f(\tau,\tau')=\frac{1}{2}\langle 0|\{\hat{\phi}^f(\tau),\hat{\phi}(\tau')\}|0\rangle$

$$\chi^f(\tau,\tau')=\frac{1}{2}\langle 0|\{\hat{\phi}^f(\tau),\hat{\phi}(\tau')\}|0\rangle$$

The resulting operaator is operaor on atomic states

$$(\delta E_{-.})_{vf}=\langle -,0|\hat{H}_A|-,0\rangle=\frac{-1}{2\hbar}\omega_0\mu^2\int_{\tau_0}^{\tau^2}\int_{\tau_0}^{\tau} d\tau\, d\tau' C^F(\tau',\tau'')\,\chi^A_{--.}(\tau',\tau'')$$

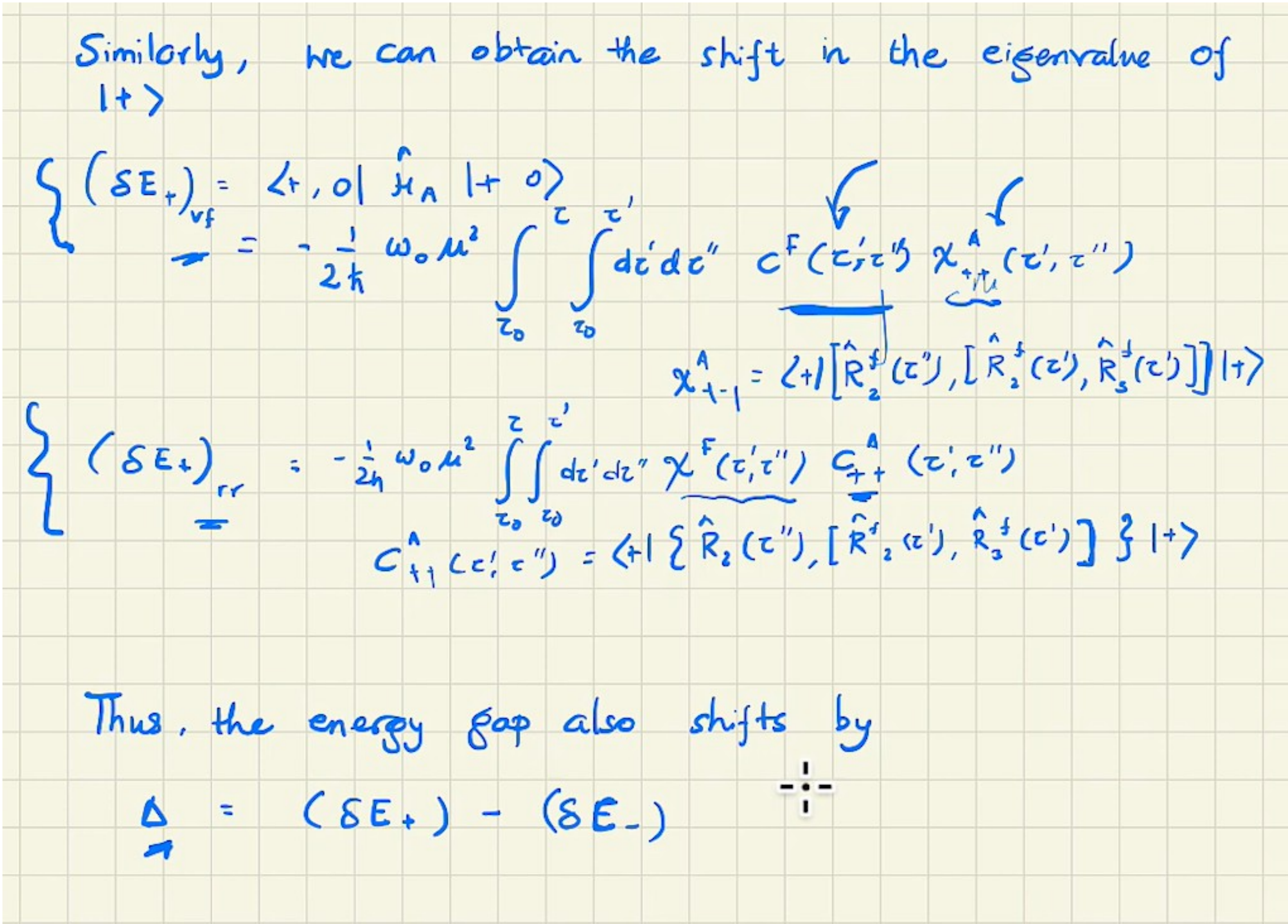
$$\chi^A_{--.}=\langle -|[\hat{R}^f_2(\tau'),[\hat{R}^f_2(\tau),\hat{R}^f_3(\tau')]]|-\rangle$$

$$(\delta E_{-})_{\gamma\gamma}=\langle -,0|\hat{H}_A|-,0\rangle=\frac{-1}{2\hbar}\omega\,\mu^2\int_{\tau_0}^{\tau^2}\int_{\tau_0}^{\tau} d\tau\, d\tau'\,\chi^F(\tau',\tau'')C^A_{--}(\tau',\tau'')$$

$$C^A_{--}(\tau',\tau'')=\langle -|[\hat{R}_2^f(\tau'),[\hat{R}_2^f(\tau),\hat{R}_3^f(\tau')]]|-\rangle$$

And previously we had discussed the free part of the d field will commute with the operator side expressions, the free part of the operator side Remember we had this confusion of operator ordering that if there is source part that has terms belonging to both atomic side and the field side So, therefore, I cannot commute them with respect to atomic side and field side operator But free parts do commute So, therefore, if you see d In the first term, ϕ^f is on the left hand side and in the second term, ϕ^f on the right is on the right hand side Similar thing should have happened for the $\hat{R}_s$ part as well If I do for the $\hat{R}_s$ part, the same thing would have happened and then I would have got a ϕ^s on the left and ϕ^s on the right That I could not have flipped But once I write down the ϕ^s itself in terms of its structure, the ϕ^s which is effectively $\hat{R}_3(s)$, $\hat{R}^s_3$ can be entirely written in terms of free field operators and free field operator of the atom So, using these two things, you can say that you can combine the various ϕ^s appearing on the right side or the left side and they can be written in this nice anti-commutator form for the field multiplying this object. Similarly, we can write down the radiation reaction correction to the Hamiltonian itself as this expression Similarly, we can write down the radiation reaction correction to the Hamiltonian itself as this expression Put everywhere, wherever you have a G, you put roughly $\hbar\omega$ times $\hat{R}_3$ here also And do the same game again Break $\hat{R}_3$ in terms of its free part and the source part And for the source part, write down the expression for $\hat{R}^s_3$ like this And this exercise yields me that the radiation reaction correction to the Hamiltonian has been nitrified So, we had seen before as well, the vacuum fluctuation has field anticommutator and atomic operators coming in commutator form, while for radiation reaction, the fields appear in commutator and that operator belonging to atomic sector come as anticommutator This becomes the structure for radiation reaction and vacuum fluctuate But this is an operator of atom which becomes collectively an operator over field as well as an atom So, that means the change in the total Hamiltonian of the atomic side becomes an operator over both Hilbert spaces of the atom as well as of the field So, it should be when we try to get the numbers out d expectation value, we have to take both the states of the atom as well as the field So, let us see first, if I take the state of the field, suppose I know that the field is in the vacuum, I have not switched on, no particle in the field is excited, it is vacuum state Then that means I have to take the expectations of this thing with the field states So, I have squeezed these operators with vacuum of the field This thing here will get squeezed with respect to the field state and here also you will get the ϕ^f over here and ϕ^f over here, they will get squeezed between the field state Now you can see only the field operators are here, these are atomic operators, they do not care about states of the field Only ϕ^f cares about the state of the field and then the second term anticommutator cares about the state of the field And in the radiation reaction, this commutator cares about the state of the field All these operators just act on the atomic Hilbert space Now, you can see when I take the expectation with respect to the field state, the first term in the vacuum fluctuation term that vanishes because I will get a vacuum expectation value of the field Vacuum expectation of the free field is 0 in the vacuum state So, the first term will drop off While for radiation reaction, you will see that the commutator is getting an expectation value and admic operators are in their anti-commutative form So, you will get χ^F, χ^F is just expectation value of anti-commutator in the vacuum mixture and the atomic operators become in anti-commutator. So, the rules flip In vacuum fluctuations, the field is in anti-commutating state, in the anti-commutating order and operators of atoms are in commutative order. While in the radiation reaction, the field is in commutative ordering, while the atomic operators are anti-commutative ordering. Anti-commutative, meaning the field operators are $\hat{R}_2$ and commutator of $\hat{R}_2$ and $\hat{R}_3$. Now, this is half our expectation in some sense because this operator is an operator over both Hilbert space, that of field as well as that of atom. So, we have just taken partial expectation with respect to the field state. Therefore, it would leave an operator as an operator over atomic Hilbert space. Therefore, this thing which is obtainable is still an operator. The field part have been put to their expectation value but the atomic parts are just living d the operator. So, they have to act on atoms state. So, we can act on let us say the ground state of the atom and try to see what is the expectation value of in the ground state of the atom. That means how fast the Hamiltonian expectation value changes due to this coupling in the ground state of the atom. So, – remember was the symbol given to the ground state and was the symbol given to the excited state. So, across the, if I take this operator d which is partial expectation with respect to field state and take the full expectation in the ground state of the atom, I will get this d So, when I take the expectation value with respect to – and – here, these things will get under expectation. Even in the first term, if I take expectation with respect to – state, these operator will get skewed between the –. So, you see in the first term, d I will get expectation of a commutator with respect to – state and in the second term, I will get expectation of an anticommutator with respect to – state So, ultimately, I will get a structure like the total change in the expectation value of Hamiltonian or you can think of it as a change in the eigenvalue in the time dependent plane in a loose fashion is made up from the anticommutator piece from the field in the vacuum getting convoluted with commutator structure of the operators from the atomic sector getting squeezed between χ_{--} state. χ_{--} is this squeezing of the commutator of $\hat{R}_2$ and this whole operator between χ_{--} . Similarly, for the radiation reaction if I do d Again the same thing will happen, it will be the first term will be the commutator now and the second term will become anti-commutator of operators from the admins sector, squeezed between the χ_{--} state. So, this would be the change in the ground state due to vacuum fluctuation and this will be the change in the ground state energy rather due to vacuum fluctuation and this will be the change in the ground state energy due to radiation reaction. The same computation you can do for excited state as well and there you will get the corresponding quantities. So, here it should have been a , here. So, , . So, you see that the roles will almost remain the same. The first piece will not change, only the χ will now be a , expectation and thus in the second term C belonging to the atom will be in the , . C is the anti-commutator, χ is the commutator. So, field state remains between the vacuum in the first term. And the second term gets squeezed between the that means excited state. And in the second radiation reaction driven correction that would be just the field state in the commutator structure and the atomic state in the anti-commutator structure. In the meanwhile, this was the rate which we had previously obtained. We can integrate this equation to obtain the net change.

So, we remember here up to this we had this equation. We have just integrated things out and we are obtaining the net change in the eigenvalue. This was the rate of change. You integrate it over data, you will get the net change.



Similarly, we can obtain the shift in the eigenvalue of $|+\rangle$

$$(\delta E_{+.})_{vf} = \langle +,0|\hat{H}_A|+,0\rangle = \frac{-1}{2\hbar}\omega_0 \mu^2 \int_{\tau_0}^{\tau^2}\int_{\tau_0}^{\tau} C^F(\tau', \tau'')\,\chi^A_{++.}$$

$$\chi^A_{++.} = \langle +|[\hat{R}_2^f(\tau'), [\hat{R}_2^f(\tau), \hat{R}_3^f(\tau')]]|+\rangle$$

$$(\delta E_{+.})_{\gamma\gamma} = \langle -,0|\hat{H}_A|-,0\rangle = \frac{-1}{2\hbar}\omega \mu^2 \int_{\tau_0}^{\tau^2}\int_{\tau_0}^{\tau} d\tau\, d\tau'\, \chi^F(\tau', \tau'')\, C^A_{++.}(\tau', \tau'')$$

$$C^A_{++.}(\tau', \tau'') = \langle +|[\hat{R}_2^f(\tau'), [\hat{R}_2^f(\tau), \hat{R}_3^f(\tau')]]|+\rangle$$

Thus, the energy gap also shifts by

$\Delta = (\delta E_{+.}) - (\delta E_{-.})$

So, this is how typically the change of eigenvalues of different eigenstates are obtained due to coupling of that to a field. Even though the field is in vacuum, the coupling is non-zero and quantum fluctuations appearing through the commutator and the anticommutator. Both of these are non-zero even in the vacuum state. commutator between the two points of the field which we have computed long back was remember $1/\delta(t^2 - x^2)$, $\delta(x^2)$ that was commutator. Anti commutator has more than some subtlety, so I will not write that over here, but both of them even in the vacuum are non-zero. So, despite the fact that there is no particle in the field. Just a coupling term is there in the Hamiltonian. All the vacuum fluctuations are able to change the eigenvalue of the hydrogen or any atoms eigen states. So, they keep changing due to vacuum fluctuation and radiation reaction. Now due to radiation reaction and vacuum fluctuation, they have become slightly different. They will change in time. The integration tell you up to what time it changes to what. So, ground state will change and become something over time, the fitted state will change and become something over time. Ultimately, you can see which we will not discuss in this course, but you can work it out in more detail and see that initial value they start with ultimately over long time they will settle into a new configuration and it will also over long time settle into a new configuration. So, the gap between the excited state and the ground state does not remain the same. Initially, it was δE and finally, it becomes $\delta E'$ after long, long time, long meaning time scale longer than 1 over δE. After long time scale, we will see they have settled into a new split difference and that change in $\delta E'$ and δE is given the symbol this delta that is the standard lamb shift in the atoms. This is happening due to vacuum fluctuations and radiation reactions only and this is just a hallmark of quantum field despite not present actively through any particle, no photon is present only vacuum fluctuations are there. They are sufficient to change the hydrogen atoms or any other atoms Eigen splitting. And this has been observed in lab that atoms do have this kind of Lamb shift predicted value. This is not, so if I do the standard hydrogen atom computation without looking for this coupling term, then I would get some number for Eigen value separation. But suppose hydrogen atom is talking to the background electromagnetic field through dipole coupling. I know what is the dipole expectation, meaning transition element for dipole. And I know what is the vacuum fluctuations or that means vacuum correlators of electromagnetic field. Put together you will get an extra num correction term, which again is the order of 10^{-5eV}. So, you will see that really if you do this computation of hydrogen atom accounting for electromagnetic vacuum. You will get the eigen space splitting between various states of atoms off by 10^{-5eV} or 10^{-6eV} or even 10^{-7eV}. In very sophisticated setups, this has been measured. Actual splitting in hydrogen atom is off by this much order which is predicted by this quantum fluctuations of field. So, indeed that is a party to the determination of the splitting between eigen states of any atom. So, this is a very robust statement about the characteristics of atom which came here. Its eigenvalues and other things are not protected quantities, they change over time and ultimately settle into a vacuum allowed new separation. Theory predicts and theory gets verified in experiments as well. So, this is one of the very robust predictions. Now, briefly I will touch upon another aspect which is generated by vacuum fluctuation and which also nowadays is getting attraction. which is people are trying to come up with observations or experiments which can measure that which goes by the name of witness of quantum entanglement between the atoms. I will just briefly touch it before ending the whole discussion session and this will not be a very rigorous but only idea I will try to convey what people try to do.

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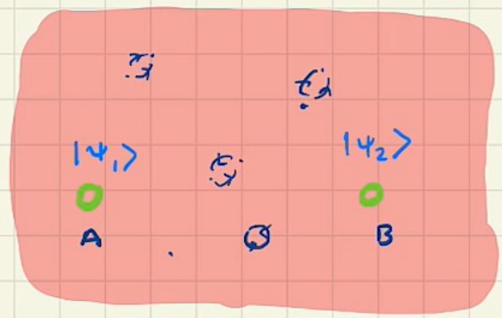
★ Entanglement

○ LOCC : Unentangled atoms remain unentangled unless interacting with some quantum channel

A tripartite description in presence of quantum fields



⇒ If initially both the atoms are in their ground state and field is in vacuum

$$\rho_0 = |0\,g\,g\rangle\langle 0\,g\,g|$$

Reduced $\rho_{AB} = Tr_\phi(\rho_0) = \sum_{\{n_k\}} \langle\{n_k\}|\hat{\rho}_0|\{n_k\}\rangle$

But since $\langle\{n_k\}|0\rangle = \delta_{n_k 0}$

$$\rho_{AB} = |g\rangle\langle g|$$

$$= \begin{array}{c|cccc} & |gg\rangle & |ge\rangle & |eg\rangle & |ee\rangle \\ \hline \langle gg| & 1 & 0 & 0 & 0 \\ \langle ge| & 0 & 0 & 0 & 0 \\ \langle eg| & 0 & 0 & 0 & 0 \\ \langle ee| & 0 & 0 & 0 & 0 \end{array}$$

Eqn

$$P_0 = |0\,g\,g\rangle\langle 0\,g\,g|$$

Reduced

$$\rho_{AB} = Tr_{\phi}(\rho_0) \sum_{\{n_k\}} \langle \{n_k\} | \hat{\rho}_0 | \{n_k\} \rangle$$

But since $\langle \{n_k\} | 0 \rangle = \delta_{n_{k0}}$

$$\rho_{AB} = |g\rangle\langle g| = \begin{matrix} \\ |g\,g\rangle \\ |g\,e\rangle \\ |e\,g\rangle \\ |e\,e\rangle \end{matrix} \begin{pmatrix} |g\,g\rangle & |g\,e\rangle & |e\,g\rangle & |e\,e\rangle \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

So, the discussion of entanglement which I am briefly going to touch upon is pertaining to this LOCC theorem sort of thing where you know the fact that two atoms, let us say A and B, if they are separated by space like system, that means causal communication between them are switched off, then they cannot develop entanglement between them, if initially they were the same entangled, if they are not talking directly through thermal quantum channel. If there is no quantum operator between them connecting them, then they cannot become entangled later on. Initially, they were disentangled. That means initially they were separable states. That is the Elasticity theorem that unless you write an operator which connects, which acts on both of them in direct way, there is no way to generate entanglement approach. Each of them will independently evolve with their own Hamiltonian and the states will remain separable. However, this picture gets challenged in the quantum field theoretic setup because even if I do not actively put any operator connecting them, there is always a background field which lives in a vacuum state. So, these atoms do talk to the background field through the monopole coupling or dipole coupling or spin coupling, whatever coupling which we have. We will just think of as a toy example of monopole coupling. And in background, constantly particles are being created and destroyed due to vacuum flux. And due to this effect, now we can see that despite there is no active operator we have put in between *A and B*, but A and B both are talking to background quantum field, a common operator which both of them are talking to. And yet, if we keep the operator in the vacuum, meaning the field conservation is put in the vacuum, that means actively I do not put any quantum particles in there. Any particles in that, despite that entanglement will get generated. And that is slightly, now that we have done with time evolution of Hamiltonian and other things and how the thematic changes, now we know how to look about. So, suppose initially I write down a disentangled state of the two atoms and let both of them be in their ground state. Also, let the background field be in the vacuum state. So, it becomes a tripartite system. So, there are three parties, the atom, the field and the second atom. So, overall the state of the full system is ground state for the first atom *g*, ground state for the second atom and vacuum for the field. And the density matrix constructed out of it, I could write the field on the left as well, so $<0\,g\,g\,|$. So, the density matrix initially for this setup would be this outer product of the state at hand. Now, if I want to write down the reduced density matrix for the atomic sector, then I will trace over the density matrix over the field configurations. That means I will trace over on the field eigen basis in some sense of complete set of states. So, complete set of states I know are the Fock space elements. So, I sum over this density matrix, trace over this density matrix over the Fock space of the quantum field. So, when I write down this ρ_0 like this, this mk ket and the bra and the ket will undergo overlap with this $|0>$ and this $<0|$. So, ultimately I will have terms like this or its complex conjugate. But we know 0 also, the vacuum also is one of the element of the fork basis and fork basis elements are also normal to each other. Therefore, it will be δ_{nk} . It will survive only if all other modes are in vacuum. If they have any excitation, it will become overlap will become 0. So, overall the trace will only survive for one term which is 0 here and 0 here which will just leave things like |*0 g g*> . outer product <*0 g g*|. So, both the states being ground state. So, this *g* is depicting both the atoms in the ground state. This will be the outer product of the reduced density matrix and in the full basis of the reduced density matrix, there are four possibilities. Both the atoms in the ground state, one atom in the ground, second in the excited, first in the excited, second in the ground and both excited. In this space, in this basis, the density matrix is written like 1 in the upper diagonal element and all other things are zero. So, this as you know it is very simple pure state density matrix which is also separable.

In this course we have not discussed about that but I hope you have heard it about from the density matrix you can compute constitute various entanglement measures. So, this state is having trivial entanglement it has no entanglement if you look at any of any operator corresponding to entanglement. Whenever the states are separable, that means two atoms states, the full density matrix can be written as separable states of the two atoms, then the entanglement is 0, which is the case here. It is the direct product of *g* of first atom and g of the second atom. Similarly, if I would have started from |*g e*>, one atom is the ground state, another atom is in excited state and field is in vacuum state, the same thing would have happened, only thing this element for example would have, this element |*g e*> would have turned 1 and all others would have remained 0. So, this is initially, if I initially try to write things up, this is tripartite system. All of them are independently evolving. This is evolving with field Hamiltonian, this is evolving with atomic Hamiltonian of first atom, this is evolving with atomic Hamiltonian of the second Hamiltonian. But as soon as there is an interaction between any two of them. In this particular case, I am going to turn on interaction between the field and the first atom and the field and the second atom. But field atom first and atom second are not interacting with themselves. Then let us see what happens.

Now, due to interaction with the field each atom will have an interaction term

$$\hat{\mathcal{H}}' = \mu_1 \hat{m}_1(\tau)\, \hat{\phi}(x_1(\tau)) + \mu_2 \hat{m}_2(\tau_2)\, \hat{\phi}(x_2(\tau_2))$$

So the change in the density matrix

$$\hat{\rho}_0 \longrightarrow U \hat{\rho}_0 U^\dagger$$

In the interaction picture

$$(\rho_0)_I \longrightarrow U_I (\hat{\rho}_{0\,I}) U_I^\dagger$$

$$U_I = \Big(\mathbb{1} \underbrace{- i \int \hat{H}_I' \, dt}_{U^{(1)}} \underbrace{- \int dt \int dt' \, \hat{H}_2'(t) \hat{H}_2(t')}_{U^{(2)}} + \dots \Big)$$

Under this

$$\rho_0 \longrightarrow \rho \equiv \rho_0 + \rho^{(1)} + \rho^{(2)} + \dots$$

$$\rho^{(1)} \text{ (order } \mu) \; : U^{(1)} \rho_0 + \rho^0 U^{(1)\dagger}$$

$$\rho^{(2)} \text{ (order } \mu^2) \; : U^{(2)} \rho_0 + \rho_0 U^{(2)\dagger} + U^{(1)} \rho_0 U^{(1)\dagger}$$

Under this evolution

$$\rho_{AB} = \mathrm{Tr}_\phi \, \rho =$$

	$\lvert gg\rangle$	$\lvert ge\rangle$	$\lvert eg\rangle$	$\lvert ee\rangle$
$\langle gg\rvert$	$1 - \mathcal{I}^\dagger$	0	0	$-\mathcal{J}^*$
$\langle ge\rvert$	0	$\mathcal{I}_{BB}$	$\mathcal{I}_{AB}$	0
$\langle eg\rvert$	0	$\mathcal{I}_{AB}^*$	$\mathcal{I}_{AA}$	0
$\langle ee\rvert$	$-\mathcal{J}$	0	0	0

Eqn

Now, due to interaction with the field each atom will have an interaction terms

$$\hat{H}' = \mu_1 \hat{m}_1(\tau_1)\hat{\phi}(x_1(\tau)) + \mu_2 \hat{m}_2(\tau_2)\hat{\phi}(x_2(\tau))$$

So, the change in the density matrix

$$\hat{\rho}_0 \rightarrow U \hat{\rho}_0 U^{\dagger}$$

In the intraction picture

$$(\hat{\rho}_0)_I \rightarrow U_I (\hat{\rho}_0)_I U_I^{\dagger}$$

$$U_I = (\underbrace{1 - i\int \hat{H}'_I\, dt}_{U^{(1)}} - \underbrace{\int dt \int dt' \hat{H}'_2(+.)\hat{H}'_2(t)}_{U^2} + \ldots)$$

Under this

$$\rho_0 \rightarrow \rho \equiv \rho_0 + \rho^1 + \rho^2 + \ldots.$$

$$\rho^{(1)}(Order\, \mu) : U^{(1)} + \rho^0 + U^{(1)} + \ldots$$

$$\rho^{(1)}(Order\, \mu^2) : U^{(2)} + \rho^0 + U^{(1)} \rho_0 U^{(1)} + \ldots$$

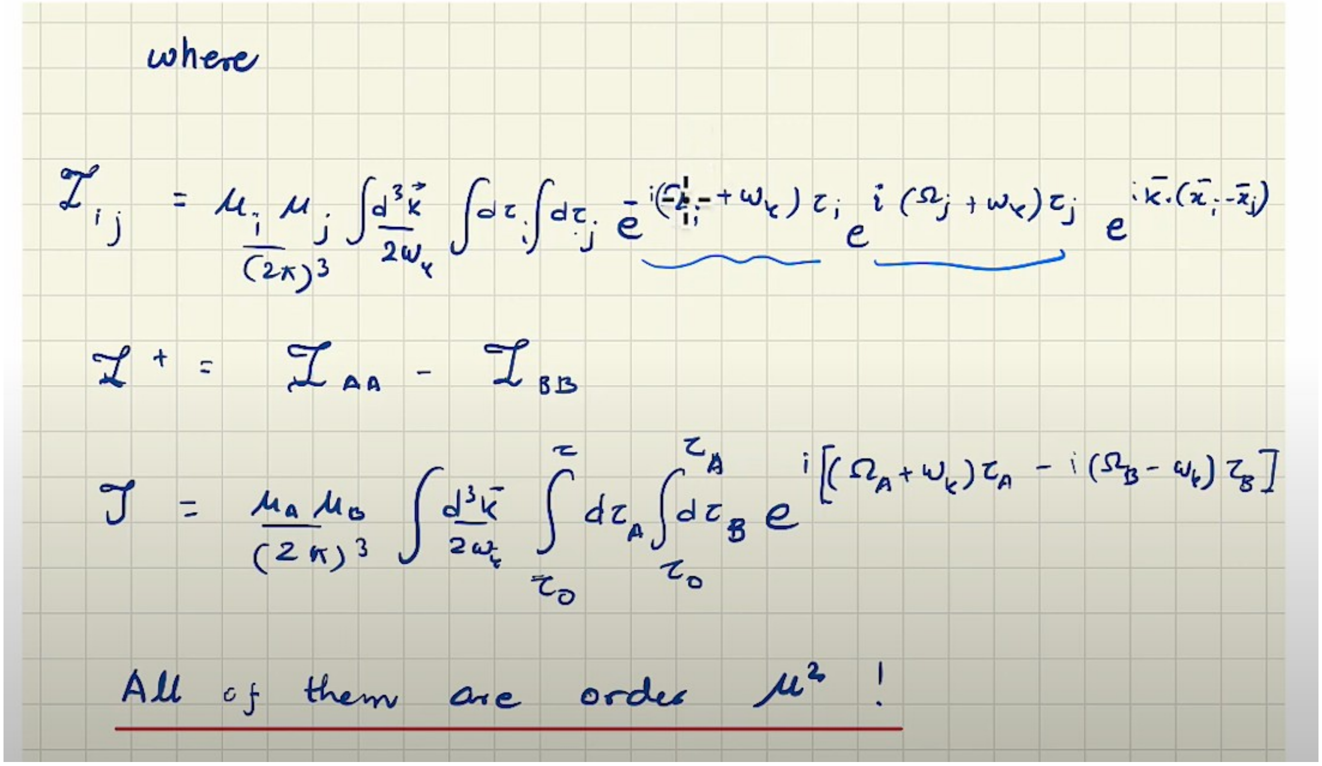
Under this evolution

$$\rho_{AB} = Tr_{\phi}\, \rho = \begin{matrix} \\ |g\,g\rangle \\ |g\,e\rangle \\ |e\,g\rangle \\ |e\,e\rangle \end{matrix} \begin{pmatrix} |g\,g\rangle & |g\,e\rangle & |e\,g\rangle & |e\,e\rangle \\ 1-\Gamma^{\dagger} & 0 & 0 & -\Gamma^{\dagger} \\ 0 & \Gamma_{BB} & \Gamma_{AB} & 0 \\ 0 & \Gamma_{AB}^{\dagger} & \Gamma_{AA} & 0 \\ \Gamma & 0 & 0 & 0 \end{pmatrix}$$

In that case, I am going to write down the coupdurling term, the interaction Hamiltonian will be the monopole of the first atom times the field at its own location x1 and monopole of the second atom and the field value at its own location, second location. So, this would be the coupling term and under this coupling we have to write down the change in the density matrix. It would be time evolution operator. So, density matrix will evolve to some unitary as we know.

However, we also know that we can do the computation in the interaction picture as well. So, I will convert every object into their interacting counterpart which we have learnt in this course how to do by just evolving things with respect to the three part of the Hamiltonian individually. So, this would be the total time evolution according to the interaction picture, where the interaction unitary is given by the only the coupling Hamiltonian, through the coupling Hamiltonian. All these things we have done. With this also you can look back at our note in week 2 or week 3, where time dependent Hamiltonian was being discussed. So, under this, the first term is identity which is trivial unitary. The second term I am going to call as order first unitary because this is made of one interaction Hamiltonian which will come with one single power of mu, H^{μ} . That means it is linear order in interaction. When I write down the second order term which is, *H* interaction and two copies of interacting Hamiltonian *H'*. When I write down the second order term which is, *H* interaction and two copies of interacting Hamiltonian H'. Two copies of interaction Hamiltonian. That means two of such terms will go with e^{+iH_0t} and e^{-iH_0t} because they have to be brought into interaction form. They will go and sit in this Hamiltonian. You can see this term will be order either μ_1^2 or μ_2^2 or $\mu_1\mu_2$. So, overall order μ_2^2, μ^2 term it is. This is second order term. Similarly, there could be third order term, fourth order term. So in this discussion, I am going to stop the theories at the second order because we are going to talk about entanglement. We will see that entanglement initiates at order λ^2 or order μ^2. So therefore, I have to retain terms up to order μ^2 if I am looking for any property of the system like entanglement which gets generated at order μ^2. I cannot just truncate it here. And say that at final stage if I get order μ^2 term, then that is correct because this is order mu, it will get multiplied with another order μ somewhere, which will generate order μ^2 term in the integral. But there was already this order μ^2 term which could have multiplied with identity to give you order μ^2 term as well. We will see it, let us retain this term up to here and then we will see why it was necessary to retain. So, let us go ahead. So if I evolve this, I put this U interaction into the left and the right and put the density matrix in between. This is how the density matrix will change. It will pick up a order 1 term which will be order μ and order 2 term correction which is order μ^2. You can write down the whole series expansion over here on the left. and its dagger on the right and start collecting terms. So, I will have a identity identity – all these terms on the left and identity all these terms on in the *i* term on the right hand side and then I start doing cross multiplication. So, identity times ρ_0 times identity will generate ρ_0 term. The first order term where this term from left takes the row not and multiplies identity on the right will give me this object U^1 row not identity. Similarly, identity from left side row not and U^1 from the right side that will generate the order μ term. If I want to look at order μ^2 term, I will have many options possible. I can take the second order term from the left row not in between and identity of the first term or of the right term versus identity of the left unitary ρ_0 in the middle and second order correction of the $U^\dagger$ on the right. This will give me $\rho_0 U^2$ dagger. Lastly, there is another possibility that order U^1 term from the left density matrix between and order U^1 term coming from the right U dagger. So, that will give you this option. So you see, this is how order by order we can compute corrections into the density matrix. So again, since we are just touching upon, I am not going to derive all this. I am going to just state that if you keep doing that, writing the interaction Hamiltonian, you take this and you write down the interaction Hamiltonian. By now, you already know how to write down this thing. It has to be squared between e^{-iH_0t} and e^{+iH_0t} . which will be put over here and when I try to write down the density matrix in this basis, you will get all these operators squeezed between these basic elements. So, I am just mentioning the rough structure you will get, you will get previously I had started with only one over here and all other entries were 0. This time under time evolution these items will turn on. There are various terms which will become turned on because of these two terms which are correcting the density matrix. Initially density matrix was just this much which was 100 everywhere.

Due to this ρ_1 and ρ_2, other terms have come about. All these curly Idependent terms have come from ρ_1 and ρ_2. Otherwise, you can see it was just 1000. What are these curly I's and curly j?



Eqn

Where

$$\Gamma=\frac{\mu_i\,\mu_j}{(2\,\pi)^3}\int\frac{d^3\vec{k}}{2\omega_k}\int d\,\tau_i\int d\,\tau_j\,e^{-i(\Omega_i+\omega_k)\tau_i}\,e^{-i(\Omega_j+\omega_k)\tau_j}\,e^{i\vec{k}(\vec{x}_i-\vec{x}_j)}$$

$$\Gamma^{+\cdot}=\Gamma_{AA}-\Gamma_{BB}$$

$$J=\frac{\mu_A\mu_B}{(2\,\pi)^3}\int\frac{d^3\vec{k}}{2\omega_k}\int_{\tau_0}^{\tau}d\,\tau_A\int_{\tau_0}^{\tau_A}d\,\tau_B\,e^{[i(\Omega_A+\omega_k)\tau_A--i(\Omega_B-\omega_k)\tau_B]}$$

<u>All of them are order of μ^2 !</u>

These are just exponential integrals which we have learned to compute many times during transition rates and other things. If you work out slightly carefully, write down the unit trees as the interaction picture Hamiltonian. This we have done for computing for transitions as well. So, this exact operator we had written when we were writing the transition rates of atoms as well and this as well. So, all these things we have already computed and you just look back at our nodes, you will realize that the all the curly I operator are very similar operator which we have already computed. So, this is some version of that only time this you have to be careful about it will talk about curly I A B will come about because you see here it is unitary of one and unitary of one multiplying here unitary of two and unitary of two unitary of one unitary of one and there is overall correction term which is coming from the second order term as well okay all right so there are various possibilities which which has come about so here also one should have u, so this is fine, this is the thing which has appeared. So, overall when I am writing this u, this unitary cares about a total interaction Hamiltonian and total interaction Hamiltonian has two terms, two monopole coupling, monopole coupling of first atom with field location at the location of first atom and monopole of the second atom with the field at the location of the second atom. So, when I write these Hamiltonians over here, the interaction Hamiltonian, I have to squeeze the whole term between – I_0 free field and free operator $e^{iH_0 t}$ and + $e^{-iH_0 t}$ from this type. And that is how we will obtain the interaction Hamiltonian in the double atom system. So, therefore, when I write down this U^1 or U^2, they are made up from both the atoms. Therefore, a joint property of atoms appear. Sometimes it is a double property of single atom, I_{AA} , sometimes it is double property of the second atom. And then there is a mixedness, *A, B,* first atom, second atom, I_{AB} and $I^{\star}_{AA}$ and this curly J, which are all obtainable in terms of these integrations, where the frequency I am writing for general ij, i can take value *A* or *B*, j can take a value *A* or *B*. Suppose I am computing for curly I_{AB} , μ_{AB} it will become here, it will become time of the first atom, time of the second atom, frequency is gap of the first atom, frequency gap of the second atom and the separation between the first atom and second atom. They are also just simple integrations of the kind which we have already done. This exponential sine functions which become a delta function in long time limit. All these integrals are of very similar kind of a structure. That you have to be just carefully be doing. Here only, the only fact that previously we were computing that the atom is at rest with respect to itself. So, separation between two different times, we could have put it to 0. Right now it is first atoms location which you can put to some coordinates, second atoms location. So, this will become non-zero. it will not be 1. Previously for atom at rest we had put it to 0. Now these are two atoms and their separation is finite. It is not 0. So therefore this will survive. So we have to just carefully do this integral. Otherwise this much of integration we have done so many times by now, which appears in curly J. So again I am briefly discussing the idea and not going for a mathematical rigor in this discussion. So you just be aware That standard monopole coupling with respect to both the atoms, even though the field is in vacuum, the density matrix changes to a more rigorous form. It is not just one element anymore. It has generated various things. And now you can see that these terms which are appearing $I + J$ or I_{AB}, I_{AA}, I_{BB} , all these terms are made from the double coupling of *A* and *B.* So, it is order square term. So, since entanglement or the density matrix is getting corrected with order square term, that is why we wanted to account for all order square term from the beginning. I do not want to miss any order square term because ultimately the final expression I want is true up to order square. It is either 0 or it becomes non-zero at order square. So, I should not miss out anything which can contribute at order square. So this is all this standard computations which you are welcome to try. But I just wanted to give a statement that all of them are order square and non-zero. All these integrals we have done so many times. So you are welcome to do it and find out how much F entanglement or at least density matrix will become this. From here you can find out. From here you can find out with this density matrix which entanglement measure gives you what result. That discussion I am not going ahead in this set of lectures.

So, I am going to end over here with this information that due to individual coupling of atoms with the background field, despite the vacuum, the field is in vacuum state, that means actively I cannot see anything communicating between *A and B* when there is no quantum operator directly connecting them. But there is quantum fluctuations which are captured in these kind of intakers. Remember, these were coming from quantum correlations and these exactly go and sit in the computation of the density matrix range as well. So, those vacuum fluctuations generate entanglement between them. So, therefore, this process sets up two species, two particles, two atoms which are initially disentangled and there is no other active operator between them. Even after that, after some time, There is a non-trivial entanglement setup between them. So, therefore, vacuum state leads to development of entanglement and this goes by the name of entanglement. So, I stop over here. So, I hope you had some good at least glance at vacuum properties of fields and how vacuum properties of field affect the nature of atom and properties of atom which are very salient and new set of prediction. Normal day-to-day quantum mechanics because there is almost a hidden sector because vacuum does not allow you to see anything directly. Only tiny fluctuations are there which are not obviously seen by any experiment. Their indirect presence is felt by changes in things, for example, transitions are possible across ground state, spontaneous emission is possible or the separation between the eigen states are possible, development of entanglement is possible. So, there are many, many, many avenues where this hidden sector of a quantum field affects the quantum analysis of atom surface. So, in this course, our mandate was to just have a good view of that and I hope it was helpful in conveying that message. So, we stop the course over here, maybe in some other future course, we will discuss in more rigor, what more treatment one can do with the interactions of atom and field in more observational kind of experimental kind of setup. This course was more oriented towards theoretical understanding.

Okay, I hope you have a good, I hope you have had a good glance at these things and it was not worthless in some sense and it was somewhat of enjoyable to see how quantum fields are affecting the structure of atoms. Okay, so thank you for your time and attention. Of course, we stop over here and I hope you would have a good time at examples.