**Foundation of Quantum Theory: Relativistic Approach**
**Change in atomic characteristics via field interactions**
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## Change in atomic characteristics via interaction with background field.
## Dalibard, Dupont-Roc and Cohen-Tannaoudji (DDC) formalism

So, in today's discussion, we will be discussing about the change in the atomic characteristics by the interaction with a background field. Till now what we have learnt about is when an atom starts interacting with a background field, there is a non-zero probability of transition from one state to another.

However, we must note that the states which we are considering are no longer the eigen states of the true Hamiltonian because the true Hamiltonian involves interaction term with the field, free field of the atom or free Hamiltonian of the atom and free Hamiltonian of the field itself. So, net Hamiltonian is made up of various pieces.

The eigen states or excited state and ground state were just the eigen states of one of the pieces of the total Hamiltonian which was the atomic part of the Hamiltonian.

Now, net Hamiltonian as we will see below will be just Hamiltonian of atom plus Hamiltonian of field and plus an interaction term.

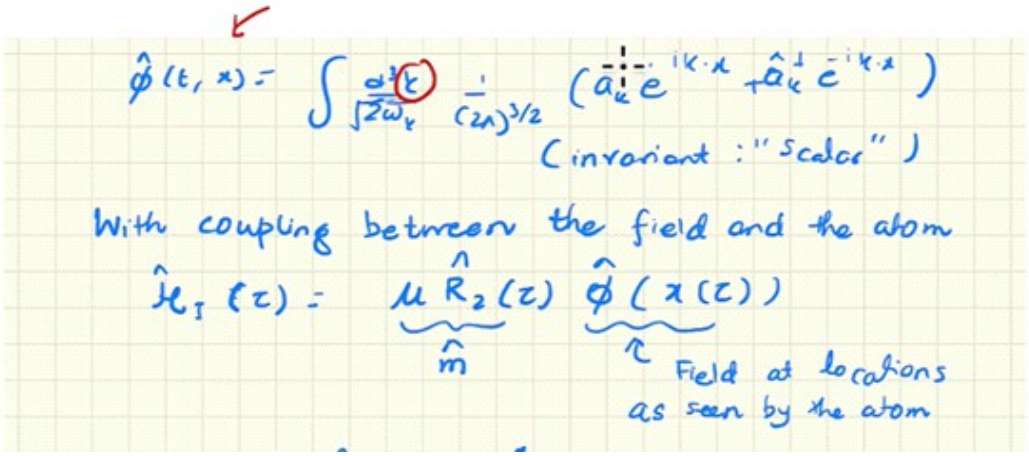
Therefore, only eigen states of the first term do not constitute the full Hamiltonian. And as we have seen, if the field is time dependent, even in vacuum state, it gives rise to an effective Hamiltonian to the atomic system, which is as good as a fluctuating quantum correlation coming into the picture.

Therefore, it will be a time dependent Hamiltonian term which would be present for the atomic parts' effective dynamics.

So that constitutes the total Hamiltonian or effective Hamiltonian rather, of the atomic system becoming a time dependent Hamiltonian and therefore there is no eigen-state of the system because this is a time development, the dynamics is there explicitly expressed in the Hamiltonian and therefore, steady state picture does not hold.

Therefore, the energy gap between the atomic states which is $\Delta E$ between excited state and the ground state was the energy gap before the atom started interacting with the field.
Once the interaction starts, the $\Delta E$ is no longer the energy gap between the eigen states of the Hamiltonian of the atom.
Therefore, we should be aware of the fact that the energy gap itself changes, due to the interaction of the field even if the field is in a vacuum through vacuum fluctuations. This goes by the name of Lamb's shift that the energy state of the eigen state difference, energy Eigenvalue difference between the atomic states keeps changing due to interactions of vacuum fluctuations. And over long time average, it will settle to a well-known expectation value that we will see in this set of equations. And for just obtaining more apparent version of how to compute this shift in the atomic spectra, we will be adopting something called as the DDC formalism, which are named after three people, **Dalibard, Dupont-Roc and Cohen-Tannaoudji (DDC) approach** which will show to deal the effects of background fields on the states of atomic shells.

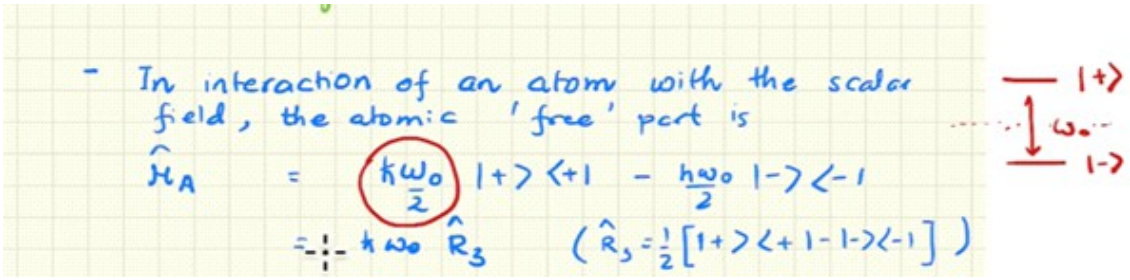
Let us get going with this understanding that now we are no longer just looking for transitions alone, that are concerned with the change in the characteristic properties of the atom itself.




Now as we know the total Hamiltonian has many terms, it has a atomic free part which is free part of free Hamiltonian of the atom $\hat{H}_A$ which in its spectral decomposition is written as $\hbar\frac{\omega_0}{2}$ $|+\rangle|+\rangle$ which is the excited state and minus $\hbar\frac{\omega_0}{2}$ $|-\rangle\langle-|$ which is the grounded state. I am just defining the structure like the grounded state is given with the symbol kth of |->, excited state is given with the symbol of k of |+> and the energy separation between them is something whose frequency is $\omega_0$ .

$$\hat{H}_A = \frac{\hbar\omega_0}{2}|+\rangle\langle+| - \frac{\hbar\omega_0}{2}|-\rangle\langle-|$$

$$= \hbar\omega_0\hat{R}_3 \qquad \left(\hat{R}_3 = \frac{1}{2}|+\rangle\langle+| - \frac{1}{2}|-\rangle\langle-|\right)$$



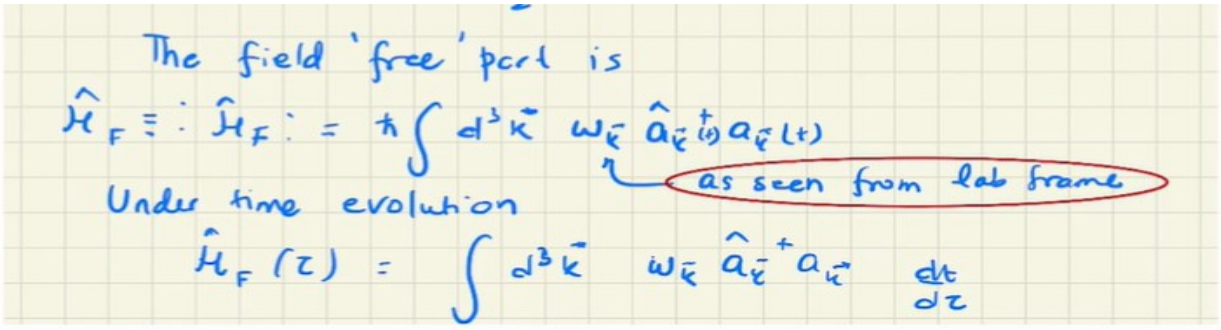

I am just setting the 0 of the energy at the mid of this separation such that the excited state is $+\bar{h}_0\frac{\omega_0}{2}$ eigenvalue and ground state is $-\bar{h}\frac{\omega_0}{2}$ eigenvalue. I can pull out $\bar{h}\omega_0$ a as a common factor and you will see I will be left with one half of $|+\rangle|+\rangle$ |-><-| gets out of the product. Therefore, this is a

compact operator representation of the atomic part of the total hemisphere. This $\hat{R}_3$ , you can remember and realize that it is very much the spectral decomposition of the $\sigma(3)$ , the third Pauli matrix.
Now, under time evolution if you want to look at this Hamiltonian will undergo a time evolution such that $\hat{R}_3$ over here will change in time.
How would I compute the total time evolution of this $\hat{H}_A$
Under time evolution

$$\hat{H}_\tau = \hbar \omega_0 \hat{R}_3$$

I will take its commutator with respect to total Hamiltonian and whenever this $\hat{R}_3$ is commutator with total Hamiltonian will arrive. I will call this $\hat{R}_3(\tau)$ . This is the total time evolution of the free part of the atom. Similarly, we can try to look at the free part of the quantum field. Similarly, we can try to look at the free part of the quantum field. Each mode is bringing energy $\bar{h}\omega_{\vec{k}}$ and how many such modes are excited, $a_{\vec{k}}^\dagger a_{\vec{k}}$

This is the number expectation? This is the standard exercise which we had done in the scalar field when we discussed and this is the normal order Hamiltonian which ignores the $\delta_0$ part coming into the Hamiltonian.
Here, let us notice once more that this $\omega_k$ which is appearing over here is the frequency of the mode as seen from the lab frame.



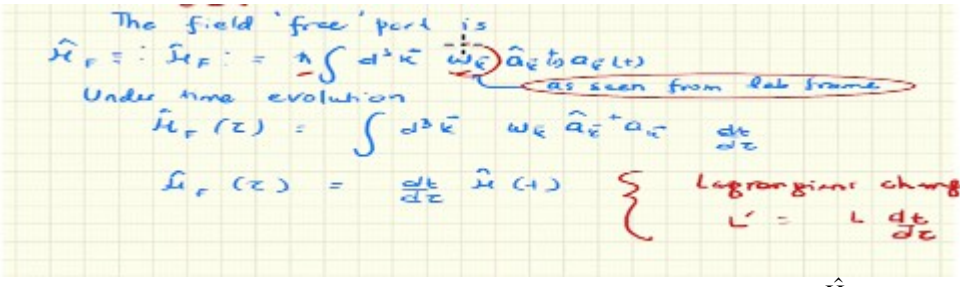

The field part is

$$\hat{H}_F = \hat{H}_F \hbar \omega_0 = \hbar \int d^3\vec{k}\, \omega_{\vec{k}}\, \hat{a}^\dagger_{\vec{k}}(t)\, a_{\vec{k}}(t) \frac{dt}{d\tau}$$

As seen from lab frame.

From the atoms frame, it is $\omega_k$ whose expression which we obtained in the previous discussion session. But right now. I have given a decomposition in terms of the left way either I can lower transform everything $d^3\vec{k}$ $\omega_{\vec{k}}$ $a^\dagger_{\vec{k}}$ and $a_{\vec{k}}$ , and obtain the Hamiltonian for the field as seen by the atom or we can

try to find out what it looks like in the eyes of the atom itself.




The atom at the description of the field is given by the Hamiltonian $\hat{H}_F$ and in frame of atom its time evolution is controlled by parameter $\tau$ which is just the previous Hamiltonian times an extra factor $dt/d\tau$. So, the free fields Hamiltonian as seen by the atom is the free field Hamiltonian as seen by the lab observer times this $dt/d\tau$.

How do I obtain that?
Just approach this problem from the structure that the Lagrangian or Lagrangian since action has to remain preserved in any frame. And Lagrangian has to transform with this relation under Lorentz transformation and therefore Hamiltonian derived from it will also inherit that relation dt/dτ factor.
Under time evolution

$$\hat{H}_F(\tau)=\int d^3\vec{k}\,\omega_{\vec{k}}\hat{a}^{\dagger}_{\vec{k}}(t)a_{\vec{k}}(t)$$

$$\hat{\phi}(t,x)=\int \frac{d^3k}{\sqrt{2\omega_{\vec{k}}}}\frac{1}{(2\pi)^{\frac{3}{2}}}(a_k e^{ikx}+\hat{a}^{\dagger}_k e^{-ikx})$$

$$\hat{H}_F(\tau)=\frac{dt}{d\tau}\hat{H}(\tau)\left(Langrangian\ changes\ as\ L'=L\frac{dl}{dt}\right.$$

Now, in this way the field was initially written in the lab frame modes, lab frame frequency, lab frame mode scale of the wave numbers and the operators $a_{\vec{k}}$ and $a^{\dagger}_{\vec{k}}$ type. That was the scalar and that is why since it is scalar field, there is no extra $dx\,\mu$ factor, the Jacobian or the λ matrix transformation which will associate it is why try to jump from one frame to another.

Now, we come to the interaction term. Previously, we had written the interaction term in terms of the lab observer. We want to cast it back in terms of the atom. Same thing will happen. One will write down the let us say monopole operator. This time I am just writing it as some parameter of $\mu$ and $R_2$ operator whose definition I am just coming in a minute and times $\phi$. This is the $\phi$ the field as seen by the atom at different location in its own coordinates.

With coupling between the field and the atom

$$\hat{H}_I(\tau) = \underbrace{\mu \hat{R}_2(\tau)}_{\hat{m}} \qquad \underbrace{\phi(x(\tau))}_{\text{Field at locations as seen by the atom}}$$

This monopole operator we have seen before. I am just writing it with some parameter $\mu$ times some operator.

"Typically," $\hat{m}$ or $\hat{R}_2$ is a transition matrix operator connecting $|+\rangle\langle -|$
This operator $\hat{R}_2$ typically is a transition matrix operator. That means only off diagonal elements survive.

It is spectral decomposition is, not spectral decomposition, it is decomposition is such that in the eigen basis of the free Hamiltonian is $\hat{R}_{+.}$ is a raising operator in some sense that it takes state, ground state and projects it to excited state, while $\hat{R}_{-.}$ is the inverse of that which takes the excited state and projects it back into the ground state.

$$\hat{R}_2 = \left(\frac{i}{2}\right)\left(\hat{R}_{-.} - \hat{R}_{+.}\right)$$

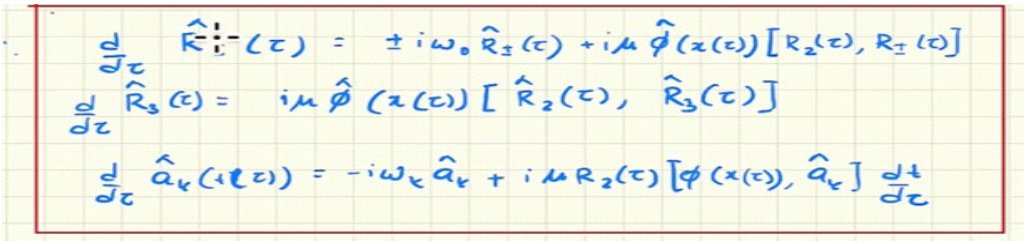
For hermiticity

Therefore, this monopole operator or the transition operator can be written like $(\hat{R}_{-.} - \hat{R}_{+.})$ with $\frac{i}{2}$ factor just to restore the hermeticity. This is just $\sigma_+$ and $\sigma_-$ if you recall in terms of Pauli matrices. This is the interaction term with respect to the field. Now we have everything in terms of atoms. Now, atom has this description of physics going along that it has one Hamiltonian $\hat{H}_A$ it has a one field around it whose description is given by the $\phi$ at location $x_\tau$ and there is an interaction term which now is being written in terms of atoms clock as well. We have everything at hand from the point of view of atoms.
Remember, we have taken this philosophy that we will compute things of our interest in atoms rest frame where all the Schrodinger mechanics for atoms happens. Therefore, the perturbation theory which we developed for non-relativistic systems keeps applying in the frame of atom. It will not be applicable if I start doing physics from the lab frame because atom is relativistically moving. So, better strategy is to jump into the atom frame and complete the task.

We write down all the relevant commutators which will appear if I try to write down the time evolution of these operators because remember time evolution of these operators is controlled by these operators commutation value with the total Hamiltonian. So, various commutators will appear in that.
Let us list down all the commutators since $\hat{R}_2, \hat{R}_3, \hat{R}_1$, $\hat{R}_{+.}$ and $\hat{R}_{-.}$ are very similar to what we have seen for the Pauli-matrices, their commutations are also following that. That means $\hat{R}_3$ and $\hat{R}_{+.}$, $\hat{R}_{-.}$ commutator will be up to $\pm R_\pm$. That means if I am taking commutator of $\hat{R}_3$ with $\hat{R}_{+.}$, I will get $\hat{R}_{+.}$. $R_3$ with $\hat{R}_{-.}$, I will get a - $\hat{R}_{-.}$. $\hat{R}_{+.}$ and $\hat{R}_{-.}$ will be twice of $\hat{R}_3$ and so on.

$$[\hat{R}_2, \hat{R}_\pm] = \pm \hat{R}_\pm \quad , \quad [\hat{R}_2, \hat{R}_\pm] = 2\hat{R}_3$$

Total $\hat{H} = \hat{H}_A + \hat{H}_F + \hat{H}_I$



This set of commutation relation is handy because now as we discussed total Hamiltonian is the collection of all the Hamiltonian of individual atom, the field and interaction all written in the atom's frame. Okay go ahead and quickly write down the time evolution of all the operators of interest. For example, if I want to write down dt/dτ of $\hat{R}_{-\cdot}$ , then I will take the $\hat{R}_{+\cdot}$ or $\hat{R}_{-\cdot}$ commutator with its total Hamiltonian.

$$\frac{d\hat{R}_{\pm}(\tau)}{d\tau} = \pm i\omega_0 \hat{R}_{\pm}(\tau) + i\mu\hat{\phi}(x(\tau))[\hat{R}_2(\tau), \hat{R}_{\pm}(\tau)]$$
$$\frac{d\hat{R}_3(\tau)}{d\tau} = 0 + i\mu\hat{\phi}(x(\tau))[\hat{R}_2(\tau), \hat{R}_3(\tau)]$$
$$\frac{d\hat{a}_k(t(\tau))}{d\tau} = -i\omega_k \hat{a}_k + i\mu[\hat{\phi}(x(\tau)), \hat{a}_k] dt/d\tau$$

Remember this $\hat{R}_{+\cdot}$ is made of Pauli-matrices which concerns only with the atomic sector. It will just commute with the pre-filled sector. This only contains field operators, atomic operators, and a mixed term, both atomic operator and field operator. When I do a commutator with $\hat{R}_{+\cdot}$ , if I take the commutator of $\hat{R}_{+\cdot}$ with the total Hamiltonian, that $\hat{R}_{+\cdot}$ operator will have a non-trivial commutation with the first term and the third term. These involve operators from atomic sector. With second term, it would be trivially commuting because field operator is different from the atomic operator. Therefore, when I try to compute the $\hat{R}_{+\cdot}$ , it will undergo $\hat{R}_{+\cdot}$ + commutator with $\hat{H}_A$ and $\hat{R}_{+\cdot}$ commutator with $\hat{H}_I$ . So, $\hat{R}_{+\cdot}$ commutator with $\hat{H}_A$ will give me this term, which is the free field evaluation, free evolution of the atomic system. But due to interaction now this new term will be generated which is $R_2$ commutator, $\hat{R}_{+\cdot}$ commutator with the interaction Hamiltonian.

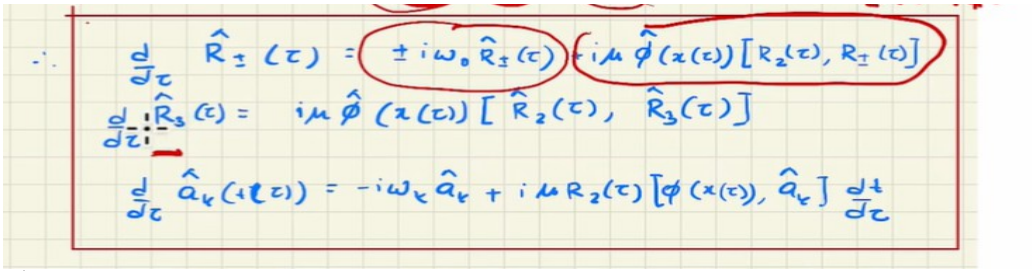
Similarly, when I try to write down the time evolution of $R_3$ which appears in the Hamiltonian of the atom. Time evolution of this if I want to obtain, I should write down the $\frac{d\hat{R}_3(\tau)}{d\tau}$ .Every time evolution is right now being done in the clock of atom because this is where the Schrodinger evolution holds. Again the same thing which we will do, take this $\hat{R}_3$ , obtain its commutator with the total Hamiltonian, $[\hat{R}_3, \hat{H}_A]$ will be 0 because $\hat{H}_A$ itself is also $\hat{R}_3$ up to a constant. $\hat{H}_A$ is also $\hat{R}_3$ , so $\hat{R}_3$ will commute trivially with $\hat{H}_A$ , $\hat{R}_3$ will also commute with $\hat{H}_F(\tau)$ because $\hat{H}_F(\tau)$ is made from different operator, field operator, they do not concern that. Only thing which will be surviving with

non-zero value is $\hat{R}_3$ 's commutator with $\hat{H}_I$ and remember what was $\hat{H}_I$ , $\hat{H}_Y$ was just $\hat{R}_2$ , some $\mu$ times $\hat{R}_2$ , so you will see that the total term on the right hand side will be the interaction Hamiltonian commutator of that with respect to $\hat{R}_3$ and that is why this term will survive.

Lastly, if I look for the field operator itself, its evolution, again the $\hat{a}$ operator will undergo the commutation relation with respect to all the three terms.

First term does not concern with the field, so it will vanish, the commutator will be 0. $[H_F, \hat{H}_A]$ commutator gives the free field This is the pre-fill time evolution of the ladder operator which we had seen previously. But now due to this interaction term alive, this $a_{\vec{k}}$ will have a non-zero commutation relation with respect to $\hat{H}_I$ , because in $\hat{H}_I$ there is not only $\mu$ but also $\phi$ term present which also have a and a † and what not. So, if you write in terms of that, I will have a non-trivial commutation relation between the a and the $\phi$ which will be appearing into the third term. And this will be the total derivative of the time derivative of the $a_{\vec{k}}$ operator. These are the relevant operators time evolution which appear in each of the pieces of the individual Hamiltonian. A Hamiltonian is made from $\hat{R}_3$ . $\hat{H}_F(\tau)$ is made from $a_{\vec{k}}$ and $a^{\dagger}_{\vec{k}}$ and $\hat{H}_I(\tau)$ Hamiltonian is made from $\hat{R}_{+\cdot}$ , $\hat{R}_{-\cdot}$ and $\phi$. Therefore, we have the temporal derivative of all the relevant operators which appear at different places in the time evolution. Therefore, now we are ready to write down the total time evolution of anything. Just to be curiously informed, you can see that every piece over here when we write down breaks into two parts. This is free part. This would have been as the usual evolution if there were no interaction plus controlled evolution due to the interaction. This is $0^+$ , $0^+$ was the free evolution. If there were no coupling of atom and film, this was zero term. Therefore, this evolution is coming from the coupling term. Lastly, the field operator also has a free part and there is a coupling part.

All the operators which we have written down can be written that they have a free field, free evolution and there is a sourced evolution in the sense of due to coupling. Free evolution and sourced evolution, free evolution and the sourced evolution. And you can see that the this is a differential equation structure.

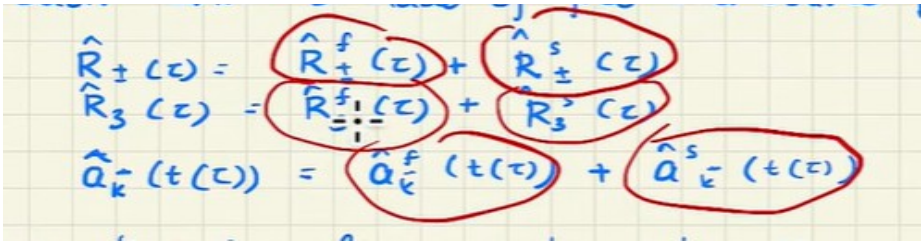


$$\frac{d\hat{R}_{\pm}(\tau)}{d\tau} = \pm i\omega_0 \hat{R}_{\pm}(\tau) + i\mu\hat{\phi}(x(\tau))[\hat{R}_2(\tau), \hat{R}_{\pm}(\tau)]$$

$$\frac{d\hat{R}_3(\tau)}{d\tau} = 0 + i\mu\hat{\phi}(x(\tau))[\hat{R}_2(\tau), \hat{R}_3(\tau)]$$

$$d\frac{\hat{a}_k(t(\tau))}{d\tau} = -i\omega_k\hat{a}_k + i\mu[\hat{\phi}(x(\tau)), \hat{a}_k]\frac{dt}{d\tau}$$

So, if I start writing all the R operator as free part plus source part, all the $a_{\vec{k}}(t)$ operators as free part and source part, then it will become an equation which will be connecting various free part and source part.

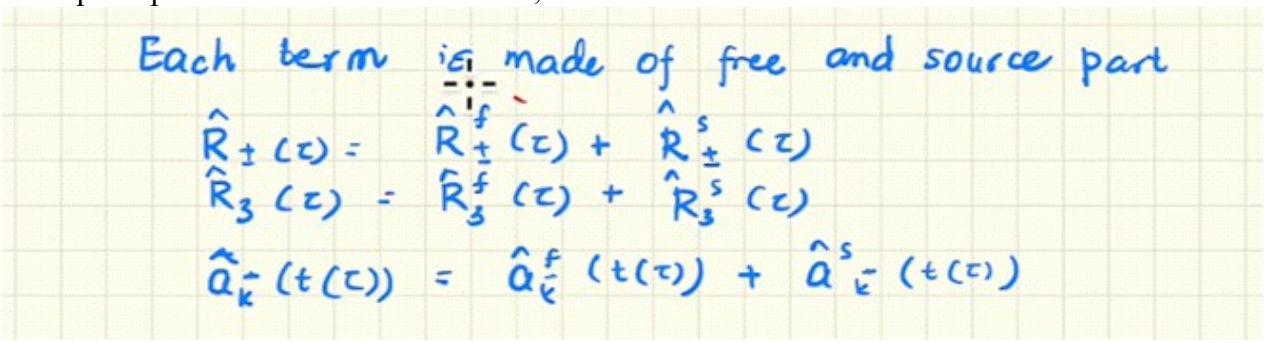


$$\hat{R}_{\pm}(\tau)=\hat{R}^{f}_{\pm}(\tau)+\hat{R}^{s}_{\pm}(\tau)$$

$$\hat{R}_{\tau}=\hat{R}^{f}_{3}(\tau)+\hat{R}^{s}_{3}(\tau)$$

$$\hat{a}_{\vec{k}}(t(\tau))=\hat{a}^{f}_{\vec{k}}(t(\tau))+\hat{a}^{s}_{\vec{k}}(t(\tau))$$

You can see if I write down here, let us say or here, it will have a $\hat{R}^{f}_{2}(\tau)$ and $\hat{R}^{s}_{2}(\tau)$ Similarly, we have $\hat{R}^{f}_{\pm}(\tau)$ and $\hat{R}^{s}_{\pm}(\tau)$ . Similarly, for $\phi$ , you will write $\hat{\phi}^{f}_{s}$ and $\phi^{f}+\phi^{s}$ . So, you will see this kind of things will appear and it will become an iterative equation. That means here $\hat{R}^{f}_{3}(\tau)$ part and $\hat{R}^{f}_{\pm}(\tau)$ sourced part will depend on the sourced part and free part itself as well. You see there is a coupling between the free part and the sourced part and so on. If I try to write down let us say free part and the sourced part up to linear order in interaction,




Each term is made up of free and source part.

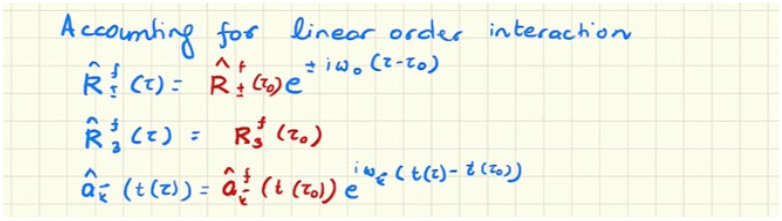
$$\hat{R}^{f}_{\pm}(\tau)=\hat{R}^{f}_{\pm}(\tau_0)\,e^{\pm i\omega_0(\tau-\tau_0)}$$
$$\hat{R}^{f}_{3}(\tau)=\hat{R}^{f}_{3}(\tau_0)$$
$$\hat{a}_{\vec{k}}(t(\tau))=\hat{a}^{f}_{\vec{k}}(t(\tau_0))\,e^{i\omega_{\vec{k}}(t(\tau)-t(\tau_0))}$$

Accounting for linear order interactions.

Therefore, I will put in this when I am writing over here, I put it $\hat{R}^{f}$ and $\hat{R}^{s}$ everywhere, and $\hat{\phi}^{f}$ and $\hat{\phi}^{s}$ . I will see there is a free parts evolution, which is controlled by free parts.

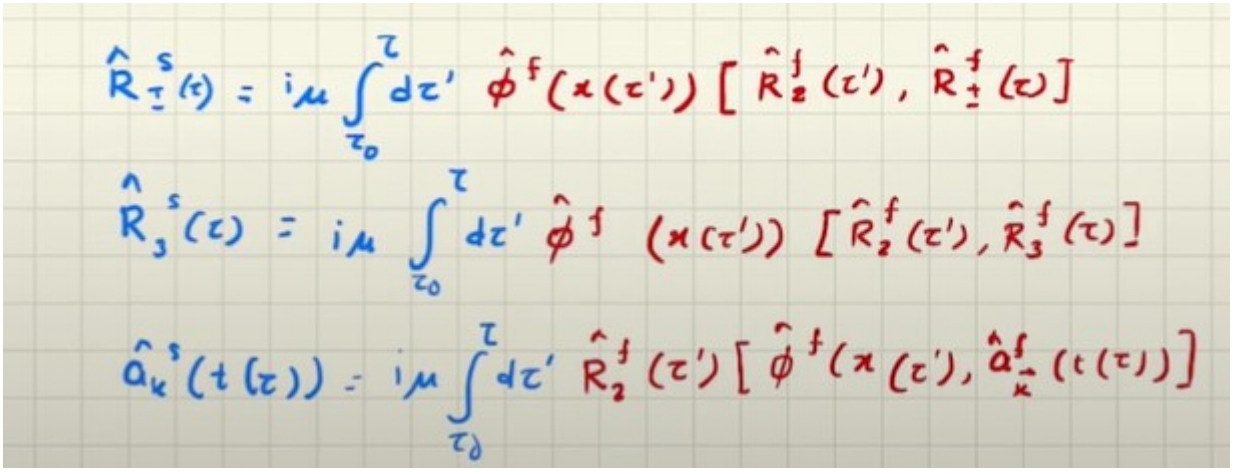
But source part contribution, if I am looking for source term evolution, that means from here if I want to write down $\frac{dR^{s}}{d\tau}$ are various terms which are coming. $\frac{dR^{s}}{d\tau}$ will be made from first term $\omega_0$ and

$\hat{R}^s$ here. Then there is a $\phi^f$ which can multiply either $\hat{R}^f$ here, $\hat{R}^s$ here. Accounting for linear order interactions.




$$\hat{R}^f_{\pm}(\tau) = \hat{R}^f_{\pm}(\tau_0)\, e^{\pm i\omega_0(\tau-\tau_0)}$$
$$\hat{R}^f_3(\tau) = \hat{R}^f_3(\tau_0)$$
$$\hat{a}^f_{\vec{k}}(t(\tau)) = \hat{a}^f_{\vec{k}}(t(\tau_0))\, e^{i\omega_{\vec{k}}(t(\tau)-t(\tau_0))}$$

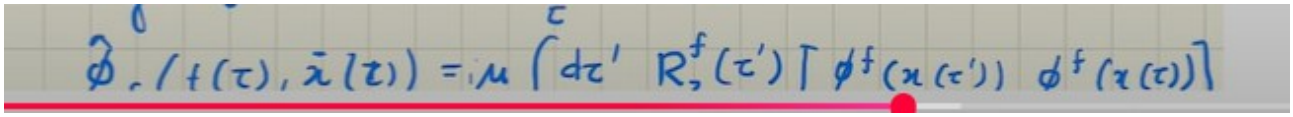
So, there are three operators which are appearing one after another. If I open the commutator and both all of the three operators can be broken into free part and sourced part, free part and sourced part. And sourced part if you see it is coming through the interaction term that means the parameter of $\mu$ appears once. So, if I want to truncate things at the leading order, only one operator of source nature should appear and two operators of free nature should appear.




$$\hat{R}_{\pm}(\tau) = i\mu \int_{\tau_0}^{\tau} d\tau'\, \hat{\phi}^f(x(\tau'))\,[R_2(\tau'), \hat{R}^f_{\pm}(\tau)]$$
$$\hat{R}_3(\tau) = i\mu \int_{\tau_0}^{\tau} d\tau'\, \hat{\phi}^f(x(\tau'))\,[R_2(\tau'), \hat{R}^f_3(\tau)]$$
$$\hat{a}_{\vec{k}}(t(\tau)) = i\mu \int_{\tau_0}^{\tau} d\tau'\, \hat{R}^f_2(\tau)\,[\hat{\phi}^f(x(\tau')), a^f_{\vec{k}}(t(\tau))]$$

Ultimately there are terms which is for example, $\phi^f$ for free and then commutator I would have $R^f_2$ and $R^s_{\pm}$ let us say. There could be term like $\phi^f$ and both of them are $R^s_2$ and $R^s_{+.}$ are sourced. Similarly, there could be a term $\phi^s$, which will be either multiplying this term or multiplying that term.

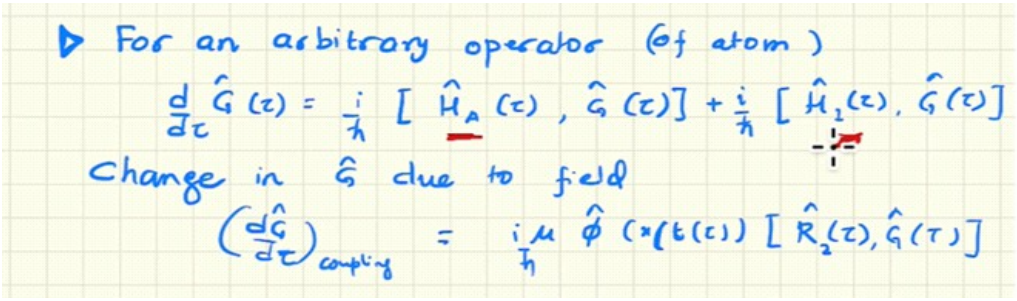
And similarly, there could be a term $\phi^s$, which will be either multiplying this term or multiplying that term. But you see, when I try to write down for $\phi^s$ again, $\phi^s$ itself will satisfy a differential equation structure controlled by the $a_{\vec{k}}$ vectors as well. If I have to keep feeding the value of $\phi^s$ or $R^s$ or the $R_{\pm}$ here and $R_2^s$, it will become higher and higher power in $\mu$ because $R^s$ becomes non-zero only when there is a $\mu$. That means whenever $\hat{R}^s$ appears, it will come with a $\mu$ and outside there is already a $\mu$. Therefore, when I have to write down the thing at the leading order, only one operator is allowed to become a forced operator and even that operator will be multiplying an extra $\mu$ to it. So, that means at the leading order thing, $\mu$ with all free, free, free part would be order $\mu$. Anything, if any of this thing develops $R^s$ or $\phi^s$ $R^s$ then basically what we are meaning that it is next order term, it is order $\mu^2$ term. Therefore, we have to be very aware of the fact that all the $R^s$ which are appearing an iterative equation of $\hat{R}^f_{+.} + \hat{R}^s_{+.}$ at the leading order $\hat{R}^f_{+.}$ should be obtained of $\mu$ that means all the operators appearing inside are just free parts and all free parts multiplying together with the coupling parameter of $\mu$ generate for you what is the source part. This is leading order result. In principle, you have to feed this relation and on and on again, there was truly $\hat{\phi}(x(\tau'))$, $\hat{R}^f_{\pm}$ and $\hat{R}^f_{\pm}$ were sitting. If you want to write down in terms of their free and source part, you will obtain many, many terms which are all higher order. This is the philosophy we do and at the leading order, all the operators appearing are made into free operators and the source operators are just derivative quantities from the free operators. So, using that we can write down the leading order source correction to the source correction to the quantum field itself as $i\mu$ times all the operators which are appearing over here we can just multiply mode function and † of that to obtain the field and it will become multiplied with $v_k$ over here do the † thing the field will come and sit over here and all the quantities from the source part will be converted into their free form. You will get this operator determining what is the source correction to the field. The free field theory was giving me the field operator as the usual this thing this is the free part to that a new correction has been added because of its interaction with atom which cares about atomic operators. Atomic operators tell the field how to modify at the leading order.
This is the correction brought to the field itself. Similarly, corrections will be brought to any operators of atom as well. Again, at the leading order, we will do all the correction term with their free forms appearing in the commutator.
Using this



$$\hat{\phi}_s(t(\tau)), \vec{x}(\tau) = \mu \int_{\tau_0}^{\tau_0} d\tau' [\hat{\phi}^f(x(\tau')), \hat{\phi}^f(x(\tau'))]$$

For instance, if I am talking about some operator $G(\tau)$ of an atom, this is atomic operator, it is not the field operator. Then again, I have to find out what is its time evolution rate. I will take its commutator with the total Hamiltonian, total Hamiltonian is made from First piece involves atomic operators, and $\hat{G}(\tau)$ might or might not commute with that. Third operator also involves atomic operator, so G might or might not commute with that. But middle term which was the free field Hamiltonian, that only involves field operators. So, atomic operator $\hat{G}(\tau)$ ill definitely commute with that, so I do not write that term. Now this is the trivial evolution under no interaction. This much was just a standard

evolution. Now, if I focus what is the change brought by coupling, and I can write down $\frac{d\hat{G}}{d\tau}$ of coupling, that is just this, that term and I open up what is the interaction term.




⇨

For an arbitrary operator (of atom)

$$\frac{d\hat{G}}{d\tau} = \frac{i}{\hbar}[\hat{H}_A(\tau), \hat{G}(\tau)] + \frac{i}{\hbar}[\hat{H}_I(\tau), \hat{G}(\tau)]$$

Change in $\hat{G}(\tau)$ due to the field

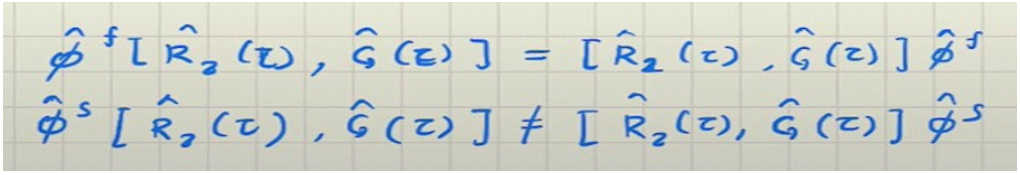
$$\left(\frac{d\hat{G}}{d\tau}\right)_{coupling} = i\frac{\mu}{\hbar}\hat{\phi}(x(\tau))[\hat{R}_2(\tau), \hat{G}(\tau)]$$

Interaction term remember was $\mu$ times the operator $\hat{R}_2 \times \phi$. It does not matter whether I write $\phi$ on the left or right of $\hat{R}_2$ because $\hat{R}_2$ and $\phi$ are different Hilbert space operators. $\hat{R}_2$ acts on the states of atom, $\phi$ acts on the state of the field. Therefore, I can write it either way.
So, as you see, again we have the same philosophy which we discussed about that this is made from $\phi^f$ and $\phi^s$

Since $\phi = \phi^f + \phi^s$, the coupling driven changes are driven by two terms separately.

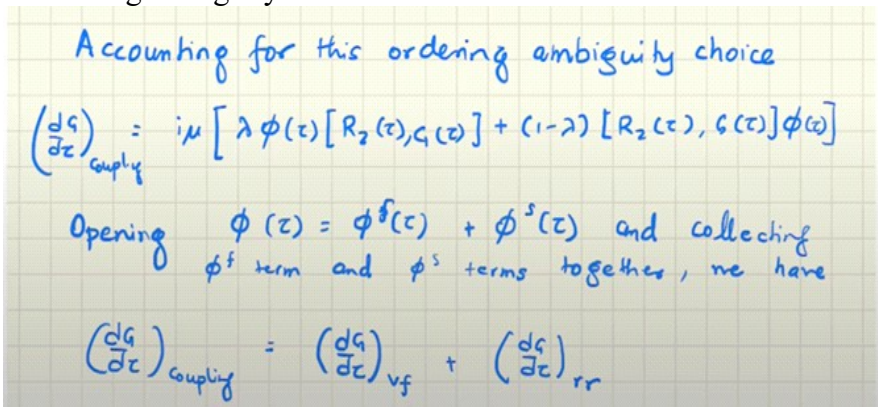
There is a $\phi^s$ sitting over here as well. Now, remember the expression of $\phi^s$ which we just derived. $\phi^s$ depends on the atomic operators as well. So, by the time when I said that $\phi$ and atoms are different operators, that was different Hilbert space operator, that was a statement for free field theory. In free field theory, $\phi$ is completely made of operators which do not care about what state atom is. But now when the interaction is on, this $\phi$ which is written over here is made up from which also contains an atomic operator. This time it is no longer true that if I write on the left or the right, it does not matter because this time post of $\phi$ going across $\hat{R}_2$.
But $\hat{\phi}^s$ also contains operators from atomic side. ⇨

Thus



$$\hat{\phi}^f[\hat{R}_2(\tau),\hat{G}(\tau)]=[\hat{R}_2(\tau),\hat{G}(\tau)]\hat{\phi}^f$$
$$\hat{\phi}^s[\hat{R}_2(\tau),\hat{G}(\tau)]\neq[\hat{R}_2(\tau),\hat{G}(\tau)]\hat{\phi}^s$$

$\hat{R}_2$ and field commutation is non-zero in absence of interaction. If interaction were not there, then $\phi^s$ part would have been vanishing and then this would have been zero, the commutator would have been zero, then writing left or right is immaterial. But since we are writing the field also get corrected due to the interaction of atoms, the is part being written on the left of this commutator or on the right of this commutator are two different things. They are not the same thing. That story was true only for the free field part. Therefore, we do not know which way to write. Either I write here or I write there or some 30% here, 70% there. So, some operator ordering ambiguity arises out of it, which way to write this product? So, we do ad-hoc splitting, λ for fraction of that is written on the left and $1-\lambda$ fraction is written on the right. λ could be chosen any number depending upon what observations nature tells us. So, a priori there is no way to write as of now that which way I should split.
Accounting for this ordering ambiguity choice



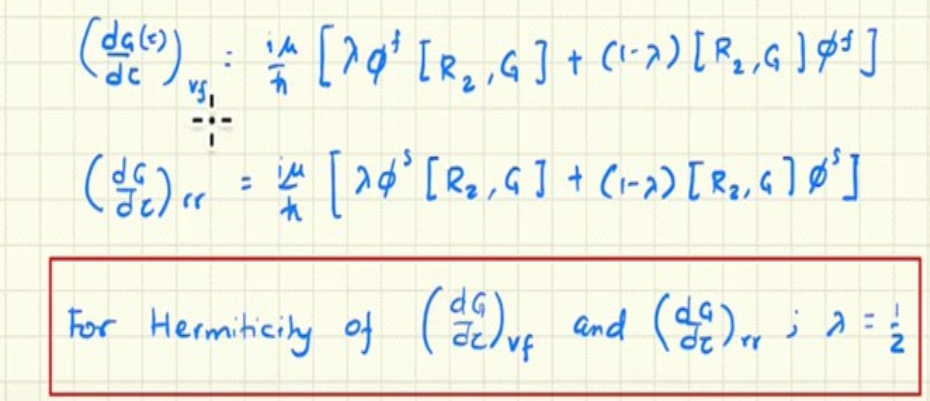

$$\left(\frac{d\hat{G}}{d\tau}\right)_{coupling}=i\mu[\lambda\hat{\phi}(\tau)[\hat{R}_2(\tau),\hat{G}(\tau)]+(1-\lambda)[\hat{R}_2(\tau),\hat{G}(\tau)\hat{\phi}(\tau)]]$$

$$\left(\frac{d\hat{G}}{d\tau}\right)_{coupling}=\left(\frac{d\hat{G}}{d\tau}\right)_{vf}+\left(\frac{d\hat{G}}{d\tau}\right)_{rr}$$

Now, again, do the same game again that if I open ϕ as $\phi^f+\phi^s$ , open this thing as $R_2^f+R_2^s$ and all the operators over here, then you will find out that the $\frac{d\hat{G}}{d\tau}$, the coupling term that also splits into two portions, one determined by field part, free field part in some sense, this is vacuum fluctuation part is called. And then there is an interaction of atom and the free, this is called radiation reaction term.

Ultimately, the coupling term also splits into two nice terms, which are like this.
The coupling term also splits into two nice terms, which are like this.
Opening $\hat{\phi}(\tau)=\hat{\phi}^{f}(\tau)+\hat{\phi}^{s}(\tau)$ and collecting $\hat{\phi}^{f}$ and $\hat{\phi}^{s}$ together we have

So, the vacuum fluctuation term is made from the free field part.
What did I do? I just split it up. I just split the field operator into its two segments, free part and the source part. Therefore, all the terms will in this expression $\phi$ appears only once. So, it is linear in $\phi$. Therefore, it will split up into its free part and its source part. Therefore, I will have two equations, one for free part and one for source part.



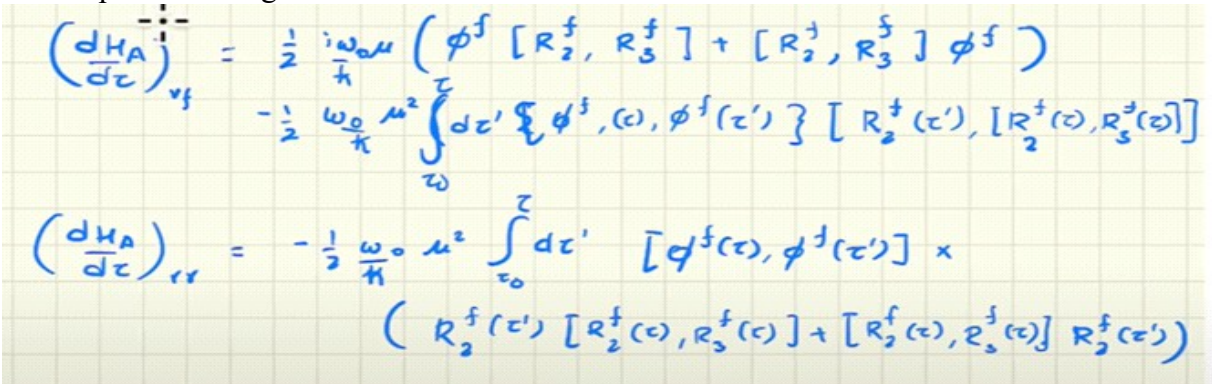

$$\left(\frac{d\hat{G}}{d\tau}\right)_{vf}=i\frac{\mu}{\hbar}[\lambda\phi^{f}[R_2,G]+(1-\lambda)[R_2,G]\hat{\phi}^{f}]$$
$$\left(\frac{d\hat{G}}{d\tau}\right)_{rr}=i\frac{\mu}{\hbar}[\lambda\phi^{s}[R_2,G]+(1-\lambda)[R_2,G]\hat{\phi}^{s}]$$
*For Hermiticty of* $\left(\frac{d\hat{G}}{d\tau}\right)_{vf}\wedge\left(\frac{d\hat{G}}{d\tau}\right)_{rr}$ ; $\lambda=\frac{i}{2}$

The source part equation is called the radiation reaction term. And the free part term is called the vacuum fluctuation term. Now, you see if I want this $\frac{d\hat{G}}{d\tau}$ which is an operator's time evolution to be Hermitian as well. At this stage, we can fix there is no way but to fix $\lambda$ to one half because you see if $\lambda$ is any other value take the † of this thing it will not come back to itself. So, this is the observation the DTC approach realized that in order to have a Hermitian description of the time evolution of operator, $\lambda$ is necessarily required to be one half. Therefore, this gives you a good description, put this this to one half as well. This gives you a good description of how a field operator or atomic operator rather would change its character due to interaction with the field itself. For instance, if I want to write down what is the change in the atoms Hamiltonian itself completely.

So, again it will be obtained from these two terms.
There will be $\frac{d\hat{G}}{dt}$ of atom vacuum fluctuation part which will be totally determined by the free field part and all the operators appearing over here will be put to their free field value and a radiation reaction part which will be made from the sourced Hamiltonian. Here the G would be written as the Hamiltonian itself and Hamiltonian remember was what? It was something times $\hat{R}_3$ say it is $\omega_0\times\hat{R}_3$.

So, wherever this you see, you will see $\hat{R}_3$ . Therefore, you will see $\hat{R}_3$ commutator will appear and their splitting one halves will appear.

If we look up at the change in Hamiltonian $\hat{H}_A$ itself.

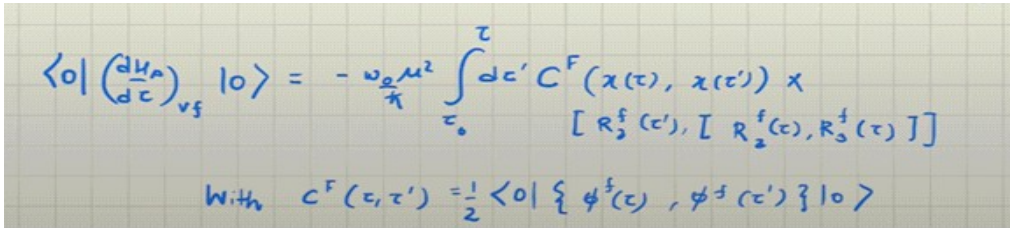


$$\left(\frac{d\hat{H}_A}{d\tau}\right)_{vf} = \frac{1}{2}\frac{i}{\hbar}\omega_0\mu\left(\phi^f[R_2^f, R_3^f] + [R_2^f, R_3^f]\phi^f\right) - \frac{1}{2}\frac{\omega_0}{\hbar}\mu^2\int_{\tau_0}^{\tau} d\tau' \{\phi^f(\tau), \phi^f(\tau')\}[R_2^f(\tau'), [R_2^f(\tau), R_3^f(\tau)]]$$

$$\left(\frac{d\hat{H}_A}{d\tau}\right)_{rr} = -\frac{1}{2}\frac{\omega_0}{\hbar}\mu^2\int_{\tau_0}^{\tau} d\tau' [\phi^f(\tau), \phi^f(\tau')] \times \left(R_2^f(\tau')[R_2^f(\tau), R_3^f(\tau)] + [R_2^f(\tau), R_3^f(\tau)]R_2^f(\tau')\right)$$

$$\left(\frac{d\hat{H}_A}{d\tau}\right)_{vf} = \frac{1}{2}\frac{i}{\hbar}\omega_0\mu(\phi^f[R_2^f, R_3^f] + [R_2^f, R_3^f]\hat{\phi}^f) - \frac{1}{2}\frac{i}{\hbar}\omega_0\mu^2\int_{\tau_0}^{\tau} d\tau'\hat{\phi}^f(\tau), \hat{\phi}^f(\tau')[\hat{R}_2^f(\tau'), [\hat{R}_2^f(\tau)\hat{R}_3^f(\tau)]]$$

$$\left(\frac{dH_A}{d\tau}\right)_{rr} = \frac{-1}{2}\frac{i}{\hbar}\omega_0\mu^2\int_{\tau_0}^{\tau} d\tau'[\hat{\phi}^f(\tau), \hat{\phi}^f(\tau')] * (\hat{R}_2^f(\tau')[\hat{R}_2^f(\tau), \hat{R}_3^f(\tau)] + [\hat{R}_2^f(\tau), \hat{R}_3^f(\tau)]\hat{R}_2^f(\tau'))$$



$$\langle 0|\left(\frac{d\hat{H}_A}{d\tau}\right)_{vf}|0\rangle = -\frac{\omega_0}{\hbar}\mu^2\int_{\tau_0}^{\tau} d\tau' C^F(x(\tau), x(\tau')) \times [R_3^f(\tau'), [R_2^f(\tau), R_3^f(\tau)]]$$

$$\text{with } C^F(\tau, \tau') = \frac{1}{2}\langle 0|\{\phi^f(\tau), \phi^f(\tau')\}|0\rangle$$

So

$$\langle 0|\left(\frac{d\hat{H}_A}{d\tau}\right)_{vf}|0\rangle = -\frac{1}{2}\frac{i}{\hbar}\omega_0\mu^2\int_{\tau_0}^{\tau} d\tau' C^F(x(\tau), x(\tau')) \times [\hat{R}_2^f(\tau'), [\hat{R}_2^f(\tau'), \hat{R}_2^f(\tau), \hat{R}_3^f(\tau)]]$$

where $C^F(\tau, \tau') = \frac{1}{2}\langle 0|\hat{\phi}^f(\tau), \hat{\phi}^f(\tau')|0\rangle$

**and**

$$\langle 0|\left(\frac{d\hat{H}_A}{d\tau}\right)_{rr}|0\rangle = \frac{1}{2}\frac{i}{\hbar}\omega_0\mu^2\int_{\tau_0}^{\tau} d\tau' X^F(\tau, \tau') \times \hat{R}_2(\tau')[\hat{R}_2^f(\tau'), \hat{R}_2^f(\tau), \hat{R}_3^f(\tau)]$$

with $X^F(\tau, \tau') = \frac{1}{2}\langle 0|[\phi^f(\tau), \hat{\phi}^s(\tau')]|0\rangle$

And here if I look at the radiation reaction part of the total Hamiltonian, a $\hat{\phi}^s$ is here which is controlling, $\hat{\phi}^s$ is appearing here as well as here. And recall the $\phi$ s itself was, the $\phi$ s itself was obtainable from the commutator between $\phi^f$ at two different locations multiplying the $\hat{R}_2$ . So, you

have to put this expression of $\phi^s$ back into the total change in the Hamiltonian.
So, radiation reaction term will be made from the commutator between the fields, their free parts commutator at two different times and then the splitting of $\hat{R}_2$ appearing across commutator of $\hat{R}_2$ and $\hat{R}_3$ in the symmetric way. So, that would be the radiation reaction change in the Hamiltonian and this will be due to vacuum fluctuation. Compactly it can be written as, you can see the structure which is appearing.

The vacuum fluctuation is made from the anticommutator of the three fields and commutators of this and that. While for the radiation reaction, the opposite happens. Commutator of the field appears and anticommutator of this object $\hat{R}_2$ and commutator of, $\hat{R}_3$ appears. So, you see $\hat{R}_2$ and commutator of $\hat{R}_2$, $\hat{R}_3$, its commutator appears here and field anticommutator appears here and the exactly opposite appears in the radiation reaction. So, we can just define quantities like field anti-commutator with $C^F$ and field commutator with respect to field commutator as $\chi^F$. And if suppose I want to know the atomic Hamiltonian rate of change if the field is in the vacuum state.

Because atomic Hamiltonian rate of change cares about which state atom is and which state field is, I am prescribing the field is in vacuum state. So, previously the anticommutator and the commutator were appearing. Now, those anticommutator and commutator will get squeezed between the vacuum states. So, that I am calling $C^F$ and $\chi^F$. So, commutator's expectation in vacuum and anticommutator's expectation in vacuum, they go and sit over here. So, I am taking expectation here and expectation in the preceding term here to obtain at least the field segment has been fixed to be the vacuum. This is due to vacuum fluctuations and vacuum commutator and vacuum anti-commutator is telling me what is the total rate of the change of Hamiltonian operator of atomic system. This is happening due to vacuum fluctuations and radiation reaction from vacuum. Both things put together generate this much change in the Hamiltonian of the atom. Now what we will do, we will take this change in the Hamiltonian and find out due to this change how much change in the Eigen space separation occurs and that is the amount of Lamb shift and atoms occur which we will evaluate in the next class.