Foundation of Quantum Theory: Relativistic Approach

Relativistic corrections in transitions

Prof . Kinjalk Lochan

Department of Physical Sciences

IISER Mohali

Field driven transitions for relativistic particles

Lecture- 40

So, today we will discuss about the relativistic effects on atoms, meaning when the atom is talking to electromagnetic or any background quantum fields and if it is moving with respect to a lab observer and moving with relativistic speeds, Schrodinger equations is not supposed to be valid in such a case. Because we know that relativistic moving particles are dictated by either Dirac equation or Klein-Gordon equation, only their limiting case is Schrodinger equation as we saw. So, therefore, how do we correct for the expressions which we obtain of transitions, weights and transition probabilities and then how to write down the Lindblad kind of formalism for certain atoms which are moving with finite velocity comparable to the relativistic speeds.

Relativistic Effects

Till now, we have discussed atoms at rest in lab frame

$$P_{a\to b} \sim \frac{|\langle a|\hat{m}|b\rangle|^2}{\hbar^2} \iint dt\,dt'\, e^{\pm \frac{i\Delta E(t-t')}{\hbar}} \langle 0|\phi(x)\phi(x')|0\rangle$$

The correlator

$$\langle 0|\phi(x(t))\phi(x(t'))|0\rangle = \int \frac{d^3\vec{k}}{(2\pi)^3(2\omega_k)} e^{-i\omega_k(t-t') + i\vec{k}\cdot(t-t')}$$

⇒ The $e^{\pm i\Delta E t}$ factor came from

$$\langle a| e^{\frac{iH_0 t}{\hbar}} \hat{m}(0) e^{-\frac{iH_0 t}{\hbar}} |b\rangle$$

This time is the time the atom feels (in NR settings : uses Schrödinger eqn.)

▷ $\int dt \int dt'$ came from Schrödinger equation

The mode function's time are the time in the lab frame.

If particles move relativistically the time felt by the atom and the time experienced by a lab clock are different

In atom frame $\quad d\tau^2 = dt_{atom}^2 - \underbrace{dx^2 - dy^2 - dz^2}_{0}$

$$= dt_{atom}^2$$

Events of interaction turning 'on' and 'off' happen at same spatial location (as per the atom)

● Relativistic Effects
Till now, we have discussed atoms at rest in lab frame

$$P_{a\to b}\sim\frac{|\langle a|\hat{m}|b\rangle\rangle^2}{\hbar^2}\int\int dt\,dt'\,e^{\pm i\Delta E\frac{(t-t')}{\hbar}}\langle 0|\phi(x)\phi(x')|0\rangle$$

The corrlelator
$$\langle 0|\phi(x(t))\phi(x(t'))|0\rangle=\int\frac{d^3\vec{k}}{(2\pi)^3 2\omega_k}e^{-i\omega_k(t-t')+I\vec{k}(t-t')}$$

⇒ The $e^{\pm i\Delta E t}$ factor came from

$$\langle a|e^{iH_0\frac{t}{\hbar}}\hat{m}(0)e^{-iH_0\frac{t}{\hbar}}|b\rangle$$

This is the time the atom feels (in NR settings : uses Scrhodinger eqn.)

► ∫dt∫dt' came from Schrodinger equation

The mode function's time are the time in the lab frame.

If particles move relativistically the time felt by the atom and the the time experienced by a lab clock are different

In the atom frame
$$d\tau^2 = dt_{atom}^2 - dx^2 - dy^2 - dz^2$$
$$= dt_{atom}^2$$

Events of intreaction turning 'on' and 'off' happen at the same spatial location (as per the atom)

So, for that, for this point to be addressed, recall till now we have just discussed atoms at rest in the lab frame. Most of the times we had written like The probability of transition of a state of an atom from A to B dictated by the monopole coupling term squeezed between the state of interest from B to A or A to B, whichever way you want to look at, that will come with a ± sign. If B is ground state and A is excited state, a – sign will appear over here. And if B is excited state, A is ground state, $+\Delta E$ will appear over here. And the two-point correlator of the background field will tell me how much is the probability of such a transition. So, the coupling with the field and the two-point structure of the field together determine through this integral power form what is the probability of transition. If either of them is 0, if there is no coupling, m is switched off or the field is having 0 correlations between x and x' at location t and t', at the time, location at the time t and t', then there is no probability. So this was true for atom at which we have computed till this time. We just want to see how it will start mattering if the particles are moving as well. So previously, we used to write down the two-point function of the field in terms of its Fourier modes. And in that sense, this two-point function is written in terms of Uk at x and Uk*at x′. And if I open in terms of t and t' and x and x', *I* will get a $e^{-i\omega k(t-t')}$ and $e^{i\vec{k}(x-x')}$.

should be $(x - x')$ as a vector, spatial vector $(x - x')$. While this exponential factor was just coming from the operator, monopole operator, its interaction version vi, remember vi is queued between the states of interest a and b. However, what was true while we were doing all this business all along, the *t* appearing here, here, here or even here, all the times appearing in this game were the same time as seen by the atom or any observer in the lab. So, that means the atom was at rest with respect to lab observer or moving very slow, that means their times were not much different. So, this is non-relativistic setting in which we could have written in this way. So, the double time integral which appears over here, remember this appeared from the mod square of probability amplitude and mod the probability amplitude itself at a single integration in *dt*, it was obtained from the Schrodinger equation in some sense, writing Schrodinger equation in the interaction picture, but ultimately the Schrodinger equation. So, therefore, now whatever we have done. The basic assumption where that the time the atom sees that means the Hamiltonian time evolution this time over here or the integration time over here while the time at which the fields evolve $e^{i\omega t}$ are the same thing. So, this is the description of the field with respect to observer in inertial lab and this is the time evolution of an atom at rest with respect to that observer. However, if the particles start moving relativistically, then we know already that the time felt by the atom and the time experienced by a lab clock in the inertial frame will become different. And they will precisely become different with respect to this Γ factor. So, just to recall things once more, let us say some event happens in the atom frame. Suppose the interaction is turned on and then it is turned off. So, some time elapses in atoms frame. So, atom C is at time some *t* is equal to 0, interaction is turned on and at time *dt* the interaction is turned off. And in this meanwhile atom is not moving at all in its own frame. So, in atoms frame its displacement in space has become 0 and the total event ds^2 or the total separation is the proper time separation between the two events of interaction turning on and turning off. According to atom, this much time might have elapsed, which is the proper time. If the same event is viewed from the lab frame in which the atom appears to be moving, as we know $d\tau^2$ is an invariant quantity, the lab observer will also ascribe the same amount of $d\tau$ has elapsed. So, there was some interaction which was turned on And after a while, when atom moves here, the interaction is killed off. In atom's frame, it is happening at its own location and this much time has elapsed. So atom's clock has shown some time displacement dp atom. While there is an observer which is sitting outside and who is witnessing this atom moving along the some direction with some velocity. So, the proper space-time interval between these two events should be same as viewed by any observer was our philosophy which we learned. So, according to an observer in the lab frame, total time has passed between these two events of interaction turning on and interaction turning off is dt_{lab}, dt_{lab}^2. But this is also true that these two events of turning on and turning off are not happening at the same place. There is a spatial displacement $dx^2 + dy^2 + dz^2$ is non-zero in this case and which is equivalent to the velocity with which atom is moving times the time relapsed in the clock of the lab. So, put together the same amount of $d\tau^2$ is generated this time in the lab frame clock by this equation and therefore, you will get that this $d\tau^2$ is nothing but *dt* of atom^2 from the previous expression which we just wrote down. and that should be equal to $(1 - v^2/c^2\ )dt_2$ of the lab. So, therefore, $dt_{\text{atom}}/dt_{\text{lab}}$ is $\surd 1/\Gamma$, where Γ itself was 1 upon $\surd(1 - v^2/c^2)$ of course. When we write down like this, c^2 division and multiplication is happening all around which to say we are working in c is equal to 1. So, you see the *dt* atom/*dt* lab is $1/\Gamma$ factor and $d\tau$ therefore, which is the proper time of the atom. and this is the same *dt* which is the same proper time is seen from the lab frame which is small *dt*. So, they are off by a factor of Γ. So, the time spent is less in the frame of the atom $d\tau$ while *dt*. *t* So, $d\tau$ is $\surd(1 - v^2/c^2)\ dt$. *t* So, whatever *dt* you see *t* In the lab frame, the atom feels only a fraction of that because $d\tau$ is $1 - v^2/c^2\ dt_{lab}$. This is a quantity which is smaller than 1. So, therefore, the time filled by atom is the smallest amount of time it feels while other atoms, other observers feel that the atom has traveled for a longer time. And in composite to have that longer time seen, they have some velocity deduction happening to generate the same amount of beta.

Therefore, the distance traveled according to the lab observer is $(x - x')$ between the time events, which is given as the vectorial difference.

In this basis

$$P_{a\to b} = \frac{|\langle a|\hat{m}|b\rangle|^2}{\hbar^2}\int_0^T d\tau \int_0^T d\tau' \, e^{-i\frac{\Delta E}{\hbar}(\tau-\tau')} \times \frac{1}{(2\pi)^3}\int \frac{d^3\vec{k}}{2\omega_{\vec{k}}} \, e^{-i\gamma\omega_k(\tau-\tau')} \, e^{ikv\gamma(\tau-\tau')}$$

$$= \frac{|\langle a|\hat{m}|b\rangle|^2}{(2\pi)^3\,\hbar^2}\int \frac{d^3\vec{k}}{2\omega_k}\int_0^T d\tau \, e^{i\left(\frac{\Delta E}{\hbar}+\omega_k'\right)\tau}\int_0^T d\tau' \, e^{i(\Delta E+\omega_k')\tau}$$

where $\omega_k' = \gamma\omega_k - \gamma k v = \gamma(\omega_k - kv)$

- Still $\omega_k' = \sqrt{\vec{k}'^2 + m^2}$
- No absorption happens still! For large times!

For spontaneous emission

In this basis

$$P_{a\to b} = \frac{|\langle a|\hat{m}|b\rangle|^2}{\hbar^2} \int_0^T d\tau \int_0^T d\tau' \, e^{-i\frac{\Delta E}{\hbar}(\tau-\tau')} \times$$

$$\frac{1}{(2\pi)^3}\int \frac{d^3\vec{k}}{2\omega_{\vec{k}}} e^{-i\gamma\omega_k(\tau-\tau')} e^{ik v\gamma(\tau-\tau')}$$

$$= \frac{|\langle a|\hat{m}|b\rangle|^2}{(2\pi)^3\hbar^2}\int \frac{d^3\vec{k}}{2\omega_k}\int_0^T d\tau \, e^{i(\frac{\Delta E}{\hbar}+\omega_k')\tau} \int_0^T d\tau' e^{i(\Delta E+\omega_k')\tau}$$

where $\omega_k' = \gamma\omega_k - \gamma k v$

$= \gamma(\omega_k - k v)$

- Still $\omega_k' = \sqrt{\vec{k}'^2 + m^2}$
- No absorption happens still ! For large times !

For spontaneous emission

for spontaneous emission

$$P_{b\to a} = \frac{|\langle g|\hat{m}|e\rangle|^2}{(2\pi)^3\hbar^2}\int \frac{d^3\vec{k}}{2\omega_k}\int_0^T d\tau\, e^{+i\left(\frac{\Delta E}{\hbar}-\omega_k'\right)\tau}\int_0^T d\tau'\, e^{-i\left(\frac{\Delta E}{\hbar}-\omega_k'\right)\tau'}$$

$$= \frac{|\langle g|\hat{m}|e\rangle|^2}{(2\pi)^3\hbar^2}\int \frac{d^3\vec{k}}{2\omega_k}\left(2\pi\,\delta\left(\frac{\Delta E}{\hbar}-\omega_k'\right)\right)\left(2\pi\,\delta\left(\frac{\Delta E}{\hbar}-\omega_k'\right)\right)$$

$\frac{d^3\vec{k}}{2\omega_k}$ is Lorentz invariant measure

$$= \frac{|\langle g|\hat{m}|e\rangle|^2}{(2\pi)^3\hbar^2}\int \frac{d^3\vec{k}'}{2\omega_k'}\left(2\pi\,\delta\left(\frac{\Delta E}{\hbar}-\omega_{k'}\right)\right)\left(2\pi\,\delta\left(\frac{\Delta E}{\hbar}-\omega_{k'}\right)\right)$$

– Same as for atom at rest !!

Eqn
In this basis

$$P_{a\to b}=\frac{\langle a|\hat{m}|b\rangle^2}{\hbar^2}\int_0^T d\tau\int_0^T d\tau' e^{-i\Delta E\tau-\tau'}\times\frac{1}{(2\pi)^3}\int\int\frac{d^3\vec{k}}{\sqrt{2\omega_{\vec{k}}}}e^{ir\omega_k(\tau-\tau')}e^{ikvr(\tau-\tau')}$$

$$\frac{\langle a|\hat{m}|b\rangle^2}{\hbar^2}\int\frac{d^3\vec{k}}{2\omega_{\vec{k}}}\int_0^T d\tau e^{-i(\frac{\Delta E}{\hbar}+\omega'_k)\tau}\int_0^T d\tau' e^{-i(\frac{\Delta E}{\hbar}+\omega'_k)\tau'}$$

where $'\omega_k'=r\omega_k-rkv$

$=\quad r(\omega_k-kv)$

- Still $\omega_k'=\sqrt{\vec{k}'^2+m^2}$
- No absorption happens still ! For large times!

$$P_{a\to b}=\frac{\langle g|\hat{m}|e\rangle^2}{(2\pi)^3\hbar^2}\int\frac{d^3\vec{k}}{2\omega_{\vec{k}}}\int d\tau e^{+i(\frac{\Delta E}{\hbar}-\omega'_k)\tau}\int d\tau' e^{-i(\frac{\Delta E}{\hbar}-\omega'_k)\tau'}$$

=

$$\frac{\langle g|\hat{m}|e\rangle^2}{(2\pi)^3\hbar^2}\int\frac{d^3\vec{k}}{2\omega_{\vec{k}}}\left(2\pi\delta(\frac{\Delta E}{\hbar}-\omega'_k)\right)\left(2\pi\delta(\frac{\Delta E}{\hbar}-\omega'_k)\right)$$

$\frac{d^3\vec{k}}{2\omega_k}$ is Lorentz invariant measure.

=

$$\frac{\langle g|\hat{m}|e\rangle^2}{(2\pi)^3\hbar^2}\int\frac{d^3\vec{k}'}{2\omega_{\vec{k}}'}\left(2\pi\delta(\frac{\Delta E}{\hbar}-\omega'_k{}')\right)\left(2\pi\delta(\frac{\Delta E}{\hbar}-\omega'_k{}')\right)$$

- Same as for atom at rest!!

Now, suppose it is moving in one particular direction, let us call it in z direction. So, it will be the *v(t – t')*, the time difference elapsed in the lab, the direction is z and velocity exceeds v. So, the spatial separation between the two events is this equation. Now, we can convert this *(t – t')* into the how much time it has elapsed according to the atom itself. We already know that this is related to *t* This is related to *Γ* times*(τ – τ')*because you see *dt* lab is *Γdt* of atom. So, therefore, this is the time separation in lab frame that will be *Γ*the time separation in atom frame. At the post of *Γ*, this velocity gets corrected and this is the event being recorded, the spatial difference as seen in the lab frame. Now, when we had written this *t* integrations, for example, or this M operators, M operators squeezing, which has given right to these exponents here, these were coming from the Schrodinger equation. This portion was coming from the Schrodinger equation and this portion was coming from the field theory as seen by Labrador. Now, if we keep writing the field description as true for, as true for *t* observer in lab, then we have to translate the two-point correlation into the rest frame of the atom because the Schrodinger equation is supposed to be true in the rest frame of atom only. Schrodinger equation will not be true to describe the time evolution of the atom from the lab frame because Klein-Gordon or Dirac equation should guide that. but in the rest frame of atom because it is not moving at all, it will feel that its own evolution is governed by the Schrodinger equation only. So, there is a dichotomy between the time felt by the atoms which are controlling this portion while the time felt by the field which is controlling, which is being controlled by the two-point correlation which should be written in the last frame. So, overall what we have to do, I have to *t* cast everything into the rest frame of atom where I have everything in Schrodinger picture because particles are not moving or hydrogen atom or atom itself is not movingSo, its own evolution is Schrodinger equation. At the post of converting the field correlator which typically we write in inertial frame of lab frame ccin atom's frame itself. *t* So, what I have to do, I have to take the two-point function which is given in terms of lab clock and convert it back into the atom clock. We have already achieved that, we already know $t - t'$, how does it look like in atom's frame and this $(x - x')\cdot t$ How does it look like in the atom scale? So, both these computations are with us right now. So, therefore, I would be able to recart the two-point function, which was appearing as $\omega(t - t')$, which now will be written as $\Gamma(\tau - \tau')$and $v(t - t')$ it was. It will also be written like $v(\tau - \tau')$. In the two point function $k\cdot(x - x')$ appears. Suppose $(x - x')$ points only along the z direction as we are considering only $\vec{k}$ dot the z component of this $\vec{k}$ dot z unit vector will survive over here which gives you a $|\vec{k}|cos\theta$. So, you see you will get not only $\vec{k}$ mod $\vec{k}$ which is the magnitude of the $\vec{k}$ vector, the$v$the Γ factor times the atoms time separation, atomic clocks time separation times$cos\theta$ because the particle is moving along the z direction according to observer in the lab or vice versa that means it is the $\vec{k}$ vectors in the space in which the d^3k has been written. So, d^3k again can be written in spherical polar coordinate systems in which this$cos\theta$ will appear that means the coordinate system is written like this. So, the coordinate system is written in a particular way where the particles motion direction is identified as the z direction. *t* Suppose initially you had chosen any z-axis, x-axis and y-axis and particle was moving in a particular direction. What you can do, you can reorient your coordinate system such that your z-axis has become the particle's direction of motion and *x* and y are just orthogonal components. So therefore, if I do in such a coordinate system, the integral expression which was over here, *t* can be obtained so integral expressions here can be obtained in this nice way where i have converted where i have converted every $(t - t')$into$(\tau - \tau')$which was already there in the exponential of the monopole operator time portal dependence and here it will become $\Gamma(t - t')\tau - \tau'$ here and this should be a cost here$cos\theta$ should appear here. So, I have converted this $(t - t')$which was appearing in the two-point function as $\Gamma\tau t\tau'(x - x')$ which was appearing in the two-point function can be written as $\vec{k}$gΓ times$(\tau - \tau')$times$cos\theta$. *t* Remember, we had chosen our z axis to be *t* along the direction of motion.

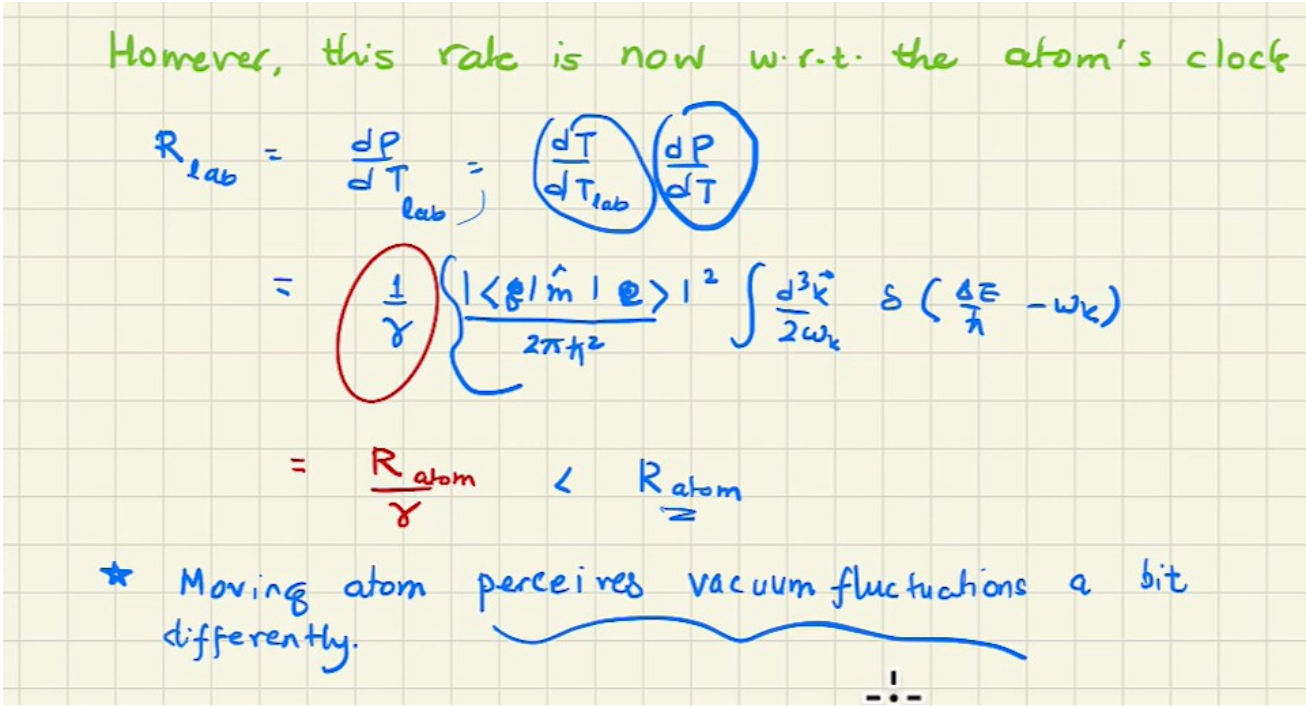
So, any arbitrary $\vec{k}$ vector will make some angle$cos\theta$ will keep changing with different, different ways. So, therefore, we have to do integration over θ as well which is hiding in this dp. So, like before what we will do, we will collect all the τ dependent terms. So, here is the τ dependency, here is another τ dependency and here is another τ dependency which we club at one place and it becomes one temporal integral 0 to capital *t t* of the τ. And what are the functions? Exponential $- i\Delta E/\hbar$ which is appearing over here $- i\Delta E/\hbar$ + whatever is appearing this $\Gamma\ \omega_k\ \tau - $ kv $\Gamma cos\theta\ \tau$ that I am collecting at one place and calling it as ω_k'. So, this is ω_k'. ω_k' is nothing but $\Gamma\ \omega_k - \Gamma$ kv$cos\theta$. Now, we can club the $\vec{k}$ and$cos\theta$ together to call it the z projection of the $\vec{k}$ vector k_z. So, effectively it becomes $\Gamma\omega_k - k_z$.

So, you see this is exactly the low range transform *t* as seen by the atom. If ω_k was the frequency as seen in the lab frame and k_z k$cos\theta$ *t* was the k_z, z projection of the special $\vec{k}$ vector in the lab frame. In the atom's rest frame, the frequency will appear low range boosted and this exact formula which we had seen in week 3 discussion as well, connects the atomic frequency, the frequency of mat mode seen by the atom rather, not atomic frequency. Atomic frequency is ΔE. The ω_k' is the frequency of the field mode as seen from the atom's frame and it is related to those quantities as seen from the lab frame. So, this is fine. So, this is all so far so good. We have very nice relation which has appeared in the exponent here. So, if I do the same time integral, again you know this is going to give me sinc function, sin of this argument divided by the same argument by t. So, and this will also give the sinc function with opposite phase, so that you will have a probability $sin^2(\Delta E + \omega_k)/(\Delta E + \omega_k')$. This is what it would give you and in long time limit we know that sinc function becomes a delta function. So, again it will become a delta of $\Delta E/\hbar + |\omega_k'|^2$and the integration is happening over d^3k', $d^3k/2\omega_k$. So, what has changed slightly is ω_k has t was appearing previously when we did for non-relative moving particle. This time its boosted value comes and sits here and similarly in the second integral. However, the integration is happening on the mode function distribution in the lab frame because that is how we had written in the field decomposition in terms of lab work. And ω_k and ω_k' and those quantities are connected by this Lorentz boost equation we had just written. So, ω_k' is different compared to ω_k while the integration is happening over $d^3k/2$r. However, whenever, despite the fact that the ω_k' and ω_k are changing from frame to frame. The relation between ω_k' and k′ and ω_k and $\vec{k}$ remains the same. That means in the lab frame I had a relation that ω_k is

$\sqrt{(k^2 + m^2)}$. Despite the loading boost has shifted their individual component value, this relation still remains true in the atomic frame. But ω_k' will still be $\sqrt{(k^2 + m^2)}$. So that means ω_k''s lower value is m and upper value is infinity. t integral whatever is appearing inside the argument of a delta function is $\Delta E + \hbar + \omega_k'$. This ω_k' is positive semi definite, this is also positive semi definite. So, therefore, the delta function is also positive semi definite and therefore, it will vanish. So, therefore, even though the atom perceives the field mode slightly differently and hence perceives the 2.4% slightly differently, the probability of absorption through vacuum fluctuation still remains 0, which was the case even for non-relativistically moving particles. Now that particles have started moving relativistically, it is suffering some amount of boost in terms of frequency and its mode functions description, but still it is not sufficient to call the transition from ground to excited. So, that feature demands true even for relativistic things. Now, what about spontaneous emission? Suppose the particle or the atom itself initially is in the excited state and then we are trying to compute the probability of going into the ground shift through the vacuum fluctuations thing. Remember this time atom perceives the vacuum fluctuations slightly differently through the change in the mode function. t What will happen? The ΔE terms would flip a sign because we know in terms of here when we are looking for spontaneous emission – sign gets replaced by +. So, therefore, the integrals what are appearing over here will just become overall ΔE term will flip a sign, all other terms will remain the same. So, you can see with this structure when ΔE undergoes a sign flip, t This exponential here becomes $e^{(i\Delta E/\hbar - \omega k')}$ and this thing is complex conjugate of that at different τ'. So, if we know the protocol already, in the large time limit it will become a double delta function with arguments appearing as $\Delta E/\hbar - \omega_k'$ and $\Delta E/\hbar - \omega_k'$,, $\Delta E/\hbar - 2\omega_k'$. However, as we see again, the integral is happening over $d^3k/2\omega_k$ while these delta functions which now can survive because this argument is not always positive. Whenever ω_k's become $\Delta E/\hbar$, then it survives. So, what I have to do, I have to put it wherever the roots of this collision over there, that will go and sit over in the twice ω_k area. ω_k has to be first written in terms of ω_k'. And ω_k' is equal to $\Delta E/\hbar - \omega_k$. And similarly, d^3k has to be converted into d ω_k'. However, we know this, so on face of it, this expression looks slightly different. In the non-relativistic case or in the lab frame, if an atom is stationary, then the expression will not have a′ sign here. It will just be the thing which we had previously obtained for an atom at risk in the lab frame. If item is moving, ω_k has changed to value ω_k'. So, integration will slightly change on face of it looks like. However, it is not going to be the case because you see even in the low range transformation, $d^3k/2\omega_k$ is a low range invariant quantity. This is low range invariant measure. So, that means $d^3k/2\omega_k$ is also equal to $d^3k'/2\omega_k'$. So, we can safely replace this $d^3k/2\omega_k$ appearing as $d^3k'/2\omega_k'$. And then you see the integral is exactly the same what we had obtained in the non-inertial case, non-relativistic case when particle was at rest in the lab plane.

This becomes just a dummy variable ω_k' is just a variable transformation in the lab frame no′s were appearing. It was ω_k, ω_k, ω_k here and ω_k here. This time all it has happened that every $\vec{k}$ thing has picked up a sign, a′. That′ is just a relabeling calling $\vec{k}$ x as x'. This is not going to change the value of the integer. So, therefore, the probability of transition will not change at all t And this is not a surprise because it is actually a computation in the inertial frame in which atom is at rest. This is atom's rest frame. So, in this frame, atom is still at rest. So, therefore, probability should not have changed. Why should two inertial frames become different if we are looking at the same exact process? And probability is a number. t Those things are supposed to be loading invariant. Probability should not, number of probability, number of some transition which is dictated by probability should not change from frame to frame. Suppose I start with a 100,000 atom in excited state and there is a 30% chance of spontaneous emission. That means after a long time I should get 30,000 atoms in the ground state and 70,000 in the excited state also. This ratio should be same in all frame, all inertial frame. It is not that the atom in a ground state appears as excited state to some inertial, some other inertial observer. Therefore, the probability should not change from frame to frame and this is exactly what we have obtained. The probability of transition of an atom at rest does not change from one inertial frame to another inertial frame. Previously, we had computed things for atom at rest. Now, we have computed things for probability as seen by the atom, which is still the computation fundamentally in an inertial frame of an atom at rest. So, therefore, this is not supposed to change. what will change would be the rate of transition. So, again if I have to write down the rate corresponding to it, I will convert this double delta functions into their sine functions for form, it is just a limiting case of sinc function if you recall.

If I take the derivative with respect to time, I will get this expression. Just a quick summary how to see that. The double delta function which has appeared can be nicely broken into a single delta function and one limiting expression. This is also a delta function, but I am writing it in a limiting fashion. And call this as sum f of ω_k, let us say. Now, use the property that $\delta(t - a)$gives you the delta back, but f can be replaced with the argument of the $\delta(0)$. So, that means I can write down the argument of $\delta(0)$, is this $\Delta E/\hbar - \omega_k$ should be 0. So, therefore, the sinc functions argument appears vanishing. So, that is what we can do. So, I can write down the twice sign upon this object sinc function like this. When I supply that argument of delta function into this function, I will just get a t outside. And again, the $\delta(\Delta E/\hbar - \omega_{k0})$ a factor half using the property of the delta function, which is this. We can write this twice of delta function with argument half went away. So, this is just a simple algebraic exercise one should be able to do to prove this.



The rate will also be the same.

$$p_{|e\rangle\to|g\rangle}=\frac{|\langle g|\hat{m}|e\rangle|^2}{2\pi\hbar^2}\int\frac{d^3\vec{k}}{2\omega_k}\delta\left(\frac{\Delta E}{\hbar}-\omega_k\right)$$

► $Since\left(2\pi\delta\left(\frac{\Delta E}{\hbar}-\omega_k\right)\right)^2=2\pi\delta\left(\frac{\Delta E}{\hbar}-\omega_k\right)\lim T\to\infty\frac{\sin\left(\frac{\Delta E}{\hbar}-\omega_k\right)}{\left(\frac{\Delta E}{\hbar}-\omega_k\right)}$

And using $\delta(x-a)\delta(x-a)f(a)$

$$\delta\left(\frac{1}{2}\left(\frac{\Delta E}{\hbar}-\omega_k\right)\right)\frac{\sin\left(\frac{\Delta E}{\hbar}-\omega_k\right)}{\left(\frac{\Delta E}{\hbar}-\omega_k\right)}=\delta\left(\frac{1}{2}\left(\frac{\Delta E}{\hbar}-\omega_k\right)T\right)=2\delta\left(\frac{\Delta E}{\hbar}-\omega_k\right)T$$

► $Since\ \delta(f(x))=\frac{\delta(x-x_0)}{|f'(x)|_{x_0}}$

However, thhis rate is now w.r.t the atom's clock

$$R_{lab}=\frac{dp}{dT_{lab}}=\frac{dT}{dT_{lab}}\frac{dp}{dT}$$

$$= \frac{1}{r}\frac{\langle g|\hat{m}|e\rangle^2}{2\pi\hbar^2}\int\frac{d^3\vec{k}}{2\omega_k}\delta(\frac{\delta E}{\hbar}-\omega_k)$$

$$= \frac{R_{atom}}{r}<\frac{R_{atom}}{z}$$

⋆ Moving atom precises vacuum fluctuations a bit differently.

So this is the rate of the transition. Again you will realize that it is exactly the same expression which we had obtained for atom at risk and no wonder this is again a rate as seen by the atoms clock. This rate *dt* is being, remember the *t* was the upper limit of time interval as seen by atoms clock. So again the rate is being calculated by an observer who is co-moving with the atom, that *t* The atom is not moving with respect to it and it is an inertial observer. So, again it is the same computation of a rate of a stationary atom in an inertial plane. So, it should not have changed and it does not. So, you get the same expression back. However, now crucial question comes that what is the rate as seen by the lab observer? This rate is seen by the atom. Atom will feel itself, will make a transition at a rate which was commensurate with non-relativistic rate.

However, if we want to obtain the date of transitions as seen by the lab observer, it will be the number of transitions or probability the derivative with respect to lab time. Use the chain rule, first obtain the derivative with respect to atoms time and then the atoms time with respect to the lab time, *dt/dt* lab and that we know it is already $1/\Gamma$. So, you will get the atoms rate multiplied by $1/\Gamma$ and since Γ is 1 over $\sqrt{1 - v^2/c^2}$, this factor is smaller than 1. So, therefore, the rate as seen by the, the value of rate as seen by the lab observer will be smaller compared to what atom sees. This is again natural recognition exists because same number of transitions are happening in large amount of time as seen from the lab time, that long. So, therefore, the rate would be slightly lower. So, this all has happened because the moving atom perceives vacuum slightly differently, but still a vacuum and quantum fluctuations and it derived the same expression for the non-relativistically moving atoms computation. However, when we go to the lab frame, we have to just do the lorry transformation from the clock rate from one frame to another frame and that is why this extra Γ factor comes about. So, this is the way we typically will work out. As we discussed this projection ψ-ψ is not a good measure of probability because its local value is the density. So, density is not a good measure in relativistic case. Klein-Gordon equation does not support this definition of probability let us say in such sense. So, therefore, the trace condition in the Lindblad and whatever we learned which was trying to protect *t* the $\tilde{\psi}_n \tilde{\psi}_{n\,t}$ trace that is suspect because this is not really enforcing correlativistic particle because they do not comply with this definition of probability. So, either we go and develop a new theory of Lindblad formalism or we jump to the rest frame of atom itself. There everything becomes non-relativistic, apply the Lindblad formalism, do all our computations and at the end of the day translate everything back into the last frame just like we did. Here we computed the rate and everything by just doing non-relativistic computations in the rest frame of the atom. And once we had the result, we transported them back into the lab frame using the low-inch transformation. So, that is the strategy most of the time it is very useful in dealing with and in not only relativistic transformation, non-inertial transformations as well which is not a mandate of this course. There also the same philosophy is applied and people try to obtain the behavior of atom *t* or the structure of transitions in non-inertially moving particle turbulence. So, we will use keep this notion in mind that first we will jump to the rest frame of atom do all our computations and then we will transport the results back into the lab. Only thing we have to be careful about when we go to the rest frame of atom is that the in the expressions of transitions and everything one bit *t* which is the monopole operators temporal evolution, the exponential with $\Delta E/\hbar$ in them. This is written in terms of atoms clock and when we write down the mode function of the two point function in terms of mode function, this is written in terms of lab clock. So, we have to properly transform these two things and write down the correct version of physics which we wanted. So, we stop over here for this class. In the next discussion session, we will see the usage of this in determining the quantities of characteristics of atom itself. Till now we have just looked for transitions, but we know that if there are fluctuating *t* Hamiltonians or time-dependent Hamiltonian, it also tries to change the eigenstate description as well. The eigenstate do not remain eigenstate anymore. Time translational symmetry is broken, then there is no eigenstate and therefore the energy gap between the atom will also keep changing. The ΔE is not a protected quantity over time because atom's eigenstates are not eigenstate of the full Hamiltonian anymore under the action of fluctuations. Those things *t* We will see in the next class. So, therefore, I stop over here.