Foundation of Quantum Theory: Relativistic Approach
Lindblad Master Equation
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Fluctuation and Dissipation in evolution
Lecture- 39

So, now that we have learnt about the interaction between atoms and background quantum fields, we can proceed forward to see what kind of effects it generates apart from spontaneous emission, simulated emission and absorption, what kind of effects we can expect on the whole collections of atoms. One good way of looking at it, it would be through the Lindblad master equation which corrects the Von Neumann kind of evolution equation for normal density matrices which depicts any quantum state of matter.

Lindblad Master Equation

For a quantum system

$$\hat{\rho} = \sum_n \rho_n |\psi_n\rangle\langle\psi_n|$$

$$i\hbar \frac{d|\psi\rangle}{dt} = \hat{H}\psi \qquad \text{(Non relativistic)}$$

$$\frac{d\langle\psi|\psi\rangle}{dt} = 0 \qquad \text{(Using current conservation equation as well) !!}$$

$$\therefore \frac{d\hat{\rho}}{dt} = \sum_n \rho_n \frac{d|\psi_n\rangle}{dt}\langle\psi_n| + \sum_n \rho_n |\psi_n\rangle \frac{d\langle\psi_n|}{dt}$$

$$= \sum_n \rho_n \left(\frac{H}{i\hbar}|\psi_n\rangle\langle\psi_n| + |\psi_n\rangle \frac{\langle\psi_n|H^\dagger}{-i\hbar} \right)$$

$$= -\frac{i}{i\hbar}\hat{H}\sum_n \rho_n |\psi_n\rangle\langle\psi_n| + \frac{i}{\hbar}\sum_n \rho_n |\psi_n\rangle\langle\psi_n|\hat{H}$$

$$\boxed{\frac{d\hat{\rho}}{dt} = \frac{i}{\hbar}(\hat{\rho}\hat{H} - \hat{H}\hat{\rho}) = -\frac{i}{\hbar}[\hat{H},\hat{\rho}]}$$

Von - Neumann eqn.

Describes reversible processes
- Schrödinger eqn. keeps evolving states and one can in principle go back to the initial

⇒ However in realistic set ups particles get scattered absorbed, emitted, changing the number count of probabilities.

Eq

Lindblad Marter Equation

For a quantum system

$$\vec{\rho}=\sum_n |\psi_n\rangle\langle\psi_n|$$

$$i\hbar\frac{d|\psi\rangle}{dt}=\hat{H}\,\psi$$ (Using current conservation equation as well) !!

$$\frac{d\hat{\rho}}{dt}=\sum_n \rho_n\frac{d|\psi_n\rangle}{dt}\langle\psi_n|+\sum_n \rho_n|\psi_n\rangle\langle\frac{d\psi_n}{dt}|$$

$$=t\quad \sum_n \rho_n\left(\frac{\hat{H}}{i\hbar}|\psi_n\rangle\langle\psi_n|+|\psi_n\rangle\frac{\langle\psi_n\hat{H}|}{-i\hbar}\right)$$

$$=\quad -i\frac{\hat{H}}{i\hbar}\sum_n \rho_n|\psi_n\rangle\langle\psi_n|+\frac{i}{\hbar}\sum_n \rho_n|\psi_n\rangle\langle\psi_n|\hat{H}$$

$$\frac{d\vec{\rho}}{dt}=\frac{i}{\hbar}(\hat{\rho}\,\hat{H}-\hat{H}\,\hat{\rho})=\frac{i}{\hbar}[\hat{H},\hat{\rho}]$$

t t t t t t t t t t t t t t t Von – Neumann eqn.
Describes reversible processes.
_ Schrodinger eqn. Keeps evolving states and one can be in porinciple go back to the initial.

For instance, suppose we are talking about a quantum system whose density matrix is given by this expression where this ρ_n or ρ_n is just the projection of in this particular state set, ψ_n ψ_n. This you might have been familiar from your earlier courses of foundations of quantum mechanics or even in quantum mechanics in which the density matrix approach is being discussed. So, for this kind of states, since these ψ_n's are supposed to be the eigenstates of the Hamiltonian, we can naturally see that if these states satisfy the Schrodinger equation, the time-dependent Schrodinger equation version of that would be this, $i\hbar d\psi/dt$ of Hamiltonian acting on ψ, there should be a ket, so there should be a $|\psi>$ over here. So this is true for non-relativistically moving atoms. As we know, Schrodinger equation is really valid for non-relativistically moving atoms. Otherwise, we have to go to either Klein-Gordon equation for relativistically moving atoms or Dirac equation. So for instance, right now we are discussing with Schrodinger equation only and later on we will see how to generalize it for relativistically moving particles through simple examples only. So, we will not do a more detailed structure analysis for example, Dirac particles analysis looking at hydrogen atom of Dirac particle and so on. But a working way of how things can be extracted out even remaining in the Schrodinger parabola in a particular frame. So, we know in Schrodinger equation the norm of the state ψ ψ is a protected quantity under time evolution. This we have known since the quantum mechanics course, the first course on quantum mechanics. But this can also be seen from the current conservation equation which we had derived in in the context when we were looking at the relativistic equations that this quantity which is ψ ψ inner product which takes a form of something called probability density and that probability density with probability current satisfies this current conservation equation. The hallmark of that current conservation equation was this. At whatever information the current conservation quantity $d\psi/dt$ *or* t $d\psi^{\star}/dt$, ψ this thing dt is equal to the current conservation divergence of J, which was also made from the spatial derivative of the wave function can combinedly written as this thing. This is a simple exercise you should be able to do or you might have already done at the level of quantum mechanics. Now if this is true, these two equations are true, then the following can be said that since ψ_n's are also some of the states which will satisfy this kind of equation, if I take the temporal derivative of the density matrix $d\rho/dt$, it will hit the terms one by one. This quantities ρ_n's or ρ_n's are supposed to be independent of time, so on the time ket and the bra of the ψ_n. So, first I will hit the ket and then in the second term I will hit the bra of the ψ_n. And use the Schrodinger equation version that $i\hbar d\psi/dt$ is Hamiltonian acting on ψ. So, whenever I see $d\psi_n/dt$ over here for example, I will replace it with H Hamiltonian acting on $\psi_n//i\hbar$. This $i\hbar$ moves to the right hand side. Similarly, for the other term, when I have a $d\psi_n/dt$ of the bra ψ_n, I will take Hermitian conjugate of this equation and I will obtain – $i\hbar$ this time because I have taken a Hermitian conjugate and $H^{\dagger}$ acting on ψ_n and since H is a Hermitian operator, $H^{\dagger}$ is equivalent to H as well. So, the second term will also become with a – sign, ψ_n outer product ψ_n and H. So, you see I have two terms, one is $-iH$, H appearing first. For example, in this term H appears first and then this ρ_n can go in with the summation inside and it will convert for itself into the density matrix. So, 1 by $i\hbar$ is a common factor. If I write it like $-i/\hbar$, that should be $-i/\hbar$. Then Hamiltonian appears and density matrix and in the second term the density matrix appears then Hamiltonian appears. So, $H^{\dagger}$ appears at the last which is equivalent to H. So, put together everything the time derivative of the density matrix can be written in a nice commutative form. All this algebra leads up to the so called the Von Neumann density evolution equation. Now, this is for unitary process as long as Hamiltonian is completely Hermitian. This describes the dynamics of reversible processes. Schrodinger equation keeps evolving the states under unitary time evolution and whatever state you land up on the final time, in principle you can do an inverse unitary transformation and get back the state which you started with. So this is a reversible process, it conserves the probability and therefore it is reversible. Only probabilities can get redistributed through a unitary transformation and that amount of So, redistribution can be back calculated by doing the inverse time translation or inverse unitary version. So, as long as probability is conserved, it is just redistributed from the final dynamic state and the unitary version which took you to this state can be inverted and you will get the state back and therefore, this process is reversible. However, relativistic particles, relativistic setups, particles do interact with various kind of things. A particle just not evolves freely under its own Hamiltonian. It will be affected by environment, some random photons coming from there, some random particles interacting it with full on potential or some other potential. So, many things go on as a particle time evolves. And many of the times we are not fully aware how many things it has interacted with. For example, we can just say that it has interacted with the photons which we can see. But there might be some photons under our resolution scheme, our detectors or our eyes are not able to detect those photons, but the atom might have interacted with them. It keeps interacting with various things and by the time we receive the final state of the atom, it is a conglomerate of not only its own Hamiltonian evolution, but as an artifact of so many interactions it underwent throughout. And since we just knew about the Hamiltonian of the atom and we did not know what exactly it interacted with, so we cannot reproduce where it started with. So, it is an example of a non-reversible process where all there are other terms which you are not fully control of and they try to control the dynamics to some extent and participate into the dynamics. And therefore, what you get at the end is a net result of all the interaction of the atom head. Therefore, it is not just for its own Hamiltonian, but other terms in the Hamiltonian should also be dictating the course and therefore, it will become a non-reversible process. For such non-reversible or irreversible processes, let us assume the dynamics is dictated by some effective Hamiltonian, H-effective. It is not only made up of atoms own interaction, H, but other Hamiltonians as well, which is for example, a photon talking to electron and other electron talking to another electron or some atom, their dipole moment talking to electron. And at the end of the day, we lose sight of all other interactions. So, I am going to assume that such effective description is controlled by two pieces.

Hamiltonian which keeps the probability of the state normalized, the norm preservation happens and then there is a non-Hermitian piece with i times some Hermitian operator such that the overall H effective does not remain a Hermitian operator. So, this new V which is accounting for all its interaction which also try to change the state from its normal evolution through H is accounted for in making the evolution non-unitary in some sense. That means it has a probability of that state would not only just get redistributed, its norm will change because there are so many interactions going on which we are not contemplating about. We are just looking at only portion of a Hamiltonian of the true Hamiltonian. So, therefore, it is open system dynamics, the state which we are access to is only that of a electron or atoms state which we are talking about. The true state is made up from all the photon, electron, secondary atom, tertiary atom and so on, which all interacted with it. So, if I look for a segment of a state in a big partite system, that state is not destined to remain non-deserved. So, therefore, the loss of other parties which played into this unitary gain. So, overall dynamics could be unitary that all particles put together, they are evolving through some Hamiltonian in unitary way. But since we are looking only at a portion of that, it is not guaranteed that the portion will also follow unitary time evolution. So, therefore, I can assume that it is controlled by some non-unitary time dynamics, which is reflected in terms of selection of some non-Hermitian Hamiltonian architecture, whose non-Hermitian part is the all interactions put together drift out which is controlling the overall dynamics.

For such irreversible processes (let us assume)

$$\frac{d|\psi\rangle}{dt} = \hat{H}_{eff}|\psi\rangle = (\hat{H} - i\hat{V})|\psi\rangle$$

Hermitian

Then $\frac{d\langle\psi|\psi\rangle}{dt} = -2\langle\psi|\hat{V}|\psi\rangle \neq 0$

In this case we can define

$$|\tilde{\psi}_n\rangle = \sqrt{\rho_n}|\psi_n\rangle$$

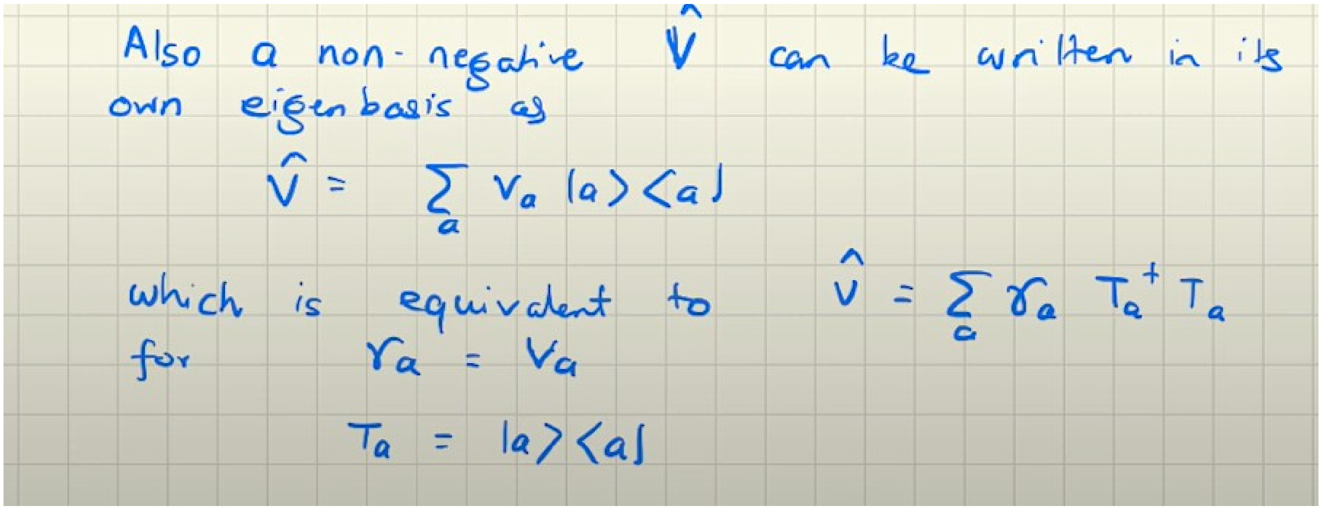
s.t. $|\hat{\rho}\rangle = \sum_n |\tilde{\psi}_n\rangle\langle\tilde{\psi}_n|$

If $\langle\psi|\psi\rangle$ has to remain between 0 and 1 at all times

$$\langle\psi|\hat{V}|\psi\rangle \geq 0$$

Thus, we can write $\hat{V} = \sum_a \gamma_a T_a^{\dagger} T_a$ for some operator T_a.

Prove: $\langle\psi|\hat{V}|\psi\rangle \geq 0$ in that case



Eq

For such irreversible processs let us assume

$\frac{d|\psi\rangle}{dt} = (\hat{H} - \hat{V})|\psi\rangle$ where $\hat{V}$ is Hermitian.

Then $\frac{d\langle\psi|\psi\rangle}{dt} = -2\langle\psi|\hat{V}|\psi\rangle \neq 0$

In this case we define

$|\tilde{\psi}\rangle = \sqrt{\rho_n} \sum_n |\psi_n\rangle$

s.t. $|\hat{\rho}\rangle = \sum_n |\tilde{\psi}_n\rangle\langle\tilde{\psi}_n|$

If $\langle\psi|\psi\rangle$ has to remain bwtween 0 and 1 at all times.

$\langle\psi|\hat{V}|\psi\rangle \geq 0$

Thus, we can write

$\hat{V} = \sum_a \Gamma_a T_a^{\dagger} T_a$ for some operator T_a

Prove : $\langle\psi|\hat{V}|\psi\rangle \geq 0$ in that case

Also a non negative $\hat{V}$ can be written in it's own eigenbasis as

$$\hat{V}=\sum_a |a\rangle\langle a|$$

which is equivalent to $\hat{V}=\sum_a \Gamma_a T^\dagger T_a$ for $\Gamma_a = V_a$

$T_a = ke$

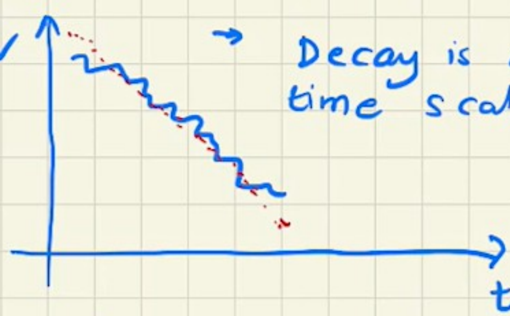
$T_a = |a\rangle\langle a|$

So, now you can readily see if the total has or effective Hamiltonian becomes non-Hermitian, its non-Hermitian piece, the *V* sitting over here controls the *dt* of the probability density, ψ-ψ model, ψ-ψ model. So, that is nothing but twice of the V, the expectation value of the *V* in the state which we wanted. If that is non-zero, then this norm preservation would not happen and particle will cease to remain with the same probability redistribution. The normalization of the state across all the Hilbert space basis will not add up to 1. That means it has some conversion in some probability of other particles also getting enhancement in their distribution, which we are not looking at. And this particles to the distribution is not only changing its projection in different basis, but also the overall probability of the state remaining in the Hilbert space of our concentration we are looking at to be smaller than what it was. *t* This is crucial that all such effects which we are not considering about that makes the system open, that it is not a closed system. We are not looking at all pieces of the game, only a finite number of piece that will make the norm not preserved. In such a case, I will first define the new $\tilde{\psi}_n$ *t* which is previous ψ_n times their ρ_n 's meaning projection in the $\sqrt{\rho_n}$, where ρ_n^2 is the probability with which the density matrix adapts them. If you remember that we have the ρ_n's or the ρ_n's. Then this density matrix will just be written in terms of $\tilde{\psi}_n$, $\tilde{\psi}_n$ out of order. However, we would want that $\psi\psi$ or even $\tilde{\psi}_n\tilde{\psi}_n$ inner product to remain between 0 and 1 because this will be self-inner product multiplied with $\sqrt{\rho_n}$ and it will just multiply a number between 0 and 1. So, therefore, if this is a valid definition of a probability, it can never become greater than 1. However, this is a differential equation which is telling how the probability density is changing in time this probability is changing in time. If it is not preserved, then it will change over time. But if it has to remain a good definition of probability that therefore, whatever happens in its dynamic, it should never cross 1 or it should never become negative. So, that can be ensured if the right hand side which is appearing $-{}_2\psi$ If that expectation of *V* is positive semi-definite, then you can show that this object over here which is appearing $d\psi$ quantity only decreases in time, first thing. Secondly, since this is a positive semi-definite quantity, ψ ψ inner product is of $|\psi|^2$ in position space integrated over. So, it cannot ever achieve a negative number. So, both the quantities, both the requirements that it remains between 0 lower bound and 1 upper bound would be realized if the expectation of *V* is positive semi definite. So therefore, in any dissipative process in which probability is leaking out into other particles, this should at least be true that whatever is controlling its non-unitary part should be a positive semi-definite operator. That means its expectation value for any state ψ should always be either 0 or positive number. It can never become negative number. Otherwise, probabilities will grow beyond 1. That is not supposed to happen. So, this is a very essential requirement that the dissipation dynamics, the *V* term should be positive semi-definite. And by that we already know if that is the case, the dissipative operator *V* can be written in terms of Hermitian operators, it should be Hermitian operator of course, but it could be written in terms of some other operators *t* and its Hermitian conjugate *t* and $t^\dagger$. You can now verify that not only *V* is Hermitian, suppose *t* is any operator, *t* does not have any property to become Hermitian or not, but since $t^\dagger$ *t* has appeared, this makes it Hermitian. And suppose more than one dissipative processes are going on, then for example, more than one particle the atom has interacted with. This total V should be sum over all such dissipative processes. So, T_a controls one dissipative process and summation over A controls all the dissipative processes. The weight factor or in this case we will see that it is the rate at which this dissipative processes are happening is controlled by Γ_a. So, $\Gamma 1 T_1 - \psi^\dagger T_1 - \psi + \Gamma_2 T_2^\dagger T_2$ are talking about two dissipative processes happening. to the atom, one with rate Γ 1 and another with rate Γ 2. From here it is easy to prove that if I have written in this particular format, it is easy to prove that we automatically satisfy the condition which we wanted for probability to remain conserved, not conserved between 0 to 1. At least the inner product is the definition of probability, that logic would be respected as long as V is positive semi-definite and this is a good description of that. Since V is a positive semi-definite operator that means a non-negative operator, this can be written in its eigen basis like that where eigen values for example are non-negative. So, in its eigen basis it can be written like that and you can identify from here quickly and clearly that this and this definitions are one and the same thing as long as we can identify Γ_a with the eigen values V_a which are positive semi definite and these therefore they are traits and T_a is identified as this A as well such that $T_a{}^\dagger T_a$ will still remain A out of product A. Hermitian operator can be decomposed in terms of its eigen basis like that and from there you can decompose it back into the terms of dissipative processes going on with certain rates, with eigen values of V controlling the rates. This is just an example of writing the dissipative Hamiltonian, very well. T_a describes some irreversible process and V is collection of all such irreversible process the atom is undergoing with.

⇒ Thus, T_a describes some irreversible process and $\hat{V}$ is collection of all such irreversible processes

⇒ In addition to these irreversible processes there could be stochastic sudden jumps which suddenly change the state of the atom with probability $(1 - \langle \psi | \psi \rangle)$
Fluctuations accompanying dissipation

(Randomness inherent in the irreversible processes)!
E.g. a small particle in a fluid experiences
→ Viscosity
→ But also random kicks by fluid particles
→ Decay is not monotonic on microscopic time scales.



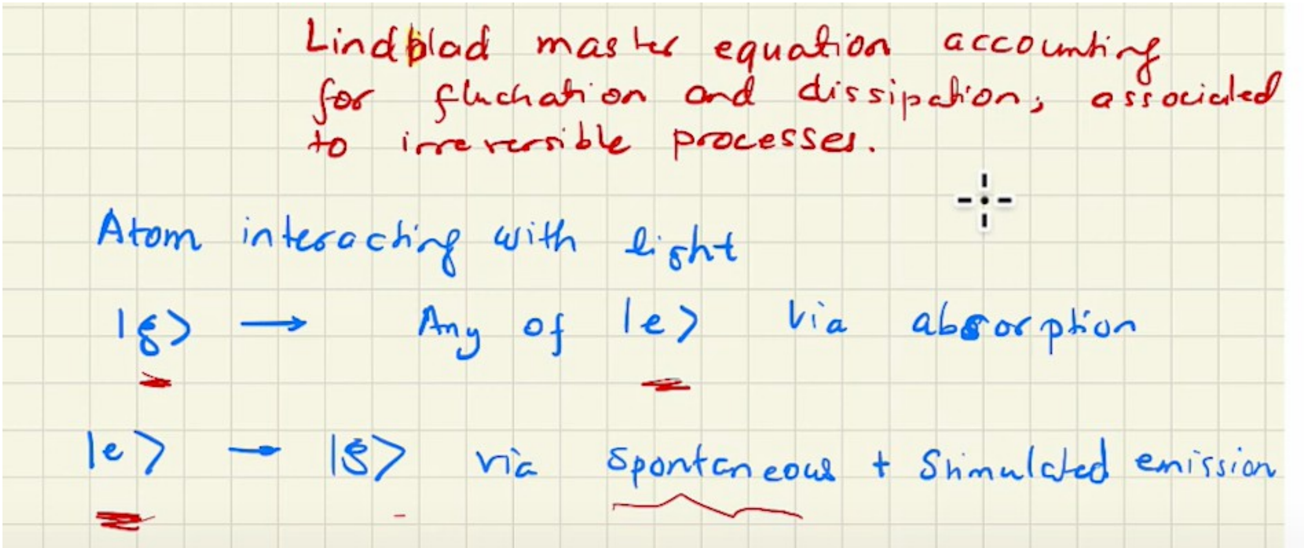
If $\hat{\tau}_a$ are such sudden change operators
$|\psi\rangle \to \hat{\tau}_a |\psi\rangle$

Then $\hat{\rho} = \sum p_n |\psi_n\rangle\langle\psi_n| \longrightarrow \hat{\tau}_a \hat{\rho} \hat{\tau}_a^{\dagger}$

Hence overall

$$\frac{d\hat{\rho}}{dt} = \frac{i}{\hbar}\left(\hat{\rho}\hat{H}_{eff} - \hat{H}_{eff}\hat{\rho}\right) + \sum_a T_a \hat{\tau}_a \hat{\rho} \hat{\tau}_a^{\dagger}$$

(T_a: Rate of random kicks)



Eq

So, now in addition to these irreversible processes, the atom also suffers from sudden stochastic jumps from its tube. So, it is not that this these irreversible processes are guiding its evolution through this kind of simple differential equation. This is one case, but most of the times the dissipative processes are also accompanying sudden or stochastic fluctuations in the wave function. It is not that only it will just dissipate out very nicely, very smoothly, its probability will go down. But most of the times, any dissipative process is happening because we are not able to account for so many interactions underwent and many of these interactions are quantum in nature. So, they are suddenly kicking up the atom in a particular way so that it is trajectory is slightly distorted and collection of a large number of such interactions or large number of such peaks makes the state suddenly jump on a microscopic time scale from one point in Hilbert space to another point in Hilbert space and so on. So, therefore, these irreversible processes, the stochastic sudden jumps should also be associated with not only just dissipation and which change the state of the atom momentarily at least with some probability $1 - \psi$-ψ. ψ-ψ remember is the probability of the state remaining ψ in some sense. If it is 1 then there is no dissipation, if it is 1 then it is no stochastic jump is going on. If it is decaying in time and $1 - \psi$, ψ will tell you how much it has decayed. But if this $1 - \psi$, ψ also keeps jumping in time up and down, that it is some higher probability, some lower probability, some higher probability, some lower probability, fluctuates about its central value. That means it is not smoothly going down. For example, a curve like this, probability is not simply going like down like this, but it is going down with a zigzag kind of motion around its central value. So therefore, most of the times fluctuations accompany the dissipation because the root cause of the both of dissipation and fluctuations are same. There are unknown degrees of freedom which interact with the atom of our choice. t So therefore, randomness is inherent in the irreversible process. You cannot get rid of that. You cannot assume that the atoms and other things which are interacting, so many things which are interacting with it are just doing a normal smoothening profile. Again, for elaboration, we can think of a particular case where a small particle is put into a fluid. So, that particle suppose it is moving in a fluid will not just experience the drag or a viscosity. Viscosity definitely will be there because it will interact with other atoms, other fluid particles, they will try to drag it down. But we know through Brownian motion that in any fluid the atom keeps getting bombardment by the random motion of the fluid particles themselves. So, therefore, there are random kicks also associated along with viscosity. So, viscosity is all the collective behavior. The

system is displaying on the motion of the atom or particle, while the random kicks is just the thermal property or the random motion of the particles or huge collection of particles is coming up here. So, viscosity will drive down the velocity to small values, but random kicks will try to make it goes zigzag. Kicks are not unidirectional, it is not that it will just increase, sometimes it will increase in this direction, sometimes it will decrease in that direction. So, overall the velocity exchange will keep happening under different, different direction and will not follow the decay profile in a uniform coherent way, but it will be zigzagging about it. So, it is not a monotonic function on microscopic time scales. Only if I zoom out of this region, it will behave, it will look to me like a smooth curve like that. But if you really zoom into this, I will see that there are tiny, tiny, tiny fluctuations. This curve is not really smooth like that. So, this is the dissipation and fluctuations accompanying. All the dissipative process are supposed to or thought to be carrying some fluctuating characteristics as well. Therefore, apart from these $T_a s$ which are just controlling the smooth downing of the probability or smooth decay of the probability. Let us assume there is some more to the openness of the quantum system. There is some sudden change operator or stochastic operators also which try to change the state from ψ to action of this stochastic operator acting on ψ.
So, the τ_a kind of operators are the peak operators which on microscopic time scales change the state of the motion, the state of the particle. So, ρ which was initially this, under this stochastic jumps or stochastic action of the τ_a operator becomes $\tau\rho\tau^\dagger$. This you can see from this ψ an$d\psi$ an$d\psi$. If ψ gets affected like that, you can prove the density matrix get affected like this. This is as simple as what we have done. So, overall in a system which is open, which is dissipative, it is necessarily having some fluctuation term as well. t And put in all, the overall dynamics is controlled by not only the effective Hamiltonian which has a V or a T_a term in it, but also the fluctuation sudden kick terms. Rate of random kick is determined by this parameter Γ_a. Small Γ_a remember was the rate of the This will try to smoothly down, smoothly take the probability to smaller and smaller value while this Γ, Γ_a are the random kicks the atom will receive at time and they will work momentarily. The density matrix will change in any direction the randomness allows. So, we keep going with this equation and we can see that in that case if I write it down, Then if I take the trace of this equation, if I take the trace of this equation, the trace on both the sides should be looked at and trace of this term as well. Then what will happen in effective Hamiltonian again remember there were two terms, one coming with V and one with the Hermitian Hamiltonian h. If ρH are put together like this $-$ $H\rho$ which was the commutator previously remember. But the next time there is an anti-Hermitian part which is $-$ iV or non-Hermitian part that does not gel in this way where you get a commutator. Therefore, that term will remain separated like this and there will be two terms, one coming from the ρ, iV here and the next term + $iV\rho$. So, if you put together, you will get the second term as twice of i –, $-2i\Gamma_a$ and then $t\rho_a t^\dagger$, $\rho a_t a^\dagger$ in summation over A. This will be true if you just do this algebra that This is the state with which ψ changes under the action of $H - iV$. You should find out how the density matrix changes and then to that change, add the third piece which we have added over here. All in all, that would be the total density matrix change. When I take the trace of the whole equation, the trace of the commutator will vanish. So, only the trace will survive with the non-Hermitian terms, because under Hermitian evolution, the commutated structure, the probability trace does not change, the trace of the density matrix does not change, speaking about the normalization of the state thinning protective. But if there are non-Hermitian pieces, that information gets compromised and therefore, the trace equation will change like this. this all non-Hermitian pieces, one was the dissipation part and this is the fluctuation part. However, if we want that under this kind of thing, if you want that the trace whatever the state gets changed to, if the trace has to remain a protected quantity in terms of probability of the whole thing meaning the normalized state. After each dissipation fluctuation term, what we were doing? We were trying to normalize the state like this. We are changing the probability of assertion into that. And if we want that under this redefinition, the total trace overall sum of all probabilities with some of all

diagonal elements of this newly defined row which absorb the definition of probability inside. The probability is changing, but with this changed wave function that trace of row if that remains has to remain the same that means the summation quantity should become 0. Remember, we are not saying that this row is the same as the initial row. In between, we have changed the state such that the ψ_n's have become $\sqrt{\rho_n}\psi_n$. And then we have rewritten the density matrix. So, this density matrix is just a projection over the normalized wave functions. The normalized wave functions carry information of how the probabilities have changed. And then in this basis, if you write down all the elements of the projection is 1 and that we want to remain as 1. So, this is just the basic statement that under this redefinition step, the stress will not change and this can only happen if the total sum on the right hand side will vanish. Now, that is summation over all the stochastic processes. However, you can think of such cases where some atom is only having one stochastic process, some atom is having only two stochastic process or maybe interaction with photons only or maybe interaction with electron only. For all those cases, individually this bracket should be zero, trace of each bracket should be zero because these are talking about different different stochastic processes and under each stochastic processes, I do not want the trace to change because I do not know whether one atom will incorporate which one of these stochastic processes. It can combine two of the stochastic process, three of the stochastic process. So, therefore, for all stochastic processes in any system, if this equation has to remain true, that means individually these stresses should be vanishing and that can only happen if the two criteria are met. Operator should be matched to each other and the rate should be matched to each other. So therefore, now we learn that from this consistency requirement, we have not assumed anything. This is all consistency. With normalized state, trace of $d\rho/dt$ t has to be 0. And for that to happen, operator should be the same and the coefficient should be the same for cancellation. So therefore, you see under this, if the system is decaying, an open quantum system is decaying, the state of the system is decaying, the decay operator or dissipation operator is also the fluctuation operator. So, this is the fluctuation dissipation condition on the time evolution of the density matrix. And overall the two pieces can be written like this. This is the unitary time evolution happening with the intrinsic Hamiltonian of the atom and all other dissipative and stochastic processes put together generate this kind of non-unitary change in the density matrix. So, this equation is called the Lindblad master equation. For instance, we can have an atom interacting with light. So, the usual evolution of the atom with its own Hamiltonian will be governed by the first piece, while the second piece which is the fluctuation and dissipation term will come through the abrupt changes in the state of the atom which can be the following, either the atom if it is in the ground state can get excited to any of its set of degenerate or non-degenerative excited state via absorption of photon. That is the state change it will bring about. Secondly, it can also happen if it is one of the excited state, it will decay to the ground state via spontaneous or stimulated emission due to the presence of the light or even the vacuum fluctuations in that. So, vacuum fluctuations and quantum state all those things with quantum fluctuation with in part this non-hermeticity and non-unitary as well as dissipative fluctuating terms to the evolution of the atom density matrix. So, this is what typically interaction of light and matter in density matrix formulation is what we can like. Most of the times we have just looked at the density, the eigenstates and the expectation values of various things or the probability of transition from here to there. But whole evolution it has to be looked at in one go, it should be looked from the density matrix approach, it contains all the information of the chain and that information is captured in terms of these fluctuating, dissipating terms which are appearing in the newly defined. master equation for this open quantum system kind of dynamics, which is the Lindblad master equation. So, till now we have discussed about the Schrodinger equation and the corresponding time evolution, because this is true for Schrodinger equation only that the time evolution happens like this, probability whether it is 0 or non-0. in presence of dissipative and non-dissipative terms is true for relativistic systems, non-relativistic systems only. Because for relativistic systems, the usage of ψ-ψ outer product, ψ-ψ inner product as a total probability itself is in question because as we remember in relativistic cases, the density, probability density itself did not have a good well-defined meaning. So, how to do this analysis whatever we have done for non-relativistic quantum systems, the next topic which we will discuss in the next class okay so I stop here