Foundation of Quantum Theory: Relativistic Approach

Spontaneous and Stimulated emission

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Deexcitation through vacuum and non vacuum states

Lecture- 38

So till now we have learned about the interaction of atomic systems with background quantum field and we have seen that the quantum fluctuations are sometimes capable of doing strange things and in principle for finite time duration operation one can cause the atom to get into excited state if it initially starts in the ground state. However, for long time duration what we have learned that the presence of photon of the same frequency is required to undergo a transition from the ground state to the excited state.

Spontaneous Emission

What happens if the atom is intially excited.

If no perturbation, then the atom will stay put in the excited state

$$H_0 \, |\varepsilon\rangle = E_\varepsilon \, |\varepsilon\rangle$$
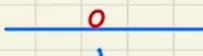
|e⟩

|ε⟩

However, $\hat{m}\,\hat{\phi}$ kind of couplings will try to disrupt this picture
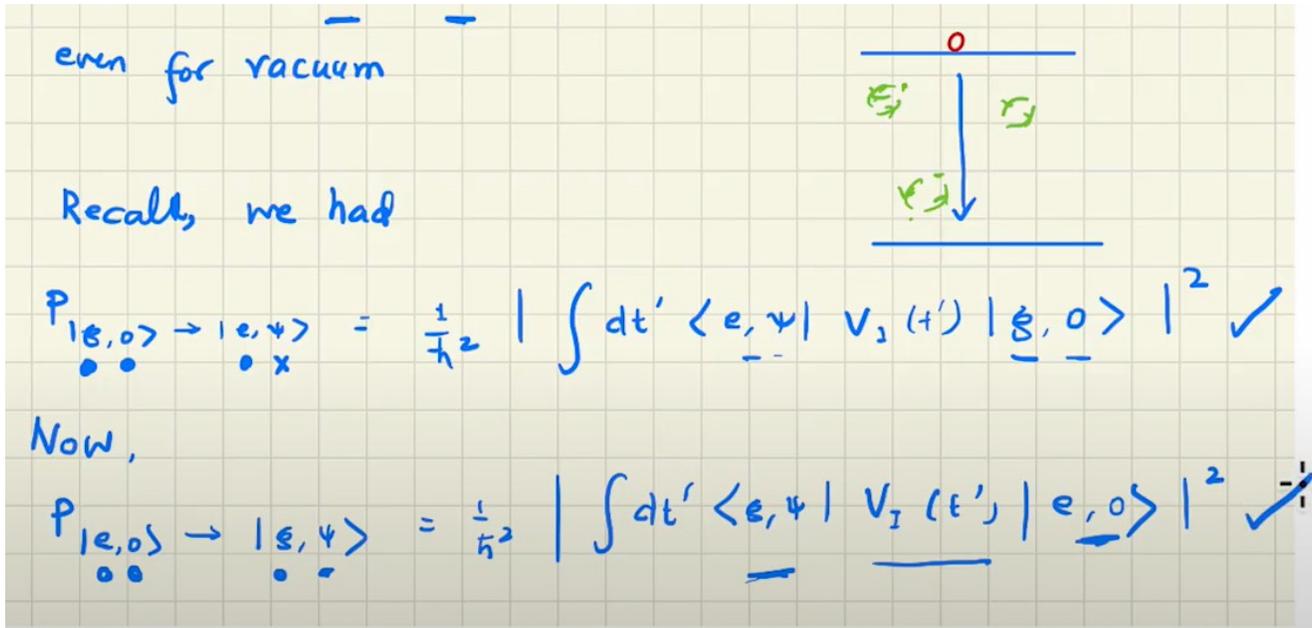
$\Rightarrow$ Even if $\hat{\phi}$ is in vacuum state !

$$\langle \hat{\phi} \rangle = 0$$

This is because transition probability is affected by $\langle 0| \, \hat{\phi}(t) \, \hat{\phi}(t') \, |0\rangle$ which remains non-zero even for vacuum

even for vacuum

Recall, we had

$$P_{|g,0\rangle \to |e,\psi\rangle} = \frac{1}{\hbar^2} \left| \int dt' \langle e,\psi | V_1(t') | g, 0 \rangle \right|^2 \checkmark$$

Now,

$$P_{|e,0\rangle \to |g,\psi\rangle} = \frac{1}{\hbar^2} \left| \int dt' \langle g,\psi | V_I(t') | e, 0 \rangle \right|^2$$

If no perturbation, the the atom will stay put in the excited stat.

$$H_0 |g\rangle = E_g |g\rangle$$

However, $\hat{m}\hat{\phi}$ kind of coupling will try to disrupt this picture.
$\Rightarrow$ Even if $\hat{\phi}$ is in vacuum state!
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Recall, we had
$$P_{|g,0\rangle \to |e,\psi\rangle} = \frac{1}{\hbar^2} |\int dt' \langle e,\psi | V_I(t') | e, 0 \rangle|^2$$

From this class onwards we will look about the reverse phenomena of that which goes by the name of spontaneous emission in which an atom is initially in the excited state of the Hamiltonian and since it is iron state of the Hamiltonian it is in principle supposed to live there forever unless perturbed.
So, what we will see that if the atom talks to the background quantum field. the quantum fluctuations within the quantum fields are strong enough to cause a downward transitions even if the state of the quantum field happens to be the vacuum. So, just like what we had discussed there was a probability of up transitions previously due to the quantum fluctuations in the background. There is also a reverse probability of getting down. Up probability was a proportional to delta function of energy gap divided by $\hbar - \omega_k + \omega_{k0.} + \omega_k$ this kind of expression was obtained for large time operation. For finite time

operation it was replaced by some *sin* function which was small but non-zero. Now we are going to talk about what would happen if the atom initially starts in the excited state and where does it land after a finite time as well as a long time. So, the premise is this thing that initially that Both ground as well as the excited state are the eigen state of the unperturbed Hamiltonian. So I have a statement that H0 acting on the ground or the excited state are going to give me the state back and with their corresponding eigen values of the energy. So in principle they would live there if they are unperturbed the electrons in the system or the atom collectively would live in those states forever and they will never change to their state. because these are steady states. However, the statement is due to coupling of the background field there can be some new correction term which we have seen and that will cause the transition downwards or upwards which we have completed. So, again for demonstration purpose we will start with a monopole coupling where the scalar field 5 is talking to the atom through a monopole operator term m. Now we have defined that the atom is going to be in the ground state in the excited state to begin with and the field, background field is in the vacuum state. The background field is in the vacuum state that means its expectation value is 0. However, we have seen while we were discussing the quantum field theory structure that despite the expectation value of the field operator being 0, the two-point correlator does not become 0 for the even for the ground state of the quantum field, even for the vacuum of the quantum field. There are inbuilt tiny quantum fluctuations coming from these quantum correlators which are present and these small fluctuations which pop up in very small amount of time and disappear, pair production happens and disappear. That makes the system dynamic and therefore, there is a time dependent perturbation which turns on and off repeatedly again and again and that causes the atom to come down. And we will compute the probability of that just like we had computed the probability of the atom going up. Only thing would be that the systems will change from the initial state was will now be excited state of the atom. Previously we had computed things of probability of going from the ground state of atom while the quantum field is in vacuum state to ground the atom goes to the excited state and the field goes to some state ψ which we later on sum over. And that expression was given by this integral mod square where the interaction Hamiltonian was squeezed between the states of interest. For this consideration, where we are considering a reverse phenomena, our excited state is the initial state. That means atom will start in the excited state, the field will still remain in the vacuum. And then we are asking for the probability that the atom goes to the ground state and field goes to some physical state ψ, which we will sum over later on. So, this time the operator of interest, the interaction Hamiltonian, interaction term will be squeezed between E0 and g ψ. Previously it was being squeezed between G0 and E ψ. So, this is the reversal which will happen. So, now we have to compute this time integral and its mod square rather than the previous one. Structure wise it is not going to be much different. We will just pay attention where do these things appear. So, recall. When we were doing the computation for excitation, this monopole operator was getting squeezed between the excited and the ground state in complex conjugate pairs. While their exponentials were coming with $e^{(+i\,\Delta E/\hbar\,t'\,-\,\Delta E\,t''/\hbar)}$. This time the role is reversed. Initial state is E, excited of the atom. Final state is g. Previously the reverse was happening. This side the g was sitting and this side the E was sitting. And that is how a + i was coming about in the exponential. This time a − i is coming about. So if you are just careful about the appearance of a state in the correct order in the bracket notation, you will just realize it is almost the same computation only for the fact that the exponentials argument will become a + i $\Delta E/\hbar$ $(t - t')$. Previously it was a − i, this time it becomes a + i. This is because the flipping of role, initial state is excited and final state is the ground. The previous time it was the other way. The initial state was ground and the final state was the excited. And that is why this change of ΔE signature comes about. Other things remain exactly the same and you do get the ϕ, the field operator squeeze between 0 and ψ and ψ and 0 like before. That does not change. That because the field state, remember initially also we were asking for the field to go from the vacuum to the ψ. Only interesting thing was happening for the atomic system, it was going from ground to excited. This time the role of the atomic system has changed, the field is doing the same bu*sin*ss. It is

going from the vacuum to some ψ. And the mod square of that would give me a complex conjugate version of this squeezing this matrix element. So, ultimately in the field side nothing changes, in the atomic side a $+ i \Delta E$ appears in the expression. And as before if we are interested in just the de-excitation of atom only without caring about what happens to the field, what we will do, we will sum over all the possible field state which appears over here, if I sum over all possible Hawke's basis elements, then this gives me just the projection operator of identity.

$$P_{|e,0\rangle \to |s,\psi\rangle} = \frac{1}{\hbar^2} \int dt' \langle s| \hat{m} |e\rangle e^{-i\frac{\Delta E t'}{\hbar}} \langle \psi| \hat{\phi}(t')|0\rangle$$

$$\times \int dt'' \langle e| \hat{m}|s\rangle e^{+i\frac{\Delta E t''}{\hbar}} \langle 0| \hat{\phi}(t'')|\psi\rangle$$

$$= \frac{1}{\hbar^2} |\langle s|\hat{m}|e\rangle|^2 \iint dt\, dt'\, e^{+i\frac{\Delta E(t-t')}{\hbar}} \langle 0| \hat{\phi}(t')|\psi\rangle \langle \psi| \hat{\phi}(t)|0\rangle$$

If we are interested in de excitation of the atom only

$$P_{|e\rangle \to |s\rangle} = \sum_{\psi \in \text{Fock basis}} P_{|e,0\rangle \to |s,\psi\rangle}$$

$$= \frac{1}{\hbar^2} |\langle s|\hat{m}|e\rangle|^2 \iint dt\, dt'\, e^{+i\frac{\Delta E(t'-t)}{\hbar}} \langle 0| \hat{\phi}(t') \sum_{\psi} |\psi\rangle \langle \psi| \hat{\phi}(t)|0\rangle$$

$$\boxed{= \frac{1}{\hbar^2} |\langle s|\hat{m}|e\rangle|^2 \iint dt\, dt'\, e^{+i\frac{\Delta E(t-t')}{\hbar}} \langle 0| \hat{\phi}(t) \hat{\phi}(t')|0\rangle}$$

Thus, vacuum correlations will cause the atom

to deexcite $\because$ Spontaneous emission

We already have

$$\langle 0|\hat{\phi}(t)\,\hat{\phi}(t')|0\rangle = \int \frac{d^3k}{(2\pi)^3\, 2\omega_k}\left[e^{-i\omega_k(t-t')}\right] \checkmark$$

for atom at rest

$$\therefore \quad P_{|e\rangle \to |g\rangle} = \frac{1}{\hbar^2}|\langle e|\hat{m}|e\rangle|^2 \iint dt\,dt'\, e^{+i\frac{(\Delta E)}{\hbar}(t-t')} \int \frac{d^3k}{2\omega_k} e^{-i\omega_k(t-t')}$$

$$P_{|e,0\rangle \to |g,\psi\rangle} = \frac{1}{\hbar^2}\int dt'\,\langle g|\hat{m}|e\rangle e^{\frac{-\Delta E}{\hbar}}\langle \psi|\hat{\phi}(t')|0\rangle \int dt''\,\langle e|\hat{m}|g\rangle e^{\frac{-\Delta E}{\hbar}}\langle 0|\hat{\phi}(t'')|\psi\rangle$$

$$= \frac{1}{\hbar^2}|\langle g|\hat{m}|e\rangle|^2 \iint dt\,dt'\, e^{\frac{+\Delta E}{\hbar}(t'-t)}\langle 0|\hat{\phi}(t'')|\psi\rangle\langle \psi|\hat{\phi}(t')|0\rangle$$

$$P_{|e\rangle \to |g\rangle} = \sum_{\psi\,\epsilon\,Fock\ basis} P_{|e,0\rangle \to |g,\psi\rangle}$$

$$\frac{1}{\hbar^2}|\langle g|\hat{m}|e\rangle|^2 \iint dt\,dt'\, e^{\frac{+\Delta E}{\hbar}(t-t')}\langle 0|\phi(t)\phi(t')|0\rangle$$

We already have

$$\langle 0|\phi(t)\phi(t')|0\rangle = \int \frac{d^3k}{(2\pi)^3\, 2\omega_k}\left[e^{-i\omega_k(t-t')}\right]$$

for atom at the rest

$$P_{|e\rangle \to |g\rangle} = \frac{1}{\hbar^2}\frac{|\langle g|\hat{m}|e\rangle|^2}{(2\pi)^3}\iint dt\,dt'\, e^{\frac{+\Delta E}{\hbar}(t-t')}\int \frac{d^3k}{2\omega_k} e^{-i\omega(t-t')}$$

So, like before if I sum over all the field, final field state where the field can go, all the good basis into which the field can be decomposed into, the probability of only, information of only the field, the atom goes from excited to the ground state is obtainable from this ψ-ψ projection summed over all ψ which is nothing but the identity. And as a result, I will again get it is yet again the two-point function of the field. This time undergoing the same Fourier kind of transform with a positive i ΔE. Last time it was a $-$ i ΔE if you remember. Therefore, it is almost the same computation but with a *sin* flip. So, we will do that and we will see again like before it was going to it is going to give me the probability of down

transition. Previously we were computing the up transition probability and this time if it survives, it is happening solely due to the quantum field two point correlated, quantum fluctuations in the vacuum state that is the source of this down transition if it is non-zero. Previously when there was a − i over here, This integration over long time interval became 0 because of positive argument of delta function. You can now anticipate since the sign of the ΔE has flipped, this time I am going to get a delta function which is not necessarily positive argument, that is why it is going to survive. So therefore, there is a vacuum structure of the field which is going to cause a de-excitation and this goes by the name of a spontaneous emission. That means atom will undergo a de-excitation, it will emit a photon and this is spontaneous, it is not caused by any active agent. You are not supplying anything to the system, only the quantum fluctuations of the background field are sufficient to do that, alright. So, this is the genesis of spontaneous emission. This, you cannot help it, this quantum fields always come with fluctuation, this is not under our control. This is no parameter where we can control and increase or decrease this inherent nature of a quantum field, it comes with a quantum two-point correlator and quantum fluctuation and no matter what, this is always going to be present and therefore, it will cause an emission of the atom on its own. So, therefore, the name is spontaneous, nothing has to be done, this is spontaneous emission. So, we go ahead and try to compute this expression. As before, we know that the two-point function can be written for atoms at rest like this. $e^{-i\omega k\,(t\,-\,t)}$. Remember, the two-point function was truly $e^{-ik\cdot(x\,-\,x')}$ *rather* $e^{+ik\cdot(x\,-\,x')}$. And when I open it up, I would get a $t - t'$ with ω with a − sign and $+\;ik\,\vec{x},$ position $x - x'$. So, if atom is rest, Then x and x' are the same quantity in the spatial locations are the same. So they will just go away from this exponential and I will only be left with this temporal part in the exponential. So the two-point function is just this. What I have to do, I have to take this two-point function and supply it back into the expression over here and as a result I get the probability of down transition as this simple integral. So, I have a one d^3k integral to compute of this exponential $\omega(t - t')$ 'and another exponential with $t - t'$ in its argument also appears. So, I will combine these like before and we will evaluate the two point correlators integral transform. So as we have done so many times by now, I will collect all the t functions at one place undergo the temporal derivative. So I will take this dt, this first exponential's time dependency and this second exponential's t dependency together and I will get some structure like this. So see what do I get? First exponential has a $+$ i $\Delta E/\hbar$. So I get $+$ i $\Delta E/\hbar$. And then $e^{-i\omega kt}$, which is coming from here. Let me erase it out so that we can see it with clarity. That $e^{-i\omega kt}$ comes over here. So, I combine it in the argument of exponential.

$$= \frac{|\langle e|\hat{m}|g\rangle|^2}{(2\pi)^3 \hbar^2} \int \frac{d^3\vec{k}}{2\omega_k} \int_0^{} dt \, e^{i\left(\frac{\Delta\epsilon}{\hbar} \pm \omega_k\right)t} \int_0^{} dt' \, e^{-i\left(\frac{\Delta\epsilon}{\hbar} - \omega_k\right)t'}$$

$$= \frac{|\langle e|\hat{m}|g\rangle|^2}{(2\pi)^3 \hbar^2} \int \frac{d^3\vec{k}}{2\omega_k} \left(\frac{2 \sin\left(\frac{\Delta\epsilon}{\hbar} - \omega_k\right)\frac{T}{2}}{\left(\frac{\Delta\epsilon}{\hbar} - \omega_k\right)} \right)^2$$

For large time

$$\Gamma_{|e\rangle \to |g\rangle} = \frac{|\langle e|\hat{m}|g\rangle|^2}{(2\pi)^3 \hbar^2} (2\pi)^3 \int \frac{2k\,dk}{2} \left[\delta\left(\frac{\Delta\epsilon}{\hbar} - \omega_k\right) \right]^2$$

$$= \frac{|\langle e|\hat{m}|g\rangle|^2}{\hbar^2} \left(\frac{\Delta\epsilon}{\hbar}\right) \delta(0)$$

$$R_{|e\rangle \to |e\rangle} = \frac{|\langle e|\hat{m}|g\rangle|^2}{\hbar^2} \left(\frac{\Delta\epsilon}{\hbar}\right)$$

* Larger the gap, more easy is for the

$$= \frac{\langle E|\hat{m}|0\rangle_n^2}{(2\pi)^3 \hbar^2} \int \frac{d^3\vec{k}}{2\omega_k} \int_0^{} dt\, e^{i(\frac{\Delta E}{\hbar} \pm \omega_k)t} \int_0^{} dt'\, e^{i(\frac{\Delta E}{\hbar} \pm \omega_k)t'}$$

$$\frac{\langle E|\hat{m}|0\rangle_n^2}{(2\pi)^3 \hbar^2} \int \frac{d^3\vec{k}}{2\omega_k} \left| \frac{\sin(\frac{\Delta E}{\hbar} - \omega_k)\frac{T}{2}}{(\frac{\Delta E}{\hbar} - \omega_k)} \right|^2$$

For large time

$$P_{|e\rangle \to |g\rangle} = \frac{\langle E|\hat{m}|0\rangle_n^2}{(2\pi)^3 \hbar^2} (2\pi)^3 \int \frac{2k}{2}\, dk\, [\delta(\frac{\Delta E}{\hbar} - \omega_k)]$$

$$= \frac{\langle E|\hat{m}|0\rangle_n^2}{\hbar^2} (\frac{\Delta E}{\hbar})$$

So, this is the whole *t* dependency in the whole. And that undergoes with a time integral of dt. Similarly, for dt', you can collect this and that together and I will have this second integral over here, which if you pay attention again is the similar thing like complex conjugate of each other, but for the fact this is computed at time *t* and while this second integral is computed at time *t'*. This was the same thing which we had seen before as well. Only thing which has changed this time is that the signature of the exponential has slightly shifted in the *ΔE*. Again like before, *I know* how to compute this exponentials, you do this exercise because we have done it before and you will be getting the same integral from both the sides, not exactly same, but up to a complex conjugate phases of each other. So, overall you are going to get *2sin(ΔE/ℏ – ωₖt/2)/(ΔE/ℏ – ωₖ /2)*, full². So, this is the simple integral. You can see that this goes. So, this the first integral goes into the whole thing with some phase and the second thing goes the whole thing again with the opposite of the phase. So, ultimately only the magnitude square survives. Phases cancel out from the integral. This is a simple exercise. You should try because we have done it before.

Previously if you recall, we were getting a + here. There was no – and here also there was a +. So, therefore I was getting + here and + here as well not this time, this time I have a sign difference. So, you see in the large time limit, in the large time range of this integral what we are discussing, we are going to get a delta function which can survive this time. I will get a *δ(ΔE/ℏ – ωk /2)*. I have done the simple opening up of the integral. This *d³k* if you remember was going to give me *4πk²dk*. For massless field k and *ωₖ* are the same thing. So, I cancel 1 power of *k²dk* with *kdk* will survive. *4π* will come about and there is a, this sign in the large time limit gives me a *2π* of this delta function as well. So, overall *(2π)³* would come about and I will get an extra 2 because there was remember there was a *4πk*. So, this is simple exercise what you should try to do and verify for yourself indeed you are going to get in the large time limit this integral. So, now you see I have a delta function which is not necessarily positive definite argument. Previously when we were looking for up transition for long time limit The delta function earned upon argument which was totally positive, hence it was 0. This time it will not be 0. Remember there is an integral going on. So, integral is changing the *ωₖ* continuously because this is a running integral in k. So, at one time the k value will be such that *ωₖ* for massless field, remember it is nothing but *kc*. So, c is equal to 1 unit, it is just k. So, one particular value of k will satisfy the argument of delta function and there it will survive. Previously when it was + no argument of, no running value of k could make the argument of *δ(0)* because k is running from 0 to infinity and argument of delta function was positive throughout and it was vanishing, this time it will survive. So,

you can see you evaluate it as a delta function, split the two delta function, $\Delta E/\hbar - \omega_k$ and another copy of that $\Delta E/\hbar - \omega_k$. Now you treat this as a function $f(k)$ and use this as a delta function. Do this integral. The result will be, it will take f of 0. f of 0 is $\delta(0)$. This k over here will take the value of $\Delta E/\hbar$ up to a c. I am putting c is equal to 1 unit. You should be very careful about that also. So you see ultimately I am going to get a result which is going to be this matrix element mod square divided by $\hbar^2$. This $(2\pi)^3$ exactly cancel with this $(2\pi)^3$ and $\Delta E/\hbar$ survives. So now there is a non-zero probability of a down transition. This is the spontaneous transition probability and this depends on the matrix element mod square and $\Delta E/\hbar$. That means if I increase the gap $\Delta E/\hbar$ becomes large, that means the separation between the ground and excited state is large, then the probability becomes high. So, therefore larger the gap, it is more easy for the vacuum fluctuations to knock the atom down. So, therefore larger the gap, it is more easy for the vacuum fluctuations to knock the atom down. The quantum fluctuations will find it easy to mediate between high between larger energy gap states so that it can come down. For smaller energy gap, this probability is slightly less. This is counterintuitive because one could have thought that vacuum fluctuation should not be too much of energetic things. So, they should cause transitions across low energy regime. But no, combined with the density, remember this $\Delta E/\hbar$ is surviving because one of the delta functions substituted the value k over here into $\Delta E/\hbar$. So, the reminiscence of This thing which was initially $4\pi k^2 dk$ and that 1 power of k was cancelled by the $/2\omega_k$ in the denominator. But anyway, this is the survival of the density of modes. So, density of modes gets increasing and with k. So, therefore, 1 power of k survives and that is the reminiscence where the $\Delta E/\hbar$ survives. Despite being low energetic stop, vacuum fluctuations do not come up with very large frequencies. But even then at large frequencies, The number of such modes are large and therefore the density times small probability becomes a larger winning function and therefore it becomes easier for vacuum fluctuations to knock the atom down. So, this is one interesting prediction that I take highly separated states, if I do nothing to the atom. Even then the atom would more occasionally be found in the ground state rather than excited state because quantum fluctuation would not leave it alone to remain in the excited state. Even then the atom would more occasionally be found in the ground state rather than excited state because quantum fluctuation would not leave it alone to remain in the excited state. So, atom could live there forever not this time. Quantum fields effects despite the field being in vacuum causes a down transition definitely and larger the energy gap larger is the probability of transition. So, therefore we have to be alert about the fact that there is a always a down transition probability which is the spontaneity. So, now we have seen a toy model in which a scalar field $\hat{\phi}$ was talking to atom through a monopole coupling term m. But in more realistic settings, we would have a more general well-known field like electromagnetic field which will be talking to atom through its let us say dipole coupling.

In case of a realistic dipole coupling

$$\mathcal{H}_{int} = \vec{d} \cdot \vec{E} \quad \checkmark$$

Then

$$P_{|e\rangle \to |g\rangle} = \frac{\langle e| d^i |g\rangle \langle g| d^j |e\rangle}{\hbar^2} \times \int dt\, dt'\, \langle 0| \hat{E}_i(t)\, \hat{E}_j(t') |0\rangle$$

In the Lorentz gauge quantization

$$\hat{E}^i = \frac{\partial A^i}{\partial t}$$

⭐ $\langle 0| A^i(x) A^j(y) |0\rangle$

$$= \frac{1}{(2\pi)^3} \int \frac{d^3\vec{p}}{\sqrt{2\omega_p}} \int \frac{d^3\vec{p}'}{\sqrt{2\omega_{p'}}} \langle 0| \left(\hat{a}^i_{\vec{p}}\, e^{ip\cdot x} + \hat{a}^{i\dagger}_{\vec{p}}\, e^{-ip\cdot x} \right) \left(\hat{a}^j_{\vec{p}'}\, e^{ip'\cdot y} + \hat{a}^{j\dagger}_{\vec{p}'}\, e^{-ip'\cdot y} \right) |0\rangle$$

$$= \frac{1}{(2\pi)^3} \int \frac{d^3\vec{p}}{\sqrt{2\omega_p}} \int \frac{d^3\vec{p}'}{\sqrt{2\omega_{p'}}} \langle 0| \hat{a}^i_{\vec{p}}\, \hat{a}^{j\dagger}_{\vec{p}'} |0\rangle\, e^{ip\cdot x - ip'\cdot y}$$

Using $\left[\hat{a}^i_{\vec{p}}, \hat{a}^{j\dagger}_{\vec{p}'} \right] = \left(\delta^{ij} - \frac{p^i p^j}{|p|^2} \right) \delta^3(\vec{p} - \vec{p}')$

In case of a realistic dipole coupling

$$H_{int.} = \vec{d} \cdot evc\, E$$

Then

$$P_{|e\rangle \to |g\rangle} = \frac{\langle e|d^i|g\rangle \langle g|d^i|e\rangle}{\hbar^2} \times \int dt\, dt' \langle 0|\hat{E}_i(t)\hat{E}_j(t)|0\rangle$$

In the Lorent guage qunatization

$$\hat{E}^i = \frac{\partial A^i}{\partial t}$$

★ $$A^i(x)A^j(y)|0\rangle = \frac{1}{(2\pi)^3} \int \frac{d^3\vec{p}}{\sqrt{2\omega_{\vec{p}}}} \int \frac{d^3\vec{p}'}{\sqrt{2\omega_{\vec{p}}'}} \langle 0|(\hat{a}_{\vec{p}}^i e^{i\vec{p}x} + \hat{a}_{\vec{p}}^{i\dagger} e^{-i\vec{p}x})(\hat{a}_{\vec{p}}^{i}{}' e^{i\vec{p}'y} + \hat{a}_{\vec{p}}^{i\dagger}{}' e^{-i\vec{p}'y})|0\rangle$$

$$\langle 0|¿$$

$$= \frac{1}{(2\pi)^3} \int \frac{d^3\vec{p}}{\sqrt{2\omega_{\vec{p}}}} \int \frac{d^3\vec{p}'}{\sqrt{2\omega_{\vec{p}}'}} \langle 0|\hat{a}_{\vec{p}}^i \hat{a}^{i\dagger}|0\rangle e^{ip\cdot x - ip\cdot y}$$

Using

$$[\hat{a}_{\vec{p}}^i, \hat{a}_{\vec{p}}^{j\dagger}{}'] = (\delta^{ij} - \frac{p^i p^j}{|p|^2})\delta^3(\vec{p} - \vec{p}')$$

So, interaction term would not be $\hat{m}\hat{\phi}$ which was just a technology demonstrated in Victorian. A realistic situation could be one of the case where the dipole moment of atom tops to the background electric field d dot e. So, this term would be mediating the transitions and let us estimate going along the same lines that what would be the down transition probability for this realistic interaction. So, you do the same steps what we have done for monopole coupling, the same step we have to repeat only time thing one has to be careful that instead of $\hat{m}$ a vector d will come and instead of $\hat{\phi}$ a vector e will come about.

When we were getting the squeezing element squeezing element $\hat{m}$ between the initial state which was excited and final state which was g this time a dipole operator will sit over there and when I do go for a complex conjugate of that the complex conjugate of the same operator but this time with a different index remember the d appearing here is getting dotted with the e appearing over here and similarly when I do a mod square the complex conjugate the d appearing here should get a dotted with a e appearing over here. So, this time the same thing structure will happen. The d and this E will come together and this D and this E will come together.

Okay. So, you will be able to write down the probability of transition like this. All right.

So now, we already know in the Lorentz gauge quantization of electromagnetic field, we can write down A0 to be 0 and therefore, the electric field will be just the temporal derivative of the spatial core vector, the vector potential Ai, okay. And we have already seen the quantization of Ai and what not. So, I am just going to recast the results which we have already seen before. The two-point function of the vector potential, the different component of vector potential Ai and Aj. could be written like this. This exercise we have done. So, this is just the, I have just copied it from our earlier discussion. This is the same slide which you have already seen. Go back to the lecture notes over there or go back to the lecture video and do verify. This is just copied from the expression which we have already seen. The two-point function between the two A's, two vector potentials Ai and Aj can be given as gij, the name

can be given to gij. And its expression is almost like the scalar field. Scalar field would have just this. But because of there is a gauge condition which we had discussed about and there are vector potentials which we are talking about. It should come with index I and j. And the gauge conditions which we were using that it is a transverse field. It has no component along the direction of propagator. It forces the delta function to be modified by δ^{ij} – pi pj upon p^2. Remember this is just the projector operator in the orthogonal field. This was for the vector potential Ai and Aj. What we need is the electric field Ei, Ej. You can take the temporal derivatives of these quantities as we have done before and obtain the electric field two-point correlator. So, using all this, this again is nothing new. You have already seen that and we have done that already. You just do the temporal derivative of each of the a's appearing in the expression. That means first I will take the temporal derivative of this a and then I will do a temporal derivative of this a. The result will be the extra $i\omega$ factors which will be thrown out.

So, the two a's will give me the two extra $i\omega$ one with level p and one with level p′, one from this derivative and one from this derivative which will come out. Again, we have seen it before and you know the results already. The two-point functions of the vector potential, the two-point function of the vector potential can be converted into two-point functions of the vector electric field like this. This is almost the same, but with the fact that here of extra ωp^2 comes about, which takes the ωp in the down to the upstairs space. This is the simple thing. Remember there was a $\delta^3(p-p')$ will come about because of the operator A and A dagger surviving only for this pair. And that will convert the P′ into P and therefore, the ωp will be uplifted.   So, the two-point function of electric field component is this. This we have seen before. So you go ahead, you compute this integral this time. So I have this extra factor which has come and sit in our place. For a scalar field, this extra factor was not there. Scalar field only this integral I was interested about for atom at rest. Again the same here, I am talking about atom at rest, that means the spatial locations are not changing. So $k \cdot (x - x')$ is going to become 0, the spatial part of that. So, only temporal part will survive.

$$\therefore \quad P_{|e\rangle \to |g\rangle} =$$

$$\frac{\langle e| \hat{d}^i |g\rangle \langle g| \hat{d}^j |e\rangle}{(2\pi)^3 \hbar^2} (2\pi)^2 \int \frac{d^3k}{2} \omega_k \; \delta_{ij} \left(1 - \frac{k_i^2}{k^2} \right) \left(\delta \left(\frac{\Delta E}{\hbar} - \omega_k \right) \right)^2$$

Further $\int \frac{d^3k}{2} \omega_k \; \frac{k_i^2}{k^2} \left(\delta \left(\frac{\Delta E}{\hbar} - \omega_k \right) \right)^2$ — is same for all i

$$= \int \frac{d^3k}{2} \omega_k \left(\frac{1}{3} \right) \left(\delta \left(\frac{\Delta E}{\hbar} - \omega_k \right) \right)^2$$

$$\therefore \quad P_{|e\rangle \to |g\rangle} = \frac{\langle e| \hat{d}^i |g\rangle \langle g| \hat{d}^j |e\rangle}{(2\pi) \hbar} \delta_{ij} \int \frac{d^3k}{2} \omega_k \frac{2}{3} \delta \left(\frac{\Delta E}{\hbar} - \omega_k \right)^2$$

$$= \delta_{ij} \frac{\langle e| \hat{d}^i |g\rangle \langle g| \hat{d}^j |e\rangle}{3\hbar} \left(\frac{\Delta E}{\hbar} \right)^3 \delta(0)$$

$$R_{|e\rangle \to |g\rangle} = \frac{\| \langle e| \hat{d} |g\rangle \|^2}{3\hbar} \left(\frac{\Delta E}{\hbar} \right)^3$$

$$P_{|e\rangle \to |g\rangle} = \frac{\langle e|d^i|g\rangle\langle g|d^i|e\rangle}{2\pi^2 \hbar^2} \int \frac{d^3\vec{k}}{2}\omega_k \left(\delta^{ij} - \frac{k^i k^j}{|k|^2}\right)\delta^3(\vec{k}-\vec{k}\,')$$

$$\times \int dt\, dt'\, e^{+i\Delta\frac{E}{\hbar}} e^{-i\omega(t-t')}$$

The k_i, k_j integral only survives if $i = j$
(Otherwise odd function under symmetric integral)

Therefore $\int d^3\vec{k}\,\omega_k \frac{k_i k_j}{|k|^2} e^{-i\omega(t-t')}$

$\approx \int d^3\vec{k}\,\omega_k \frac{k_i^2 \delta_{ij}}{|k|^2} e^{-i\omega_k(t-t')}$

$$P_{|e\rangle \to |g\rangle} = \frac{\langle e|d^i|g\rangle\langle g|d^i|e\rangle}{2\pi^2 \hbar^2} \int \frac{d^3\vec{k}}{2}\omega_k \left(\delta^{ij} - \frac{k^i k^j}{|k|^2}\right)\delta^3(\vec{k}-\vec{k}\,')$$

Further $\int \frac{d^3 k}{2}\omega_k \frac{k_i^2}{k^2}\left(\delta\left(\frac{\Delta E}{\hbar}-1\right)\right)^2$ is same for all i.

$$= \int \frac{d^3 k}{2}\omega_k \left(\frac{1}{3}\right)\left(\delta\left(\frac{\Delta E}{\hbar}-1\right)\right)^2$$

$$\therefore P_{|e\rangle \to |g\rangle} = \frac{\langle e|\hat{d}^i|g\rangle\langle g|\hat{d}^j|e\rangle}{(2\pi)\hbar}\delta_{ij}\int d3\vec{k}\,\frac{\omega_k}{2}\frac{2}{3}\delta\left(\frac{\Delta E}{\hbar}-\omega_k\right)$$

$$= \delta_{ij}\frac{\langle e|\hat{d}^i|g\rangle\langle g|\hat{d}^j|e\rangle}{3\hbar}\left(\frac{\Delta E}{\hbar}\right)^3 \delta(0)$$

$$R_{|e\rangle \to |g\rangle} = \left\|\frac{\langle e|\hat{d}^i|g\rangle\langle g|\hat{d}^j|e\rangle}{3\hbar}\right\|\left(\frac{\Delta E}{\hbar}\right)^3$$

So, therefore, I have to just do the same computation one more time, but for the fact that one has to be alert about this presence of an extra term which is handy. So, in order to do that, I have to just be careful about couple of things and then we will be done because this is nothing serious integral. Its base structure is very similar to what we have done. So, let us focus on the second term first, the second extra term, let us say this term. I have a $k_i k_j$ integral and there is a d^3k integral I have to do. d^3k integral means dk_x, dk_y, dk_j, the three-dimensional spatial integral. Now, ω_k depends on the magnitude of k_x, k_y, k_z as in ω_k is magnitude of vector k, which is^2 root of $k_x^2 + k_y^2 + k_j^2$. That means it is an even function of k_x or k_y or k_j. The magnitude mod k^2 is also an even function of k_x, k_y and k_z. . And last k dependency is also here in terms of ω_k, which is also an even function of k_x, k_y, k_z. . So that means as long as I and j are different numbers, suppose I am computing k_1, k_2. k_1, k_2 is meaning k_x, k_y. So that means I will have an integral which is dk_x, dk_y, dk_z. Even function of k_x, k_y, k_z. , even function of k_x, k_y, k_z. , even functions of k_x, k_y, k_z. . Only a single appearance of k_x here and a single appearance of k_y over here. That means I will overall have an odd function under symmetric integral. Remember in this d^3k, k_x, k_y, k_z. all are running

from – infinity to + infinity. So as long as I and j are different quantities, they can be separated out in terms of their integrals and therefore it will become an odd integral, odd function under symmetric integral that will vanish. So, therefore this second term can only survive when I is equal to g. So, therefore I can write down the integral which is the second piece, this piece. $d^3k/2$, forget about the 2, $d^3k \, \omega_k / |k|^2 \, e^{-i\omega(t-t')}$. That I have written over here. This can become non-zero only when I is equal to j. And when I becomes j, it becomes k_i^2 or k_j^2. So, this condition can be written as $k_i^2 \delta^{ij}$. So now you see both the terms are proportional to δ^{ij}. First term was anyway proportional to δ^{ij}, second term by operational means we can show that it is only proportional to δ^{ij}. So that means I can pull out a δ^{ij} as a common thing. Then the first term in the whole expression over here becomes 1, δ^{ij} has been pulled out. Second term is $\delta^{ij}/|k|^2$ multiplied by k_i^2. So I am going to write as $1 - k_i^2 \, k^2$. This is fine. Now, you can further realize if I look, pay attention for this integral. If I look for this integral, I have a $d^3k /2$, I have a ω_k, I have a k_i^2 upon k^2 and then this delta function square. Again remember the t and t' integral. This integral we have already done for vacuum fluctuation k_ind of the spontaneous emission case. So, for scalar field, this is the same integral. So, this is again going to give me a delta function squared. Now, you pay attention that this k_i^2, I can take value 1, 2 or 3.

If it is k_x^2, then that means I have a dk_x, dk_y, dk_z. . So, dk_y, dk_z. integrals will be done, but k_x integral is k_x^2 and the symmetric functions of that running from – infinity to + infinity. had it been a k y^2 then also the same integral is being performed, one quantity k y^2 running from – infinity to infinity and similarly for k z. So, that means what is appearing over here k i^2 k upon k^2 is same for all i. If it is kx^2 upon mod k^2 then also the integral will be the same. If it is k y^2 upon k^2 then also the result is same and similarly for k z^2. That means this together is k i^2 the integral value can be added together, then it will become k_x^2 + k_y^2 + k_z. ^{2}, that means that is 3 times one of the ki ^{2}. So, if I add it together, then the operational part is k_1^2 + k_2^2 + k_3^2 divided by k^2. And that should be equal to 3 times the same integral with k_x^2, because k_y^2, k_z. ^{2} integral should be the same. So, that means just the k_x^2 integral is one third of this integral. And remember k_x^2, k_y^2 + k_z. ^{2} divided by k^2 is just the k^2 itself. So, this will cancel out. So, that means I can write instead of this 1 by 3. So, that is what we do for k_i ^{2} upon k^2 as 1 by 3 that means the integral which we went after $(1-k_i^2)/k^2$ is 1 + 1 by 3 that is to say it is just 2 by 3. So, I have the same integral ΔEj has been pulled out 2 by 3 over here and you have to do this delta function integral which is very trivial to do. This time d^3k integral will again give me $4\pi k^2 dk$. Remember for scalar field there was a denominator ω_k which reduced the power of k by 1. This time for electric field ω_k is in the numerator, so it multiplies to the power 1. So k^2 here becomes kq and the delta function becomes $\Delta E/\hbar^3$. For scalar field it was $\Delta E/\hbar$ to the power 1. So you see for realistic case the probability of transition is cubically proportional to the frequency for the electromagnetic field. So, this is one of the robust reasons that the atomic spontaneous emission for realistic settings is cubically dependent on the frequency gap. It reacts very vigorously for high energy separation. Previously, for a scalar field, we had seen that the probability goes linearly proportional to the energy gap. This time it is going third power to the energy gap. That means it is even more possible for the vacuum fluctuation to knock down the electron if it is in excited state, more and more high in excited state. So, vacuum fluctuations for electromagnetic field are much more powerful than the vacuum fluctuations of the scalar field. So, you get this probability of transition rate. the probability of transition from the excited to the grounded state. This is the spontaneous emission probability. A $\delta(0)$ comes about for the long time operation. Remember long time operation here again means larger than any internal time scale, but smaller than the time scale over which perturbation theory will break down. That means this is the total time of operation where the perturbation theory approach works out. It is not a real time infinity. So, if I divide by that time, or take a derivative with respect to that time, I will get a rate which is this. The rate is also cubically proportional to the admittance. energy gap. This is one of the famous result of the spontaneous emission for heuristic system that the frequency proportionality, cubic frequency proportionality is a hallmark of a spontaneous emission for atomic system. Most of the atoms react very strongly with interact very strongly with electromagnetic field and the cubical spontaneous emission rate is a

fundamental result which one is always looking for that this has been viewed, realized in the lab settings in many experiments as well. But this is one of the hallmark signature or hallmark prediction of a quantum block equation that there is a tendency of down transition with a cubic dependency. So, that was for the vacuum part. Now, suppose we want to know about the process what happens if the field comes up with a Instead of being vacuous, if the field is in some excited state, let us say $1k_0$ or 1 photon of a frequency k_0 is present, what happens to the de-excitation?

If the de-excitation is assisted with the presence of a photon, the process is known as the stimulated emission. In this case, we have to know the process probability that when the initial state of the field is 1_{k_0} and not vacuum, what is the down state, down transition probability of the atom? the process which we are asking for, the initially atom is in excited state, field is in 1_{k_0} state and then after the process end, the atom goes to the grounded state and the field goes to anywhere it likes, ψ, which we will sum over. The process which we are asking for, the initially atom is in excited state, field is in 1k knot state and then after the process end, the atom goes to the grounded state and the field goes to anywhere it likes, ψ, which we will sum over.

And the probability of down transition takes this form where now the $\phi(t)$ and $\phi(t)'$ gets squeezed between initial state and the final state ψ and final state and the initial state ψ and then the ψ is summed over. So, this is just the identity operator of the field. The ψ, outer product ψ-ψ is just the projector operator of the full basis element which is just identity. So, therefore, we will get the two-point function of the field between the initial state of the field which is 1k of this type. This is exactly the same result which we had obtained for the case of absorption as well. Only thing if you could notice is the sign change of ΔE which is coming about, which is exactly coming from this term where the initial state is excited and the final state is ground. Previously we had the flipped row, we had the $\hat{m}$ squeezed between g and E from right-hand left respectively and therefore a $-\Delta E$ was coming about, this time a $+ i\Delta E$ is coming. So, what we have to do? We have to write down the two-point function of the field in the excited state of 1_{k_0} and then put it over here to undergo the integral to obtain the down transition probability.

$$\therefore \quad \langle 1_{k_0} | \hat{\phi}(t) \hat{\phi}(t') | 1_{k_0} \rangle$$

$$= \frac{1}{(2\pi)^3} \int \frac{d^3 k_1}{\sqrt{2\omega_{k_1}}} \int \frac{d^3 k_2}{\sqrt{2}\omega_{k_2}} \left[\left(\delta(k_1 - k_2) \, e^{i(k_1 \cdot x - k_2 \cdot x')} \, \delta(0) \right. \right.$$

$$+ \, \delta(k_0 - k_2) \, \delta(k_1 - k_0) \, e^{i(k_1 \cdot x - k_2 \cdot x')}$$

$$+ \, \delta(k_0 - k_1) \, \delta(k_2 - k_0) \, e^{-i(k_1 \cdot x - k_2 \cdot x')}$$

$$= \frac{1}{(2\pi)^3} \int \frac{d^3 k_1}{2\omega_{k_1}} e^{-i k_1 \cdot (x - x')} \, \delta(0)$$

$$+ \frac{1}{(2\pi)^3} \left[\frac{e^{i k_0 \cdot (x - x')}}{2\omega_{k_0}} + \frac{e^{-i k_0 \cdot (x - x')}}{2\omega_{k_0}} \right]$$

Thus,

$$\frac{P_{g \to e}}{\langle 1_{k_0} | 1_{k_0} \rangle} = \frac{|\langle e | \hat{m} | g \rangle|^2}{(2\pi)^3 \hbar^2} \iint dt \, dt' \, e^{-i \frac{\Delta \varepsilon}{\hbar}(t - t')}$$

$$\frac{1}{\delta(0)} \left[\int \frac{d^3 k_1}{2\omega_{k_1}} e^{-i k_1 \cdot (x - x')} \, \delta(0) + \frac{e^{i k_0 \cdot (x - x')} + e^{-i k_0 \cdot (x - x')}}{2\omega_{k_0}} \right]$$

$$e^{i k_0 \cdot (x - x')} = e^{-i \omega_{k_0}(t - t') + i \vec{k}_0 \cdot (\vec{x} - \vec{x}')}$$

$$= e^{-i \omega_{k_0}(t - t')} \qquad \text{for atom at rest}$$

$$\frac{P_{g \to e}}{\langle 1_{k_0} | 1_{k_0} \rangle} = \frac{|\langle e | \hat{m} | g \rangle|^2}{(2\pi)^3 \delta(0) \, 2\hbar^2 \, (2\omega_{k_0})} \int dt \, dt' \left\{ e^{-i(\frac{\Delta \varepsilon}{\hbar} + \omega_{k_0})(t - t')} + e^{-i(\frac{\Delta \varepsilon}{\hbar} - \omega_{k_0})(t - t')} \right\}$$

$$\underbrace{\qquad}_{T \to \infty} \qquad \underbrace{\qquad}_{T \to \infty}$$

Thus,

$$P_{|e\rangle \to |g\rangle} = \frac{|\langle g|\hat{m}|e\rangle|^2}{(2\pi)^3 \hbar^2} \left[\int \frac{d^3k}{2\omega_k} \left| \int dt\, e^{i\left(\frac{\Delta E}{\hbar} - \omega_e\right)t} \right| \delta(0) \right.$$

$$\left. + \frac{1}{2\omega_{k_0}} \left| \int dt\, e^{i\left(\frac{\Delta E}{\hbar} - \omega_{k_0}\right)t} \right|^2 + \frac{1}{2\omega_{r_0}} \left| \int dt\, e^{i\left(\frac{\Delta E}{\hbar} + \omega_e\right)t} \right|^2 \right]$$

For long time (normalized w.r.t. state)

$$\frac{P_{|e\rangle \to |g\rangle}}{\langle 1_{r_0}|1_{r_0}\rangle} = \frac{|\langle e|\hat{m}|g\rangle|^2}{(2\pi)^3 \delta^{(3)}(0)\, 2\hbar^2} \left[\int d^3k \, (2\pi)^2 \left(\delta\left(\frac{\Delta E}{\hbar} - \omega_k\right)\right)^2 \delta^3(0) \right.$$

$$\left. + \frac{(2\pi)^2}{2\omega_{k_0}} \left(\delta\left(\frac{\Delta E}{\hbar} + \omega_{k_0}\right)^2 + \delta\left(\frac{\Delta E}{\hbar} - \omega_{k_0}\right)^2 \right) \right]$$

Surviving terms are from vacuum and photon resonance.

$$\therefore \langle 1_{k_1}|\phi(t)\phi(t')|1_{k_1}\rangle = \frac{1}{(2\pi)^{3/2}} \int \frac{d^3\vec{k_1}}{\sqrt{2\omega_{\vec{k_1}}}} \int \frac{d^3\vec{k_2}}{\sqrt{2\omega_{\vec{k_2}}}} \Big[\delta(k_1 - k_2) e^{i(k_1 x - k_2 x')} \delta(0) + \delta(k_2 - k_1)\delta(k_1 - k_2) e^{i(k_1 x - k_2 x')} + \delta(k_0 - k_1)\delta(k_2 - k$$

$$\frac{1}{(2\pi)^3}\int \frac{d^3\vec{k}_1}{\sqrt{2\omega_{\vec{k}_1}}}e^{-ik_0(x-x')}\delta(0)+\frac{1}{(2\pi)^3}\left[\frac{e^{ik_0(x-x')}}{2\omega_k}+\frac{e^{-ik_0(x-x')}}{2\omega_k}\right]$$

$$\frac{P_{g\to e}}{\langle 1_{k_0}|1_{k_0}\rangle}=\frac{|\langle e|\hat{m}|g\rangle|^2}{(2\pi)^3(\hbar^2)}\iint dt\,dt'\,e^{\frac{-i\,\Delta E(t-t')}{\hbar}}$$

$$\frac{1}{\delta(0)}\left[\int \frac{d^3\vec{k}}{\sqrt{2\omega_{\vec{k}}}}e^{-ik_i(x-x')}\delta(0)+\frac{e^{ik_0(x-x')}+e^{-ik_0(x-x')}}{2\omega_k}\right]$$

$$e^{-ik_i(x-x')}=e^{-i\omega_{k_0}(t-t')+I\vec{k}_0(\vec{x}-\vec{x}')}=e^{-i\omega_{k_0}(t-t')}$$

$$\frac{P_{g\to e}}{\langle 1_{k_0}|1_{k_0}\rangle}=\frac{|\langle e|\hat{m}|g\rangle|^2}{(2\pi)^3\delta(0)(2\hbar^2)}\left\{\iint dt\,dt'\,e^{\frac{-i(\frac{\Delta E}{\hbar}+\omega_{k_0})(t-t')}{\hbar}}+e^{\frac{-i(\frac{\Delta E}{\hbar}-\omega_{k_0})(t-t')}{\hbar}}\right\}$$

(Requires photons in
the field)

(Happens through both
vacuum and photons)

These rates are sometime called the Einstein's
coefficients

$\Rightarrow$ Spontaneous Emission $\qquad A_{80}$

$\Rightarrow$ Absorption $\qquad B_{12}$

$\Rightarrow$ Stimulated Emission $\qquad B_{21}$

Total up transition rate : $\qquad B_{12} \, n_1 \, \rho(\nu)$

Total down transition rate : $\qquad A_{21} \, n_2 + B_{21} n_2 \, \rho(\nu)$

In case of detailed equilibrium

The two-point function in the ground state was already known, in the vacuum was already known. Even for the 1_{k_0} state we had computed it before in the last week. You can go back to the last week exercise or in the last lectures exercise if you can look at. There you can verify we had obtained this expression for the two-point function of the field. This first term was just the vacuum part if you recall times the normalization of state because you recall 1_{k_0} state is not normalized to unity, but it is normalized to the three-dimensional data function in the case space with the value 0. And the two remaining terms are coming from the state being non-vacuum. Collectively you can employ the delta functions in the integrals. I have a double integral of d^3k_1 and d^3k_2 with the various delta functions appearing here and there and you can verify. This we have done before. We could not have any problem in verifying this that I would have an expression at the end of the day in terms of this vacuum term which is a running k integral with plane wave structure at hand and a normalization of the state + two terms which are state dependent correction. There are two terms which are all complex conjugates of each other and they do know about the frequency k_0 at which the field is having the photon. The initial state of the field is not vacuous, but an excited state at frequency k_0. This information goes into the two extra terms which come in the two-point function. The first term does not care about k_0. This term is independent of k_0. This is just a vacuum structure. On top of vacuum structure there is a state dependent correction which are these two terms. So, therefore, these two terms will go and sit at the appropriate place in this integral over here and we will get the total down transition probability. We

have done this exercise before for absorption. This time the same integral has to be repeated again only changes this exponential serviment is $+ \delta(u)$. Now by now we are too accustomed to do all these integrals. So we do it piece by piece. First term is the vacuum term over here. This goes and sits over here. We perform the time integrals by collecting all the t parts and t' parts at one place undergoing the transitions integrals and I get this as a result. This is the vacuum part and times the normalization of the state. This is coming from the first term over here. The second term will also undergo the same process and we will get this term. And the third correction term which is over here will sit in the appropriate integral over here and give rise to this term. Remember all the time we are computing the things for atoms at rest. Therefore, whether in this or in this or in that only the temporal part survives. The spatial $x - x'$ is 0 for any time because atom is at rest, it is not moving. So, at any time x is going to be the same. So, x and x', the spatial locations of that are going to be the same, they will drop out from the exponential and only temporal part will survive which is undergoing the integral and therefore we have these states. This is for finite time.

If I want to do for a long time value of the probability of transition, then I enhance the integrals limit from 0 to really large p, which is approximately infinity, but not quite as we have discussed so many times. It is operational infinity, that means much, much larger than internal time scales, but smaller than the perturbation series breakout time. And then I divide the whole probability by the normalization of the n state. Remember the initial state was not properly normalized. So, one good way of getting probability is to divide the whole probability process by the normalization. So, this integral over here converts itself into the delta function square as we have seen and these two integrals over here also converted themselves into delta functions. Actually, this $i(\Delta E - \omega_{k0})$ becomes the third term and this becomes the second term. So, all three of them becomes delta functions of their arguments in the exponential. Only thing is that the first term is running with the k integration. This is the vacuum part. It does not care about where the field is excited. This is just the spontaneous emission part we have just seen. While these two remaining terms are the state-dependent correction. It talks about whether or not the field is excited at appropriate frequency. ω_{k0} is the frequency at which the photon is present in the field. Second term you can see already it has a positive argument in the delta function, so it is going to drop out. Only the first and the third term are going to survive. I have divided the whole expression by $\delta^3(0)$, so this $\delta^3(0)$ will be killed off by this $\delta^3(0)$ in the first term and the delta^2 integral will just become $\Delta E/\hbar \ \delta(u)$. So, I have missed out $/2\omega_k$ most likely over here, so $/2\omega_k$ should have been here, which converts itself into $\Delta E/\hbar$. Remember this is the spontaneous emission process which we are talking about. This $4\pi k^2 dk /\omega_k$ gave me this linear frequency proportionality. So, this is just the vacuum part. This third term which survives over here is the new part, which is, this is product of our delta function, again we do the splitting as we have discussed before. Use one of the delta function as $\delta(0)$ and the remaining term is this. So, this is again asking whether the photon is coming with the appropriate frequency or not. If the excited state of the field has a different frequency, this delta function dies down. But if it so happens that $2\omega_{k0}$ is the same value as $\Delta E/\hbar$, the third term will survive and it will contribute this much to the down transition process. This $\delta(0)$ is coming from the square of this two delta functions in^2 are appearing delta x whole square is equal to $\delta(x) \ \delta(0)$. So, that is how we have written these two things and this division by $\delta^3(0)$ is coming from the state norm. So, you see that there are two terms which are causing the down transition if the field is already excited. First is the vacuum part which anyway comes. This does not care about whether the field is excited or not, this is the vacuum spontaneous emission + this time a state dependence, photon dependent correction term which is the stimulated emission. So, you see the total down transition probability is vacuum + the stimulated part. There is a vacuum fluctuation which is also causing the down transition and then there is the active photons present at frequency k_0 which is same as $\Delta E/\hbar$. But if that happens, then also this gets down transitions and as a result, you get two photons out. One was initial photon which was sent in and one was due to down transition the atom enters. If there is no time initial photon is sent in, then also the

vacuum fluctuation causes it to go under forward transition and photon will be in. So, in spontaneous emission you send in nothing but a photon is obtained, in stimulated emission you send one photon in but get two photons out. One last thing one should be careful about is this, if you look at this expression, forget about the vacuum part, you will realize it is the same expression which you obtained for absorption probability, it had to match the frequency and the coefficients are exactly the same. So, stimulated emission part, not the spontaneous + stimulated, only the stimulated emission part probability is the same as the absorption probability. So, whatever is happening due to presence of photon happens with equal probability from up to down and down to up. They are just symmetric things. Only extra contribution comes from the vacuum part. Vacuum part was absent for absorption. But it turns alive for emission. Spontaneous emission assists the down transition. Up transition is only possible if a photon is there, down transitions have both the things. Even if a photon is present, then it will down transition, then the probability are the same. But in addition, there are vacuum fluctuations which also cause the down transition. So, only looking at stimulated part, one can ascertain that it is equal to absorption part as well. So, this is just a nice figurative summary of the process. Absorption is only possible if there is an active photon present in the field. Emission on the other hand is assisted by vacuum fluctuation as well as the presence of the photon in the field. Now, these rates which we have computed for spontaneous emission, the rate was proportional to $\Delta E/\hbar$ for scalar field or it was proportional to $\Delta E/\hbar^3$ for electromagnetic field. And this is the same probability which is the stimulated emission probability or absorption probability and divided by the $\delta(0)$ which is the operational time capital T. If I divide by $\delta(0)$, the remaining term will be given with the rate. There are three rates in the game. One is the spontaneous emission rate, absorption rate and stimulated emission rate. So, these are sometimes called the Einstein's coefficients in the process. A_{21} is the spontaneous emission rate. It takes the state 2 excited state to the ground state 1. Absorption, it takes you from 1 to 2 and then the stipulated emission which is B_{221}. It is so enough that B_{121}, B_{221} should be the same as we have seen. But anyway, so total up transition rate can be given in terms of B_{121}. Absorption will happen with this rate.

It will be proportional to how many atoms are in there in the ground state N1 and what is the frequency of the photon present. It should be directly proportional of the density of the photons present. So, this is the total up transition rate, total down transition rate is this B_{21}, B_{21} is taking you down from 2 to 1 and it will proportional to how many states atoms are there in the excited state N_2 and again the number density of photons surrounding this atom. But in addition to this stimulated emission there is A_{21}. A_{21} is spontaneous emission wave. This will also care about how many atoms are present in the excited state. If there are no excited state atoms present, there will be no downturn. But it will be independent of photons density because this is happening due to vacuum. It does not require a photon to be present in the main. Therefore, the first term in the down transition is insensitive to presence of ρ. ρ means the density of photon. But in absorption as well as in stimulated emission part, Most of thewhen atom attains an equilibrium under this up transition, down transition kind of thing, we do get this kind of detailed equilibrium condition that total up transition rate and total down transition rate should balance each other, this and that should balance each other. So, we have just computed one case where the down transition, up transitions are all mediated by a monopole operator for our case or dipole operator for electromagnetic field or electric field rather, for magnetic field it could have been a $\vec{S} \cdot \vec{B}$ kind of term. So, those kind of terms decide you what would be the spontaneous emission rate, what would be the stimulated emission rate, what would be the equilibrium condition corresponding to that and this equation should be satisfied for detail. So, this is by and large most of the atomic discussions in atom-light interaction people do about and try to see the transition probabilities of this with the interaction of light or background field which we had covered. In the next set of lectures, we will see artifacts of this transition. So, you see that there is a probability of transition caused by the electromagnetic or any background quantum fields. Now, we will see in the next lecture onwards what other quantum properties are present switched on due to this interaction. You see that even due to vacuum that there is

a probability of atom going to down state. For finite time operation there was a non-zero but small probability of going it up as well, but large time there is no up transition probability, it becomes a delta function of positive argument from being a *sin* function. So, therefore, on its own atoms are supposed to do this transitions due to the vacuum nature of the fluctuation nature of the quantum beam. That means if I take many, many, many atoms some of them will definitely undergo one of these transitions. So, as a collection what is the ensemble property of a collection will be heavily dependent upon what kind of field is surrounding that. And therefore, overall how many atoms remain in one phase, how many atoms become a different phase, by phase I mean which state they are initially starting with and which state they ultimately end up with. So, in a collection what happens, what is the global collective properties, if there is an entanglement between atoms, what happens to that, if there are corresponding properties between the atoms which are initially set up, how do they behave in the light of the presence of the background field, that we will start to see in the next lecture on this. So, I stop over here and then we will move on to look at other properties which we can infer out.