

# Foundation of Quantum Theory: Relativistic Approach

## Excitation of atom through field interaction

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## Excitation through quantum fluctuations

Lecture- 37

So till now we have discussed about the interaction of atoms with background quantum fields and we have seen that as a result the atom makes transitions across its excited and ground states with certain probabilities and we can define a transition rate for this procedure to happen and we obtain the expressions like this where the mod square of the transition element the monopole coupling term  $\hat{m}$  get squeezed between excited and the ground state of the atom mod square and times this  $\sin$  functions at the argument  $\Delta E/\hbar + \omega_{k0}$  and  $\Delta E/\hbar - \omega_{k0}$  is the transition probability if the background field is present with a particle in it having a momenta  $k_0$ .

Till now,

$$R_{g \rightarrow e} \rightarrow \frac{|\langle e | \hat{m} | g \rangle|^2}{(2\pi)^3 \hbar^2 \omega_k} \left[ \frac{\sin^2 \left( \frac{\Delta E}{\hbar} + \omega_{k0} \right) \frac{T}{2}}{2 \left( \frac{\Delta E}{\hbar} + \omega_{k0} \right)} + \frac{\sin^2 \left( \frac{\Delta E}{\hbar} - \omega_{k0} \right) \frac{T}{2}}{2 \left( \frac{\Delta E}{\hbar} - \omega_{k0} \right)} \right]$$

In large time limit

$$R_{g \rightarrow e} \rightarrow \frac{|\langle e | \hat{m} | g \rangle|^2}{(2\pi)^3 \hbar^2 (2\omega_{k_0})} \delta\left(\frac{\Delta E}{\hbar} - \omega_{k_0}\right)$$

If suppose more than one excited (degenerate) states are there [for instance  $2p_x, 2p_y, 2p_z$  of H]

$$R_{g \rightarrow e} \rightarrow \sum_e \frac{|\langle e | \hat{m} | g \rangle|^2}{(2\pi)^3 \hbar^2 (2\omega_{k_0})} \delta\left(\frac{\Delta E}{\hbar} - \omega_{k_0}\right)$$

If the initial state of the field was also a spread in  $\vec{k}$

$$|\psi_{in}\rangle = \int d^3\vec{k} f(\vec{k}) |\vec{k}\rangle$$

$$\begin{aligned} \text{then } & \langle \psi_{in} | \hat{\phi}(t) \hat{\phi}(t') | \psi_{in} \rangle \\ & = \int \frac{d^3\vec{k}}{2\omega_{\vec{k}}} e^{-i\omega_{\vec{k}}(t-t')} \int d^3\vec{k}' |f(\vec{k}')|^2 \\ & + \left( \int \frac{d^3\vec{k}}{\sqrt{2\omega_{\vec{k}}}} f_{\vec{k}} e^{i\vec{k}\cdot\vec{x}} \int \frac{d^3\vec{k}'}{\sqrt{2\omega_{\vec{k}'}}} f_{\vec{k}'} e^{-i\vec{k}'\cdot\vec{x}'} \right) + c.c. \end{aligned}$$

In large time limit

$$R_{|g\rangle \rightarrow |e\rangle} = \frac{|\langle e | m | g \rangle|^2}{(2\pi)^3 2\hbar^2 \omega_{k_0}} \left( \delta \frac{\frac{\Delta E}{\hbar} - \omega_{k_0}}{2} \right)$$

If suppose, more than one excited (degenerate) states are there

$$R_{|g\rangle \rightarrow |e\rangle} = \sum_e \frac{|\langle e | m | g \rangle|^2}{(2\pi)^3 2\hbar^2 \omega_{k_0}} \left( \delta \frac{\frac{\Delta E}{\hbar} - \omega_{k_0}}{2} \right)$$

If the initial state of the field was also a spread in

$$|\psi_{in}\rangle = \int d^3\vec{k} f(\vec{k}) |\vec{k}\rangle$$

$$\begin{aligned} \text{Then } & \langle \psi_{in} | \hat{\phi}(t) \hat{\phi}(t') | \psi_{in} \rangle = \int \frac{d^3\vec{k}}{2\omega_{\vec{k}}} e^{-i\omega_{\vec{k}}(t-t')} \int d^3\vec{k}' |f(\vec{k}')|^2 \\ & + \left( \int \frac{d^3\vec{k}}{\sqrt{2\omega_{\vec{k}}}} f_{\vec{k}} e^{-i\vec{k}\cdot\vec{x}} + \int \frac{d^3\vec{k}}{\sqrt{2\omega_{\vec{k}}}} f_{\vec{k}} e^{-i\vec{k}\cdot\vec{x}'} \right) + c.c. \end{aligned}$$

So, this  $\omega_{k_0}$  which we are seeing over here is the frequency of the photon available in the field. Now in infinite time limit by that we mean the time scale larger than the internal time scales but smaller than the second order corrections to kick in. This these expression *sin* functions approach the delta functions behavior such that the only second delta function which will come from the *sin* function will survive otherwise it would become 0. So, if the frequency of the photon matches with the frequency gap of the atom, then only a transition is possible. Otherwise, the atoms will not get excited. So, probability of ground to excited transition or a rate of ground to excited transition would be 0 if in infinite time limit, if  $\omega_{k_0}$  is not commensurate with  $\Delta E/\hbar$ . That we have seen in the previous discussion sessions as well. So, this is the expression we obtained for large time limit. By infinite time, I again repeat that we mean really large time scale, larger than the internal time scales, but smaller than the time scale where perturbation theory breaks off. Now, there is another interesting set of events which happens if there is additional set of states at the excited level. Let us say the degenerate states. Suppose there is a one ground state, but excited states are multifold. They all come with the same energy. and they have degenerate structure. So then we need to ask the question of transition probability across each of them. Suppose I want to ask what is the transition probability across ground state to excited state number 1. Then the similar kind of expression like this will come about where the transition element, the matrix element of  $\hat{m}$  squeeze between the ground and the excited state of our choice  $E_1$  let us say. That mod square will decide the rate of transition. Similarly for second excited state Second excited state in the sense of same energy, but out of the many excited states, the second one, the transition rate will be determined by the M<sup>2</sup>d between  $E_2$  and the g mod square and so on. For a casing point, we can think of a hydrogen atom where if we are thinking about an unperturbed hydrogen atom, then  $2p_x, 2p_y, 2p_z$  of hydrogen atom, all of them come with the same energy eigenvalue. However, depending upon what kind of operator we are putting in, some operators can survive between the ground state and  $2p_x$  or ground state and  $2p_z$ , but may not survive between ground state and  $2p_y$ . So in principle, these kind of things has to be computed for all the excited states. And then when I sum over all the possible excited states, what do I get as an expression would be the total probability of migrating to any of the excited states And therefore, this rate which we have written over here is telling us about as the rate of possible transition across any of the excited states on the ground. Again, the matrix element here would control the transition probability, but a meaningful or a necessary condition would be forced by the delta functions in large time limits. The photon frequency should match. The photon means the particles present in the background field. Its frequency should match the energy gap. Otherwise, no transition will take place.

Okay. Now, this is for the case where the photon in the field has one definite frequency. That means the state I am talking about of the background field is  $I_{k_0}$ . Now suppose the case where I do not know the state in the field force with surety. That means when I am writing this  $I_{k_0}$ , I definitely know that there is only one particle and that one particle is necessarily present with the frequency  $k_0$ . No other frequency is allowed. However, that is not the case physically we expect. Always we will have some spread of probability of having this frequency or that frequency. So in principle, I would only know that the photon or the field has a particle which is whose frequency is most likely to be  $k_0$ , but it could have some spread about  $k_0$ . That typically is captured by a spreading function  $f(k)$ . So, I write a well-defined state as  $\psi$  which will have an integration  $d^3k$ , your uncertainty of knowing which frequency it definitely has. So  $f(k)$  will tell you the spread of frequency and then what are the possible kits associated with that. So this is a superposition over all single particle excitation with different frequency. One of the frequency might be tightly supported, most of the frequency will not be so strongly supported. But this is a well-defined meaningful state. We have seen through previous examples and assignments that this state  $I_{k_0}$  is not a normalizable state. Its normalization is a  $\delta(0)$ , that means it is a non-normalizable state. However, this kind of a state as you might have worked out in the assignment, you know that the

normalization of this state is controlled by the integral property of  $|f(k)|^2$ . So,  $d^3k|f(k)|^2$  has to be smaller than infinity for normalizability. So, this is a physically meaningful state and in this if I want to compute what is the state What is the probability of transition and what is the rate of transition? I need to compute the two-point function of the field. Remember when we were doing this transition probability business, what we realized? There is some kind of Fourier like looking integral transforms of the two-point functions comes about. In  $1_{k0}$  when I had put in, this is how the delta functions of these two kinds or the *sin* functions came about. Now that we have replaced the in state or the initial state of the field with this expression over here, we should recompute what is the two-point function of the field in this new state. This is a simple easy exercise for you. So, I am just writing it out. You should be able to prove it that the two-point function of this state is, there is a one piece which is exactly like the vacuum two-point function times the normalization of the state. This is the state whose normalization is controlled by the mod square integral of this  $f(k)$ . So this normalization comes here. If the state is fully normalized, that means this integration  $e^{-ikx} / \sqrt{2\omega_k} \cdot |f(k)|^2$ . Is 1, the integral piece attaching here will become 1. Then it becomes just the two-point function of the vacuum for stationary particles. Remember there was a  $ik(x - x')$  thing which I had put to 0 because I was looking for an atom at rest whose  $x$  and  $x'$  at different times will be the same. In addition to this vacuum piece now you will have two more pieces which are of the complex conjugate in nature. First one will be the  $f(k)$ . modulated with the mode function That and similar thing for  $f^*(k)$ ,  $f^*(k')e^{-ikx'}$ . You can see these two are almost complex conjugate of each other, this and that. However, one thing is computed with  $x'$  at location  $x'$  and one thing is computed at location  $x$ . So, they are not exactly complex conjugate of each other but they are complex conjugate and time apart or position apart. So one piece is this and another piece the complex conjugate of that will come about that will totally determine what is the two points function of the field in this state. This is a homework exercise for you which will be a part of assignment as well that you should be able to work out. So now in this state, we compute the probability as before. You have to do this integral transform for the two-point function. We have rewritten the two-point function for the state like the vacuum piece. Assuming the state is fully normalized, that means the integral which we were talking about, this  $d^3k|f(k)|^2$  integral is 1.

In that case when we compute the probability or transition rate

$$P_{g \rightarrow c} = \frac{|\langle c | \hat{m} | g \rangle|^2}{(2\pi)^3 \cdot \hbar^2} \int dt dt' e^{-i\frac{\Delta E}{\hbar}(t-t')} \times \left[ \int \frac{d^3k}{2\omega_k} e^{-i\omega_k(t-t')} + \int \frac{d^3k}{\sqrt{2\omega_k}} f_k e^{-i\omega_k t} \int \frac{d^3k'}{\sqrt{2\omega_{k'}}} f_{k'}^* e^{+i\omega_{k'} t'} + \int \frac{d^3k}{\sqrt{2\omega_k}} f_k^* e^{+i\omega_k t} \int \frac{d^3k'}{\sqrt{2\omega_{k'}}} f_{k'} e^{-i\omega_{k'} t'} \right]$$

Under long time integration

$$P_{g \rightarrow c} = \frac{|\langle c | \hat{m} | g \rangle|^2}{(2\pi)^3 \hbar^2} \left[ \int \frac{d^3k}{2\omega_k} \pi^2 \delta\left(\frac{\Delta E}{\hbar} + \omega_k\right)^2 + \int \frac{d^3k}{\sqrt{2\omega_k}} f_k \pi \delta\left(\frac{\Delta E}{\hbar} + \omega_k\right) \int \frac{d^3k'}{\sqrt{2\omega_{k'}}} f_{k'}^* \pi \delta\left(\frac{\Delta E}{\hbar} + \omega_{k'}\right) + \int \frac{d^3k}{\sqrt{2\omega_k}} f_k^* \pi \delta\left(\frac{\Delta E}{\hbar} - \omega_k\right) \int \frac{d^3k'}{\sqrt{2\omega_{k'}}} f_{k'} \pi \delta\left(\frac{\Delta E}{\hbar} - \omega_{k'}\right) \right]$$

$$= \frac{|\langle c | \hat{m} | g \rangle|^2}{(2\pi)^3 2\hbar^2} \frac{\pi^2}{2\omega_{k_0}} |f(\omega_{k_0})|^2 \quad \text{where } \omega_{k_0} = \frac{\Delta E}{\hbar}$$

In that case when we compute the probability of transition

$$R_{|g\rangle \rightarrow |e\rangle} = \frac{|\langle e|m|g\rangle|^2}{(2\pi)^3 2\hbar^2 \omega_{k_0}} \int dt dt' e^{\frac{-i\Delta E(t-t')}{\hbar}} \times$$

$$\left[ \int \frac{d^3 \vec{k}}{2\omega_{\vec{k}}} e^{-i\omega_{\vec{k}}(t-t')} + \int \frac{d^3 \vec{k}}{\sqrt{2\omega_{\vec{k}}}} f_k e^{-i\omega_{\vec{k}}(t)} \int \frac{d^3 \vec{k}'}{\sqrt{2\omega_{\vec{k}'}}} f_{k'}^* e^{i\omega_{\vec{k}'}(t')} + \int \frac{d^3 \vec{k}}{\sqrt{2\omega_{\vec{k}}}} f_k^* e^{i\omega_{\vec{k}}(t)} \int \frac{d^3 \vec{k}'}{\sqrt{2\omega_{\vec{k}'}}} f_{k'} e^{-i\omega_{\vec{k}'}(t')} \right]$$

Under long time integration

$$R_{|g\rangle \rightarrow |e\rangle} = \frac{|\langle e|m|g\rangle|^2}{(2\pi)^3 2\hbar^2} \left[ \int d^3 \vec{k} \frac{4\pi^2}{\omega_{k_0} 2\omega_{\vec{k}}} \delta\left(\frac{\Delta E}{\hbar} + \omega_{\vec{k}}\right)^2 \right.$$

$$+ \int \frac{d^3 k}{\sqrt{2\omega_k}} 2 f_k \pi \delta\left(\frac{\Delta E}{\hbar} + \omega_k\right) \int \frac{d^3 k'}{\sqrt{2\omega_{k'}}} 2 f_{k'}^* \pi \delta\left(\frac{\Delta E}{\hbar} + \omega_{k'}\right)$$

$$\left. + + \int \frac{d^3 k}{\sqrt{2\omega_k}} 2 f_k^* \pi \delta\left(\frac{\Delta E}{\hbar} - \omega_k\right) \int \frac{d^3 k'}{\sqrt{2\omega_{k'}}} 2 f_{k'} \pi \delta\left(\frac{\Delta E}{\hbar} - \omega_{k'}\right) \right]$$

$$\frac{|\langle e|m|g\rangle|^2}{(2\pi)^3 2\hbar^2} 4\pi^2 \frac{|f(\omega_{k_0})|^2}{2\omega_{k_0}} \quad \text{where } \omega_{k_0} = \frac{\Delta E}{\hbar}$$

So under that assumption, first piece becomes just a vacuum piece, vacuum two point functions + state dependent corrections. These two terms are complex conjugate of each other out of which the product is almost complex conjugate as we discussed only for the fact that they are at two different times. Otherwise they are exactly complex conjugates of each other. But the extra term which is obtained is exactly the complex conjugate of the whole thing. All right.

So this is the vacuum piece. All the information that it is not the vacuum state but some excited state with a supercondition of frequency comes through these correction terms with  $f(k)$  dependencies. Recall the  $f(k)$  tells you how much spread in the frequency of the excited photon is. If there is no photon  $f(k)$  is 0 let us say in that case here only the vacuum piece will survive. The state will be in vacuum state.

Okay. So so far so good. Let us do the computation once more for this normalized state. Again the first term as we have discussed so many times by now will give me a  $\delta^2$  of the delta function with positive argument. That is how in vacuum there was no probability of going to excited state. So the term over here after this integral transforms that we have done so many times by now so that you should know how it becomes or actually it should become twice of  $\pi^2$  times, actually not twice,  $4\pi^2$  which I have written. So, let me erase it out. So, there is a  $4\pi^2$  which I had written but could not see. So, this is correct expression. So, the first term comes from the integration of the vacuum piece with the time integrals over here. You will get that term and this identically vanishes because the argument of a delta function is a positive quantity. So, it will not survive giving you 0 transition probability for vacuum piece. However, now there are two more terms complex conjugates of each other which will give me two kind of integrals which are over here. The operative piece if you look for atom at rest, here only  $e^{-i\omega t}$  comes about, here  $e^{+i\omega k' t'}$  appears and the complex conjugate of these quantities appear over here. So

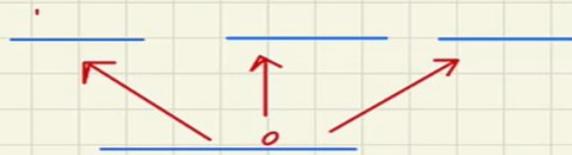
when I go to do this time integral again like before you can see you can club out all the exponentials of time at one place. This integral over here gives you this delta function, positive argument again and the integral over here with  $t'$  combines with the integral over here in  $t'$  again gives you a delta function with yet again a positive quantity in argument. So, again the first term, the first correction term of the non-vacuumness, the excited state information comes in two pieces as we have seen.

The first piece also gives you terms which are proportional to delta function with positive arguments. That means they are also going to go out, they are not going to contribute anything. However, the last correction term, which is complex conjugate of this term over here, that survives. You can see that if I look at the  $t$  integral, the  $t$  integral has  $e^{-i\Delta E/\hbar t}$  over here and  $e^{i\omega_k t}$ . So, there are sign difference between that. That is why when I do the long time integration, I will get a delta function with integration Potentially vanishing argument  $\Delta E/\hbar - \omega_k$ . Remember what was  $\omega_k$ ?  $\omega_k$  was a running frequency. We had said that I do not know what is the state. It could have any frequency with some distribution  $f(k)$ . So therefore  $\omega_k$  becomes frequency of a photon. All the photons of different different frequency are present determined by their strength from the function's value at that frequency  $f(k)$ . So  $\omega_k$  is a running frequency as the  $k$  changes  $\omega_k$  changes. So there is a possibility that one of the photons would have the same energy as  $\Delta E/\hbar$  and that photons contribution  $f(k)$  decides how much would be the transition probability. So you see the third term over here has exponential with proper signs so that the integrations over time gives you a delta function which can survive this time. It will survive only if There is a non-zero value of the function  $f$  at the frequency of resonance. So you do this integral again like before. This is a simple enough integral we have seen so many times. You will be able to show. This time I am going to get an answer. Again the mod square quantity of the monopole operator squeezed between excited and the ground states. Everything like before. Make use of the two delta functions in computing the  $d^3k$  integrals. Everything like before. Make use of the two delta functions in computing the  $d^3k$  integrals. And from there, we would be able to show that, you will be able to show from here that this is, this delta function takes care of the first  $d^3k$  integral and this delta function takes care of the second  $d^3k$  integral. So, here there is a  $2\pi$  from here, there is another  $2\pi$  which is coming from here, which will go and fight with the  $2\pi$  over here. The  $d^3k$  integral is also  $4\pi k^2 dk$  if you recall. So, there will be another  $4\pi^2$  from these two integrals. One integral will give me  $4\pi k^2 dk$  and this integral will also give me  $4\pi k^2 dk$ . So, there will be  $4\pi^2$  from here and there will be  $4\pi^2$  from here as well because the  $d^3k$  integral and then I have a  $k^2 dk$  integrals. If you look pay close attention to that I have a  $k^2 dk$  and this delta function therefore will give me  $\Delta E/\hbar^2$  factor extra from the result which I had written and divided by the twice  $\omega_{k0}$  which is the correct quantity. So, there is an extra piece of  $\Delta E/\hbar$  whole square which will come from writing this  $d^3k$  integral which is  $4\pi k^2 dk$  and then there is a  $dk$  integral is same as the  $d\omega_k$  integral. And therefore, the result what I have written should be corrected for another  $\Delta E/\hbar$  whole square term. So, now you see the correct answer is of transition for this time is determined by all the transition matrix  $M$ 's squeezing value. But crucial quantity is this  $f|\omega_{k0}|^2$ . So it asks the state of your choice. This is the state of your choice. What is the value of  $f(k)$  at the transition frequency gap?  $f(\omega_{k0})$  being the same as  $\Delta E/\hbar$ . So it asks what is the value of the function? It is mod square comes about. And that determines what is the transition probability. So, again let me write down in this color there is a let us call it  $\omega$  not<sup>2</sup> term as well,  $\omega_k$  not<sup>2</sup> term as well. As we discussed that is coming from  $4\pi k^2 d k$  integral, there was a  $4\pi^2$  quantity here as well. So, actually it would be  $4\pi\omega_0$  4 because this will give me  $k_0^2$  and this will also give me  $k_0^2$ . So, forget about this extra frequency factor that is not the crucial quantity we want to discuss over here. The crucial quantity is the information of the states, the information of the states contribution at the resonance frequency. If  $f$  of  $\omega_k$ ,  $f(k)$  has a value 0, that means there is no photon which is excited with this much of frequency gap, then the transition probability becomes 0. So, it necessarily asks for the probability of finding a photon at appropriate frequency. Remember, if you compute the probability of finding a photon at different-different frequency,  $|f(k)|^2$

tells me that. So, ultimately the transition gap that the transition probability looks for a presence of photon of appropriate frequency and mod square of that, that means the probability of that photons occurrence determines the probability of transition as well. So, that is natural to expect and that is why the computations also suggest it, all right. And as we discussed if there are more than one such excited state we should sum over all such possible excited state as well. So, again here also there is a  $4\pi\omega$  not<sup>2</sup> whole square kind of correction term should be there. But anyway do not worry about this factor too much we can absorb that  $4\pi\omega$  not<sup>2</sup> in the redefinition of function  $f$  as well. But the crucial point is this everything now depends on the transition gap transition gap frequencies photon.

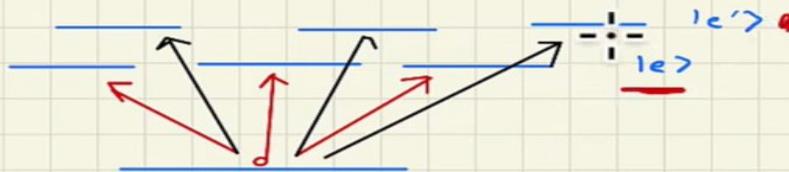
Suppose we have degeneracy of excited states, then

$$P_{g \rightarrow \{e\}} = \sum_e \frac{|\langle e | \hat{m} | g \rangle|^2}{(2\pi)^3 \hbar^2} \frac{(2\pi)^2}{2\omega_{e_0}} |f(\omega_{e_0})|^2 (4\pi\omega_0^2)^2$$



For any of excited states  $E_1$  or  $E_2$  apart  $\{ |e\rangle \}$  or  $\{ |e'\rangle \}$

$$P_{g \rightarrow \{e, e'\}} = \sum_e \frac{|\langle e | \hat{m} | g \rangle|^2}{(2\pi)^3 \hbar^2} \frac{(2\pi)^2}{2\omega_{e_0}} |f(\omega_{e_0})|^2 + \sum_{e'} \frac{|\langle e' | \hat{m} | g \rangle|^2}{(2\pi)^3 \hbar^2} \frac{4\pi^2}{2\omega_{e'_0}} |f(\omega_{e'_0})|^2$$



For many such excited set of states if  $|\langle e | \hat{m} | g \rangle|^2$  is same across all  $|e\rangle$ ,

$$P_{g \rightarrow \{e, e'\}} = \sum_{e_i} n(e_i) \frac{|\langle e_i | \hat{m} | g \rangle|^2}{(2\pi)^3 \hbar^2} \frac{4\pi^2}{2\omega_{e_i}} |f(\omega_{e_i})|^2$$

If the photon with the right frequency is present, the probability of transition will happen. And if there are more than one excited states present, the transition probability we are computing for all of them. So

$E_1, E_2, E_3$  and for all of them we should be able to compute the quantity which is over here upstairs and that should give me the probability of transition. Now suppose we are asking for not only degenerate excited state, but also excited state with slightly different frequency. Suppose there is some set of degenerate excited states  $E$ , three of them let us say, which have all the same energy. And there is another set of excited state  $E'$ , which are slightly different in energy compared to  $E$ , but they are also degenerate, three of them of that kind as well. Then if I ask for the probability of transition from ground state to any of the excited state, then the same computation would be carried out for all these excited state. Only thing that I will separate out the summation into two sets of excited state. Excited state with the energy  $E_1$ , let us say,  $\omega_{k0}$ . Energy gap is  $\omega_{k0}$ . okay and similarly here is  $\omega_{k0}^2$  whole square. So, the first set of terms talks about excited state with energy gap  $\omega_{k0}$  and the mod square of finding a photon of that probability counts while the second summation counts over the probability of finding a photon with the energy gap corresponding to this transition  $\omega_{k0}'$ . So, together all across different excited state if the photon of any of these frequencies are present one of the transitions will take place. And in case there are many, many, many such excited states suppose there is another set of excited state four of them  $E''$  and so on. Then the same computation will be done for that and summation over  $E''$  will also come out. So, ultimately if all the transition elements let us say this is same for all  $E_1, E_2, E_3$  even  $d^2, d^3$  if that is same across all excited states then what will only happen that this probability will become the same for all the same level energy eigen states. So, it would just count how many number of degeneracies are there at level  $e$  at level  $e'$  and so on and that is how we write the total transition probability would be summation over all the excited states times their degeneracies. So, this expression again with a correction factor of  $4\pi\omega_{k0}$  I this time I am writing because I is the summation index for the excited state as well. So, all this for all different times types of excited state the degeneracy of that kind that excited state kind comes into the game and this is the transition probability across one of such excited states. And this is how the summation over degeneracy times all the possible excited states gives me the total probability of transitions across any of the excited states. So, this is as straight forward as we can see and that is how we can get the probability as well as the rate of transitions as well. If we carry on the game for the continuous case in the same fashion, then the summation over here over the discrete set of excited states converts itself into integration and  $N_e$  over here which was counting the degeneracy at each discrete excited state becomes the degeneracy density function  $g$ . And then the usual integrals up to a  $4\pi\omega_k^2$  whole square multiplication. So this is how the continuum case is also dealt with. If I have a set of a continuously distributed excited state like this and I have a ground state, the probability of transition across any of these from the ground state will be computed like this. One crucial assumption is that for each excited state with energy  $E$ , the excited state with energy  $E$ , the matrix elements of transition is the same. That is why the  $g(E)$  function has been multiplied. If different excited states meaning at energy  $E$ , let us say there are more than one excited states and then there is a degeneracy like that. Then across all of the excited states the transition matrix element is the same. Then only this number multiplication or degeneracy multiplication works out. Now, just for notational convenience, I am going to call the ground state of the atom as  $0A$ .  $0A$  is the ground state of the atom and excited state is  $E$  such that I do not have to write  $\Delta E$  all the time. I will just write  $E$ ,  $E$  means energy gap between the ground and excited state. I am redefining the base value as the ground state value of the atom, ground state energy value of the atom. This is just for notational convenience. Now all this  $4\pi\omega_k^2$  whole square thing was coming from this  $d^3 k$  integral which I had not completely written correctly, which was coming from this  $d^3 k$ ,  $4\pi k^2$  from this and  $4\pi k^2$  from this. So if I go one step back, that means I do not do this  $d^3$  integral and look for only finite time integral. These three terms which were surviving here, this was giving me a delta function. This was giving another set of delta function with positive argument. Only the third term was giving me the proper delta function with non-positive argument. These things for finite time integration convert themselves into *sin* integrations that we have seen in the previous class that you know by now as well. These integrals are just the *sin* functions.

So, let us focus on the last term only for these two terms for time, for this time.

In the continuum case  $\left\{ \begin{array}{l} \text{Setting } |B\rangle = |0\rangle_a \\ \{|B\rangle\} = |E\rangle \end{array} \right.$

$$P_{|B\rangle \rightarrow |E\rangle} = \int dE \mathcal{G}(E) \frac{|\langle E | \hat{m} | 0 \rangle|^2}{(2\pi)^3 \hbar^2} 4\pi^2 \frac{|f(E)|^2}{2E} \quad (4\pi\omega)$$

★ For finite time

$$P_{|B\rangle \rightarrow |E\rangle} = \int dE \mathcal{G}(E) \frac{|\langle E | \hat{m} | 0 \rangle|^2}{(2\pi)^3 \hbar^2} \left| \int \frac{d^3k}{\sqrt{2\omega_k}} f(k) \frac{\sin\left(\frac{E - \omega_k}{\hbar} \frac{t}{2}\right)}{\left(\frac{E}{\hbar} - \omega_k\right)} \right|^2$$

$$= \int \frac{d^3k}{\sqrt{2\omega_k}} f(k) \int \frac{d^3k'}{\sqrt{2\omega_{k'}}} f(k') \int dE \mathcal{G}(E) \frac{|\langle E | \hat{m} | 0 \rangle|^2}{(2\pi)^3 \hbar^2} \frac{\sin\left(\frac{E - \omega_k}{\hbar} \frac{t}{2}\right)}{\left(\frac{E}{\hbar} - \omega_k\right)} \frac{\sin\left(\frac{E - \omega_{k'}}{\hbar} \frac{t}{2}\right)}{\left(\frac{E}{\hbar} - \omega_{k'}\right)}$$

If  $f(k) = \sqrt{2\omega_k} \delta(k - k_0)$

$$P_{|B\rangle \rightarrow |E\rangle} = \int dE \mathcal{G}(E) \frac{|\langle E | \hat{m} | 0 \rangle|^2}{(2\pi)^3 \hbar^2} \frac{\sin^2\left(\frac{E - \omega_{k_0}}{\hbar} \frac{t}{2}\right)}{\left(\frac{E}{\hbar} - \omega_{k_0}\right)^2}$$

Defining  $\left(\frac{E}{\hbar} - \omega_{k_0}\right) \frac{t}{2} = \beta$

$$\frac{2\hbar}{t} d\beta = dE$$

$$P_{|B\rangle \rightarrow |E\rangle} = \frac{2\hbar}{t} \int d\beta \tilde{\mathcal{G}}(\beta) \frac{|\langle E | \hat{m} | 0 \rangle|^2}{(2\pi)^3 \hbar^2} \frac{t^2 \sin^2 \beta}{4 \beta^2}$$

$$= \frac{t}{4\hbar} \int d\beta \tilde{\mathcal{G}}(\beta) \frac{|\langle E | \hat{m} | 0 \rangle|^2}{(2\pi)^3} \frac{\sin^2 \beta}{\beta^2}$$

$$R_{|B\rangle \rightarrow |E\rangle} = \frac{1}{(2\pi)^3 4\hbar} \int d\beta \tilde{\mathcal{G}}(\beta) |\langle E | \hat{m} | 0 \rangle|^2 \frac{\sin^2 \beta}{\beta^2}$$

In the continuum case { setting  $|g\rangle=|0\rangle_a$   
 $\{|g\rangle\}|E\rangle$

$$\int dE g(E) \frac{\langle E|\hat{m}|0\rangle_n^2}{(2\pi)^3 \hbar^2} 4\pi^2 |f(E)|^2 4\pi\omega$$

★ For finite time

$$P_{|g\rangle \rightarrow |e\rangle} = \int dE g(E) \frac{\langle E|\hat{m}|0\rangle_n^2}{(2\pi)^3 \hbar^2} 4\pi^2 \left| \int \frac{d^3 k}{\sqrt{2\omega_k}} f(k) \frac{\sin\left(\frac{E-\omega_k}{\hbar} \frac{t}{2}\right)}{\left(\frac{E}{\hbar} - \omega_k\right)} \right|^2$$

$$= \frac{\int d^3 k}{\sqrt{2\omega_k}} f(k) \int \frac{d^3 \vec{k}'}{\sqrt{2\omega_{\vec{k}'}}} f^*(k') dE g(E) \frac{\langle E|\hat{m}|0\rangle_n^2}{(2\pi)^3 \hbar^2} 4\pi^2 \frac{\sin\left(\frac{E-\omega_k}{\hbar} \frac{t}{2}\right)}{\left(\frac{E}{\hbar} - \omega_k\right)} \frac{\sin\left(\frac{E-\omega_{k'}}{\hbar} \frac{t}{2}\right)}{\left(\frac{E}{\hbar} - \omega_{k'}\right)}$$

If  $f(k) = \sqrt{2\omega_k} \delta(k - k_0)$

$$P_{|g\rangle \rightarrow |e\rangle} = \frac{2\hbar}{t} \int d\beta \tilde{g}(\beta) \frac{\langle E|\hat{m}|0\rangle_n^2}{(2\pi)^3 \hbar^2} \frac{t^2 \sin^2 \beta}{4 \beta^2}$$

$$= \frac{t}{4} \hbar \int d\beta \tilde{g}(\beta) \frac{\langle E|\hat{m}|0\rangle_n^2 \sin^2 \beta}{(2\pi)^3 \beta^2}$$

$$P_{|g\rangle \rightarrow |e\rangle} = \frac{1}{(2\pi)^3 4\hbar} \int d\beta \tilde{g}(\beta) \langle E|\hat{m}|0\rangle_n^2 \frac{\sin^2 \beta}{\beta^2}$$

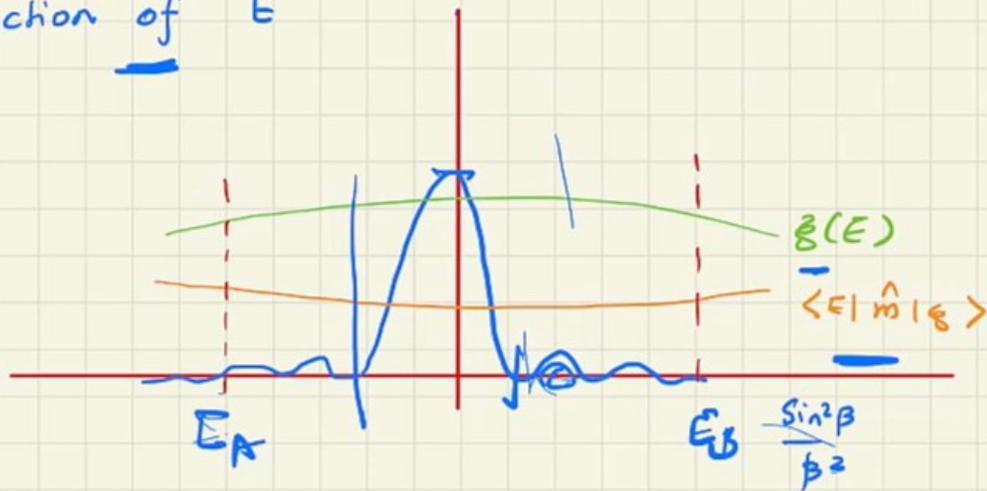
The first two terms anyway in large time limit they become suppressed functions of a positive delta argument.

So, therefore I am not going to pay much attention to them even for finite time despite they might have non-zero value, but that would be subdominant corresponding to the last term. So, let us first focus on the last term for finite time duration and let us see what is the structure we get to see as a development. So, in the finite time integral the last term will convert itself into sin integral, the  $|\sin|^2$  integral and I will have a sign  $\Delta E/\hbar - \omega_k$  only thing is  $\Delta E$  is being written as  $E$  this time because of my redefinition of energy. So, otherwise I will have this integral as coming from the last term of the transition program. Again to remind you there are for finite time the first vacuum term and even the first correction term is also non-zero, but for the time being I am just looking at the third term which is anyway much larger than the two corresponding terms. So, if I rewrite this mod square over here, this is the integral whose mod square is being computed. So, let me write down the integral and its complex conjugate.

So, this is just the rewriting of the same expression. I have just split apart the mod square term. mod square is, mod square is  $zz^*$ . So, I have written the first integral and its complex conjugate integral separately. Again be careful that there is always a  $4\pi\omega E^2$  thing which was initially missing this time they have gone into  $d^3k$ . So, this is not a wrong statement this was missed only for

large time duration when I made use for the delta function. So, this expression is fine despite the previous expression missing out on this  $s a 4\pi\omega E^2 t$ . If I take this  $f(k)$  over here and actually let me make correction over here, let me write down divided by  $4\pi\omega_k^2$ . So, if I take this  $f(k)$ , then you see the integral over here will just become  $dk\delta(k - k_0)$ . This  $\omega_k^2$  will exactly cancel the  $4\pi k^2$  thing coming in the  $d^3k$  integral and this root  $2\omega$  will exactly cancel the denominator root  $2\omega E$  into So what will happen if I make  $f(k)$  if I choose to be of this kind? Then two things happen. First, my state will become a single frequency state. That means the state which we were discussing initially over here will just become a  $k_0$  state up to a multiplication of  $E$ , probably by  $2\omega k_0$  something. But anyway, it would be just a single frequency state. And as a result, the integral which we are looking at at the moment over here will just become an integration over delta of  $k - k_0$ . That means it will convert all the  $\omega_k$  appearing in the expression as  $\omega_{k_0}$ . And the second integral will convert all the  $\omega_k$ 's appearing in the expression into  $\omega_{k_0}$ . And the second integral will convert all the  $\omega_k$ 's appearing in the expression into  $\omega_{k_0}$  where all the running integrals  $d^3k$  and  $d^3k$  times have been performed to the use of delta function and I have got this expression. This is simple enough expression to deal with. You can just do a variable transformation. I can define the argument of the sign which is appearing over here as  $\beta$ . Whatever appears as the argument of sign I am going to call it as  $\beta$ . and then the simple variable transformation game the  $dE$  will be converted into  $2\hbar d\beta T$ . I am keeping the  $\omega_{k_0}$  as I am treating the  $\omega_{k_0}$  as a constant because this is the single frequency excitation there is no uncertainty now So, once you do this variable transformation it is very straightforward to show you show that this the resulting expression is this. where  $de$  has been replaced by  $2\beta \cdot 2\hbar/T$  factor. The  $g(E)$  becomes some other function of  $\beta$ . I am just writing the mod square term as  $e$ , but ultimately  $e$  is a function of a  $\beta$  as well from this expression. And the sinc function which was appearing over here becomes  $\sin^2\beta/\beta^2$  with an extra  $t^2$  by 4 because there was not by 2 in the denominator. So, when I write in terms of this, this will become  $p/\hbar - \omega t_0$  will just be  $\beta/2$  and whole square of that will throw me  $\pi^2/4$ . So, you can see there is a tradeoff happening between the  $t$ . So, there is a  $t$  in the denominator here and a  $t^2$  in the numerator over here which leaves me a  $t$  in the numerator. So, you can see there is a tradeoff happening between the  $t$ . So, there is a  $t$  in the denominator here and a  $t^2$  in the numerator over here which leaves me a  $t$  in the numerator. So you can see there is a trade-off and as a result what I get is the probability of transition becomes linear in time This is one of the observations Fermi had which leads to something called a Fermi's golden rule as we will see as we go along. That the probability for this specialized case where I have a single frequency and I have this kind of conditions then I would get for finite time a linear rise, a linear rise coming into the game. The remaining things are integrals are independent of  $T$ ,  $\beta$  is just a running index. I can compute the  $\beta$  for the true range of the spread of the eigen states or in principle many of the times it is extended up to infinity as we will see. But anyway the crucial factor is this, after the variable transformation all the  $t$  dependency has come out. All the HOA dependency has come out and therefore I have a rate which is, I have a probability of transition which is linearly proportional to time with some integral value which will give me some number but at the end of that day linear rise in time and therefore the rate will be just derivative of this or division by  $t$  which both of them are same spontaneous rate or average rate is the same thing this becomes this nice expression the integral of  $d\beta$  degeneracy frequency degeneracy functions degeneracy functions version in  $\beta$  parameter and this transition element one has to write in terms of  $\beta$  as well and  $\sin^2\beta/\beta^2$  so this integral one has to perform otherwise this whole thing  $1/(2\pi)^3 4\hbar$  times this integral is the rate, the rate of transition for finite time coming from the third term. whole thing  $1/(2\pi)^3 4\hbar$  times this integral is the rate, the rate of transition for finite time coming from the third term. You can verify for yourself those are not highly contributing things. So, dominant rate of transition is this. So, now let us go ahead and compute this quantity as accurately as we can. That would give me the exact expression for the rate of transition. However, there is a cute trick one can play off, which is from the realization that the degeneracy of states  $g(E)$ , the function which is appearing over here,  $\tilde{g}(\beta)$  is just the re-expression of  $g(E)$  in terms of  $\beta$ .

If  $\delta(E)$  and  $\langle E | \hat{m} | \xi \rangle$  are slowly varying function of  $E$



$$R_{|\xi\rangle \rightarrow |E\rangle} = \frac{\delta(E_\xi) |\langle E_\xi | \hat{m} | 0 \rangle|^2}{(2\pi)^3 4\hbar} \int_{-\infty}^{\infty} d\beta \frac{\sin^2 \beta}{\beta^2}$$

$$= \frac{\delta(E_\xi) |\langle E_\xi | \hat{m} | 0 \rangle|^2}{(2\pi)^3 4\hbar}$$

And this thing which is appearing over here  $\langle E | \hat{m} | 0 \rangle$ , it is squeezed between, which is excited and grounded, the grounded state of atom. These things are typically very slowly varying functions of  $E$ . However, the function which is appearing over here  $\frac{\sin^2 \beta}{\beta^2}$  sinc function that is highly oscillatory

function of  $\beta$  or  $E$ . So, if I plot these things, the blue one over here is a sinc function  $\beta \sin^2\beta / \beta^2$ . That will have a very high maxima at  $\beta$  is equal to 0, this is value 1 then it will be quickly dampen out. then it will be quickly dampen out. then it will be quickly dampen out. You do not get to see much of oscillations. This quickly dies down. But both of these typically,  $g(E)$  remember is  $4\pi k^2 dk$  in free space and those kind of things, the degeneracy factor. or in hydrogen atom it will become not very rapidly rising function across very tiny energy gaps or in tiny energy setting that will change marginally. So you can see both emg and  $g(E)$  both these quantities do change in energy but very slowly compared to this sinc function. So what you can see here when we were doing this integral. Initially the integral was over the energy  $T_e$  from where to where this was from let us say  $-a$  to  $+b$  which is the energy gap of the excited states. So, figuratively let us say this was the ground state and I was talking about energy let us say  $E_a$  up to  $E_b$ . From here to here the energy integration was done not  $-a$  let us call it. Correspondingly the  $\beta$  integral will run for some finite range as well. So, let me draw the  $E_a$  over here and  $E_b$  over here. So, initially I wanted to compute the integral between  $E_a$  to  $E_b$  and in this there were three energy dependent function one was  $g(E)$  other was this squeezing of the matrix element and then squeezing of the monopole operator. and then the sinc function. Now, either you can go ahead and do this exact computation by knowing the exact form of  $g$  and doing the exact form of the matrix element and do this integral or what can be done just to get a quick ballpark value is to approximate this integral with something. So, what I see the sinc function quickly dies down to nothing that means most of the integrals contribution is coming from the first peaks region only. In the second peak there is some contribution but that is highly small compared to the first peak's contribution. So what I do, I just do not use this integral but I can use the corrected approximation. I am going to think that this emg, this orange kind of thing over here is almost a constant and similarly degeneracy of state is also almost a constant. And then I leave the same function as it is. Of course, I am doing an error in this. However, most of the error is not coming in this region. Here I am treating it as exact constants here, which is roughly the case. Here on here, they do not change much. Much of change will come from here up to here. If I had computed the integral fully, then the value of  $g(E)$ , let us say here, and  $g$  of energy here will be much different. So my approximation will become bad across large energy separation, but sinc function comes to save me. Wherever the approximation becomes bad, the sinc function has gone down to almost nothing. That means the weightage of the error portions are very small. So, I can safely approximate this correct integral with this. They are not very apart. Most of the difference between this exact thing and this approximate thing lies in this regime, which where anyway the sinc function will not allow you to get any meaningful number of. So, therefore, what I can do, I can take  $g(E)$  and this matrix element as constants of  $e$  and pull out from the integral. So, I had the thing which is inside. I am saying that they are not changing much.

So, I can define a central value of them. I can take a base central value  $E^*$ . I can compute  $g$  at this value and say that at remaining values of  $E$ , running value, it is not much different from  $g(E^*)$  of. Similarly, the matrix element I can compute at  $E^*$  and I can claim that the matrix elements value for the remaining integral is not much different from the matrix value at  $E^*$ . So, I replace the  $g$  and the matrix values with their  $E^*$  value, where  $E^*$  is one of the energy eigenstate in the excited set. And saying that remaining things are exactly the same, not exactly, but very much close to being the same value. So, therefore, I can pull it out from the integration and then I am left with  $d\beta$  Anyway, now I can extend this integral from  $-\infty$  to  $\infty$ . I am adding on this tail and that tail, but sinc function is allowing me to only contribute very marginally. So, just for getting a order of magnitude unsat, I actually wanted to compute this quantity, but after these approximations, the statement is it is not going to be much different from this integral. This is one useful way of computing things. And ultimately you know that the sinc function integral is very trivial, this is standard  $\text{sinc}^2$  integral which you can work it out or look it up, it will just give you some  $2\pi$  factor which will be or which will give you some up to  $2\pi$  factor. So, you will see that rate of transition in this case is directly proportional to one of the energy excited state value and the degeneracy function over there and times the matrix element. So the matrix element mod

square times the degeneracy factor divided by  $\hbar$  up to some numerical constants give you the rate of transition that is Fermi's golden rule. This is very useful in looking at atoms and other things because you can quickly look at the degeneracy. This is the degeneracy of the field state. How your fields are distributed, how many states are there for the field at energy, You can just compute that, you can look from the  $d^3k$  integral that how many states are there for a 3 star and you have to just compute one of the transition matrix element and then by assuming most of the excited state transition matrix element are the same, you can quickly multiply these two things, divide it by  $\hbar$  and then  $2\pi Q$  and what not and you will quickly get the rate of transition for those atoms. So, this is very powerful tool in knowing the transition rates for atoms dealing with photons present in the field. This is ballpark gives you very right value, it is not 100 percent accurate but it is not It is good approximation up to 99 percent as well. You can compute exact value, you will find out this is up to 99 percent it works fine. So, we can take it as a thumb rule that I will just count the degeneracy of fields and I will just multiply with the matrix element mod square and then other things will just give me small order unity. value constants which have to be multiplied. The ballpark central value will be obtained from there. So, for any atom in order to get their transition probability and rate this Fermi s golden rule is a quick mechanism to get an order of magnitude estimate. I tell you what is the matrix element you will be able to directly tell me what is the transition rate for this atom. So, let us stop here and in the next class we will try to see one more feature of quantum field So, vacuum which goes by the name of spontaneous emissions. Okay, so I stop here.

