

Foundation of Quantum Theory: Relativistic Approach

Matter-Field interaction 1.2

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Unruh DeWitt Detectors

Lecture- 34

So today we will discuss about the Unruh – de Witt detectors as we learned in the previous class that generically if I try to couple or try to use the quantum field theoretic language on background fields which are talking to atomic systems invariably we lead into a situation where we have a Hamiltonian which is made up of operators coming from the both the sectors atomic sector as well as the field sector. So we have seen in the previous class we obtained in the non relativistic settings a Hamiltonian which is made up from $\vec{p} \cdot \vec{A}$ or $\vec{A} \cdot \vec{p}$ kind of term or $\vec{p} \cdot \vec{A}$ in terms times kx term in the dipole approximation.

Unruh - de Witt detectors

◇ An atom talks through some operator to a background field

$$H_{int} = \vec{p} \cdot \vec{A} \quad \text{or} \quad \vec{r} \cdot \vec{E}$$

Representatively a monopole operator \hat{m} couples with scalar field $\hat{\phi}$

$$\hat{H} = \hat{H}_{atom} + \hat{m} \hat{\phi} + \hat{H}_{field}$$

State	Atom	Field
	$ g\rangle$	$ 0\rangle$
	$ e\rangle$	$ 1_{k_0}\rangle, 1_{k_1}\rangle, 1_{k_2}\rangle, \dots$
		$ 1_{k_0, 2_{k_1}}\rangle, 1_{k_1, 2_{k_2}}\rangle, \dots$
	Eigenstates of \hat{H}_{atom}	Eigenstates of \hat{H}_{field}

Initial state

$$|g\rangle \otimes |0\rangle$$

$$|g\rangle \otimes |1_{k_0}\rangle$$

$$|e\rangle \otimes |0\rangle$$

$$|e\rangle \otimes |1_{k_0}\rangle$$

Unruh -de Witt detectors

★ An atom talks to a background field operator through some operator.

$$H_{\{nl\}} \quad \begin{aligned} & \hat{\vec{p}} \cdot \hat{\vec{A}} \quad \text{or} \quad \hat{\vec{r}} \cdot \hat{\vec{A}} \\ & \hat{\vec{p}} \cdot \hat{\vec{A}}(\vec{k} \cdot \hat{\vec{x}}), \vec{d} \cdot \vec{E} \\ & \vec{E} \cdot \vec{B} \end{aligned}$$

Redrepresenting a monopole operation \hat{m} couples with scalar field ϕ

$$\hat{H} = \hat{H}_{atom} + \hat{m}\hat{\phi} + \hat{H}_{field}$$

Atom

Field

State

$|g\rangle$

$|0\rangle$

$|l\rangle$

$|1k_0\rangle, |1k_1\rangle, |1k_2\rangle, \dots$

$|1k_0, 1k_0\rangle, |1k_1, 1k_2\rangle$

Initial state

$ g\rangle \otimes 0\rangle$ $ g\rangle \otimes 1k_0\rangle$ $ e\rangle \otimes 0\rangle$ $ e\rangle \otimes 1k_0\rangle$
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Remember we expanded e^{ikx} as $1 + ax$. One term will just multiply with $\vec{p} \cdot \vec{A}$ to give me $\vec{p} \cdot \vec{A}$ kind of term or kx which is appearing extra will go and combine with the whole sector. So, I have a two operators from the quantum atomic systems p and x and one operator a which is coming along with the field description. Together with some manipulations we can write down this term as $d \cdot e$ term as well, where d is the dipole moment of the atom and e is the electric field of the field. Similarly, if there is a spin which is talking to magnetic field, again if I do a consistency analysis, we will get up a Hamiltonian which is having an interaction term of the sort $\vec{S} \cdot \vec{B}$, where S is the spin sector coming from the atomic side and B is the background magnetic field. So, generically, most of the times, all the interaction terms are made up from one operator from field side, one operator from atomic side. Or more operators from atomic side one operator from field side typically we avoid writing more operators from field side and one operator from atomic side because we want to discuss things in the ranges or limits where magnetic field or background fields are not very strong so therefore their higher powers are typically ignored or propped up but there is no mathematical hard and fast proof that they should be dropped if you are interested about conditions where you have a strong electromagnetic field or strong

background field of whatever source you should include higher order terms as well. In this course we are just contained with the linear order leading order coupling between the atom and the field I would just consider a linear by linear coupling one operator from the field side one operator from the atomic side just like $\vec{p} \cdot \vec{A}$ $\vec{d} \cdot \vec{E}$ or $\vec{S} \cdot \vec{B}$, these kind of couplings we will be more generally using these are the lower order couplings where fields are not very strong. So, we want to test the quantum properties of fields along with the matter and those quantum properties are typically feeble to be witnessable directly. So, we want to test the quantum properties of fields along with the matter and those quantum properties are typically feeble to be witnessable directly. And as we discussed, one very representative way of dealing with such a system is through something called the Unruh-de Witt detectors which has some operator, we call it monopole operator m , which talks to a background scalar field ϕ . So, in terms of quantum field background, I am not talking about electromagnetic field, but the scalar field ϕ , real scalar field. This is the most simple quantum system in the field theoretic domain we can think of. And we couple these two terms linearly, bilinearly, meaning $\hat{m}\hat{\phi}$ term I write down in the total Hamiltonian. The full Hamiltonian is made up of three pieces. One is the Hamiltonian of the atom, the free atom. One is the free Hamiltonian of the free field. If they are just isolated, total Hamiltonian will be the Hamiltonian of atom and Hamiltonian of the field. But since they are talking to each other, this is the coupling between them we have envisaged for $\hat{m}\hat{\phi}$. Now this atom is supposed to be a two level atom just for simplistic consideration we are not dealing with ideal atom or we are not dealing with the realistic quantum fields we are just taking a representative atoms atom with two interesting lines excited state and a ground state and a field which is scalar most of the discussion which we are going to do will be carried over to more realistic settings of $\vec{p} \cdot \vec{A}$, $\vec{d} \cdot \vec{E}$ or $\vec{S} \cdot \vec{B}$ kind of couplings without much of a issue as we will see along so just for a good representation this two level atom talking to a scalar field is a unruh david system unruh dv detector system and through that we can anticipate what would happen for realistic settings as well so let us go ahead with this demonstration kind of exercise and then we will see what kind of rich physics it has to offer all right. so now I have Hamiltonian of the system fully given like this and then there is a quantum state of the full description atomic sector as we discussed is just a two level system that means I am just hypothesizing there are only two eigen state one is the ground state lower energy state one is the excited state higher energy state these are the two levels. typically hydrogen atoms or any other realistic atom will have many many many many lines but just for consideration as of now I am just considering two there is nothing which stops me from having more states but just for simplification and to see the exercises with clarity I am just hypothesizing only there are two interesting states which I am concerned about in the atomic sector. The field sector I have simplified myself by going to the scalar field. I am not going to do anything about the states of a scalar field. I will take the full spectrum of the scalar field. It has a vacuum all the fork basis element single excitation momentum k_0 or single momentum at excitation k_1 or k_2 or one excitation at k_0 one excitation and k_1 one excitation of k_1 and one excitation k_2 two excitations for k_0 and so on so all full fork basis is available to the scalar field so I have an atom with two levels and a scalar field with as many fork basis elements as it comes with So therefore, I can start my initial state, I can prepare my initial state by taking one state from here and one state from here and just writing them as a tensor product. So therefore, I can start my initial state, I can prepare my initial state by taking one state from here and one state from here and just writing them as a tensor product. That would be something called a separable state. Towards the end, most likely we will be discussing about non-separable entangled kind of states if time permits. But otherwise right now, I'm just starting my business with initial states of the kind where I have taken one state from this sector and one state from the field sector so it could be either both the systems are in the ground state atom is in the ground state field is in the ground state which is vacuum or atom is the ground state field is in one of the excited state one k_0 remember many many excited state could be there or the ground the atom is

in the excited state field is in the ground state which is excited times vacuum or both of them are in excited states and since fields comes with many many many different types of excited state you could choose many many of the possible slots to fill in the field sector. So, one of the states I can take as initial state and then start my computations or analysis under perturbation theory when the system is perturbed by this term. I have ignored higher order coupling ϕ^2 and other thing that means I am saying that fields are weak, the coupling is weak and that is the perturbation Hamiltonian. If suppose this term was not there in the total Hamiltonian then the full system will be governed by this atomic Hamiltonian and the field Hamiltonian. So, atomic Hamiltonian will evolve only the states in the atomic side and field Hamiltonian will evolve only the state in the field side. There will be nothing which will be common meaning they will just separable, the Hamiltonian is also separable, state is also separable. So, therefore, their individual evolution will also be separable. So, nothing interesting will happen only. field will evolve with its own Hamiltonian, atom will evolve with its own Hamiltonian and no coupling means they will just move along or evolve along as they do in standard quantum theory. But as soon as this term comes about, the total Hamiltonian does not remain. So, this any of this state, if you see this, this any of this state if you take, They are eigenstate of $H_{\text{atom}} + H_{\text{field}}$. If the interaction term is not there, whatever state I am writing over here is an eigenstate of $H_{\text{atom}} + H_{\text{field}}$ system. That means under time evolution, they will remain like there. They will not evolve. Only phase will be picked up if I take one of these states. Superposition, of course, would develop non-trivial phase dependency and they will not remain as they are initially proposed like.

However, yet if I take one of these separables, structure, then unless this perturbation Hamiltonian comes about, they will evolve independently picking up phases. That means the state will remain there. The probability of going to any other state would be nil. So, as soon as this perturbation or the coupling is turned on, these states do not remain eigenstate of the full Hamiltonian. Therefore, they will change from being this. And then we would compute under the quantization of the field and the understanding of the atomic part m . Where does the state go?

So, in order to do this discussion, I will just recapitulate some discussions which we already had in the time dependent perturbation theory approach, namely the Schrodinger picture versus interaction picture analysis. I will just recollect the essential things which we had already seen. So, if you want to brush up you go back to week 3 discussion or week 2 discussion where time dependent perturbation theory particularly the interaction picture was discussed and just recollecting the essential formulae from there. So, remember I have a interaction Hamiltonian which is a perturbing Hamiltonian and is a time dependent. You see the ϕ field ϕ and \hat{m} over here that we will see both of them have a time dependent. field changes from position to position and time to time so therefore this is a time dependent perturbation theory approach where my initial state is given as one of the possible let us see the state which is both the field as well as the atomic system are in the ground state and then let us evolve the system with the full Hamiltonian now the full Hamiltonian if it evolves the initial state $E\psi$ in will migrate to some new state $E\psi$ fwhich will be obtainable from the schrodinger picture description through this e to the power the integration of the full Hamiltonian between the initial time to time of interest where you want to know the system above acting upon the initial state $E\psi_0$ this is the Schrodinger evolution but recall we had discussed that the same analysis can be done the same time evolution can be understood in terms of something called interaction picture and interaction states interaction operators in terms of these new equations where the timed evolved Schrodinger wave function can be written in terms of something called interaction wave function times interaction wave function which is being acted upon the, which is acted upon y the $e^{-ih_0t/\hbar}$ where h_0 is just the free field part without perturbation part of the Hamiltonian. So this total Schrodinger time evolved state can be obtainable from the free Hamiltonian, unperturbed Hamiltonian's action on the something called the $\psi_i(t)$, the interaction time evolved version. This $\psi_i(t)$ that the interaction picture time evolved wave function itself is obtainable from some unitary time evolution operator in the interaction picture. This we all have done in the week 2 lectures and it can be obtained from initial time evolution, initial $\psi_i(t_0)$

which is again related to initial ψ Schrodinger wave state with $e^{iH_0 t}$ where t would be the time where the state is stuck, the state was taken as the initial state. So, this is how the time evolution happens.

Since H_{int} is a perturbing Hamiltonian we can use time-dependent perturbation theory

Let $|\Psi_{in}\rangle = |8\rangle \otimes |0\rangle$ ✓

$|\Psi_S\rangle = |\Psi_S(t)\rangle = e^{-i\int_0^t \hat{H} dt'} |\Psi(0)\rangle$

✓ $|\Psi_S(t)\rangle = e^{-\frac{i\hat{H}_0 t}{\hbar}} |\Psi_I(t)\rangle$ -

$|\Psi_I(t)\rangle = \hat{U}_I(t, t_0) |\Psi_I(t_0)\rangle$

$\therefore |\Psi_S(t)\rangle = e^{-iH_0 t/\hbar} \hat{U}_I(t, t_0) e^{iH_0 t_0/\hbar} |\Psi_S(t_0)\rangle$

$\therefore |\Psi_S(t)\rangle = U_S(t, t_0) |\Psi_S(t_0)\rangle$

where $U_S(t, t_0) = e^{-iH_0 t/\hbar} U_I(t, t_0) e^{iH_0 t_0/\hbar}$

With $\hat{H} = \hat{H}_0 + \hat{V}$

$\hat{U}(t, t_0) = 1 + \sum_n \left(-\frac{i}{\hbar}\right)^n \mathcal{T} \left(\int dt' \hat{V}_I(t') \right)^n$
 $= \mathcal{T} \exp \left(-\frac{i}{\hbar} \int_{t_0}^t \hat{V}_I(t') dt' \right)$

$$\langle \Psi_S(t) | \hat{O} | \Psi_S(t) \rangle = \langle \Psi_I(t) | e^{+iH_0 t / \hbar} \hat{O} e^{-iH_0 t / \hbar} | \Psi_I(t) \rangle$$

$$= \langle \Psi_I(t) | \hat{O}_I | \Psi_I(t) \rangle$$

One could compute expectations in the interaction picture as equally

Moreover

$$\langle \phi_I(t) | \Psi_I(t) \rangle = \langle \phi_S(t) | e^{iH_0 t / \hbar} e^{-iH_0 t / \hbar} | \Psi_S(t) \rangle$$

$$= \langle \phi_S(t) | \Psi_S(t) \rangle$$

All inner products can also be computed in the interaction picture

Thus, what is the probability that the state will be

$$|e\rangle |0\rangle$$

$$P_{|e,0\rangle \rightarrow |e,0\rangle} = \left| \langle e; 0 | \Psi_S(t) \rangle \right|^2$$

$$\left| \langle e; 0 | \Psi_I(t) \rangle \right|^2$$

Since H_{int} is a perturbing Hamiltonian we can use time dependent perturbation theory

Let $|\psi_{in}\rangle = |g\rangle \otimes |0\rangle$

$$|\psi_f\rangle = |\psi_s(t)\rangle = e^{-i \int_0^t \hat{H} dt'} \psi$$

$$\psi_s \quad e^{-iH_0 \frac{t}{\hbar}} \quad \psi_I$$

$$\psi_s(t) \quad \psi_I$$

$$|\psi_s(t)\rangle = e^{-iH_0 \frac{t}{\hbar}} \hat{U}_I(t, t_0) |\psi_s(t_0)\rangle e^{iH_0 \frac{t}{\hbar}}$$

where $\hat{H} = \hat{H}_0 + \hat{V}$

$$\hat{U}(t, t_0) = 1 + \sum_n \left(\frac{-i}{\hbar}\right)^n \int dt' \hat{V}_I(t')$$

$$= \exp\left(\frac{-i}{\hbar} \int_{t_0}^t \hat{V}_I(t') dt'\right)$$

One could compute expectations in the interaction picture as equally.

Moreover

$$\langle \phi_I(t) | \psi_I(t) \rangle = \langle \phi_s(t) | e^{\frac{+iH_0 t}{\hbar}} e^{\frac{-iH_0 t}{\hbar}} | \psi_s(t) \rangle = \langle \phi_s(t) | \psi_s(t) \rangle$$

All inner products can also be computed in the interaction picture.

Thus, what is the probabilities can also be computed in the interaction picture.

Thus, what is the probability that the state will be

$|e, 0\rangle$

$$P_{|g,0\rangle \rightarrow |e,0\rangle} = \frac{|\langle e; 0 | \psi_s(t) \rangle|^2}{|\langle e; 0 | \psi_I(t) \rangle|^2}$$

We had known previously this time evolution operator is written in terms of something called a time ordered product of time ordered exponential of V interaction potential.

$V(I)$ which is equivalent to \hbar interaction in the interaction picture for us we will see in a minute but just recollect go back to your notes of week three week two rather and recollect the formulae that time evolution of the interaction picture wave function is done through a unitary operator which is time

ordered exponential this where $V(I)$ is obtainable from the perturbation potential V or H interaction for our case squeezed between the unitary of free Hamiltonian $e^{iH_0 t/\hbar}$ and $e^{-iH_0 t/\hbar}$ from the right. So, this constitute for us the whole time evolution in the interaction picture. Once I know in the interaction picture where has the state landed, I can obtain through this action where is the state in the Schrodinger picture. This is as trivial. Just for completeness let me write down the full schrodinger time evolved state can be obtainable from the initial schrodinger state through this exponential time integral in the exponential which I can write as $U_s(t, t')$ the schrodinger unitary. Now, there is a relation between this unitary of interaction picture and Schrodinger unitary which I have written over here. This is a simple exercise for you to prove this. That us of t, t_0 is just interaction picture Hamiltonian squeezed between the free Hamiltonians at different times. Left hand side is unitary at time t and this is the unitary at time t_0 according to the free Hamiltonian. Again, this is nothing new which we do not know about. This is just rewriting of the formulae of week 2. Let us proceed further. In that week, we had already learned that once I am in the interaction picture, any operator's expectation value, either I can go to the Schrodinger picture analysis, take the operator, find out what is $\psi_s(t)$ and squeeze it between and that will give me the expectation value of that operator. But we have learned that everything can equivalently be done in the interaction picture where I just obtain what is $\psi_i(t)$ s and squeeze the operator between the interaction version of the operator which is related to the normal operator like this. This is the interaction OI. So all the expectation value can equally be computed in the interaction picture or as in the Schrodinger picture. and similarly all the inner products as well can be computed either in the interaction picture or in the schrodinger picture both of them have the same information okay the benefit of going to interaction picture is that the unitary which we had write for which we had written for the for the interaction case only depends on the perturbing Hamiltonian. v is just the perturbation Hamiltonian squeezed between the free particles Hamiltonian, free particles unitary. So, ultimately the interaction picture, the interaction Hamiltonian plays a central role and the information captured by them are as good as the Schrodinger picture. So, we will see how the Unruh-David detector makes use of that.

Okay so we know that in the schrodinger picture we are initially where the initial state let us say it could be so I want to know that initial state was ground state and vacuum for the field this was my initial state I want to know after a time evolution what is the likelihood that the state has become The state has become excited in the atom while field is still in the vacuum so I am asking what is the probability that state has become this after time evolution so what I should do I should time evolve the state initial state to $\psi_s(t)$ and then take the overlap with the state of my choice that I want to know the projection on this state E_0 so this mod square will tell me what is the probability that the final state is E_0 . This I can compute in the Schrodinger picture as well as in the interaction picture as we have just learned. Only thing is that I have to write down the E_0 here which was previously in the Schrodinger picture in the interaction picture E_0 's i. And remember again any interaction wave function is related to the Schrodinger wave function through the $e^{iH_0 t/\hbar}$. So, let us just see that equation once more. Any interaction picture wave function is related to the Schrodinger picture wave function times, so this $e^{-iH_0 t/\hbar}$ can be brought on to the left hand side. So, $e^{iH_0 t/\hbar}$, H_0 is the free Hamiltonian acting on Schrodinger state gives me the interaction wave function. So, that is the information which we have to be alert about. So, here when I am writing E_0 interaction in the interaction picture, this is at time t . That means it is obtainable from the Schrodinger e_0 through $e^{iH_0 t/\hbar}$, that is all. And similarly for $\psi_s(t)$. So, that mod square will give me the probability of getting a state E_0 if I started from $g(0)$. Similarly, we can ask a different question, what is the probability that both atom as well as the field get excited, the field gets excited at some Let us say Ik , the same computation I will do, only the final state of interest will become EIk . I am asking the probability that the final state is Ik , not 0 for the vacuum. Similarly, I

could have asked another different vacuum, another different excited state for the field. E2k, I could ask this question, what is the probability that the initial state was ground state and vacuum. After a time evolution, atom has become excited and the field has become doubly occupied at a momentum k. The computation should tell me what is the likelihood. If it is 0, it should come out to be 0. If it is 1 half, the answer should be 1 half and whatever we do. Similarly I can ask another question what is the probability of $1k 1k$ excitation in the field sector and excitation state of the atom again this projection will tell me the answer one can further ask what is the likelihood that the atom remains non-excited atom remains ground and the field only excites all these things can be obtained from $\psi_I(t)$ and taking its various projections so therefore.

$$P_{|g,0\rangle \rightarrow |e; 1_k\rangle} = \left| \langle e; 1_k | \psi_I(t) \rangle \right|^2$$

$$P_{|g,0\rangle \rightarrow |e; 2_k\rangle} = \left| \langle e; 2_k | \psi_I(t) \rangle \right|^2$$

$$P_{|g,0\rangle \rightarrow |e; 1_k, 1_k\rangle} = \left| \langle e; 1_k, 1_k | \psi_I(t) \rangle \right|^2$$

In general

$$P_{|g,0\rangle \rightarrow |e, \psi\rangle} = \left| \langle e, \psi | \psi_I(t) \rangle \right|^2 = \left| \langle e, \psi | \hat{U}_I | \psi_I(0) \rangle \right|^2$$

$$= \left| \langle e, \psi | e^{-\frac{iH_0 t}{\hbar}} \hat{U}_I e^{\frac{iH_0 t}{\hbar}} |g, 0\rangle_s \right|^2$$

$$\hat{U}_s = e^{-\frac{iH_0 t}{\hbar}} U_I e^{\frac{iH_0 t}{\hbar}}$$

$$= e^{-\frac{iH_0 t}{\hbar}} \left[\mathcal{T} \left(\exp \left(-\frac{i}{\hbar} \int_{t_0}^t V_I(t') dt' \right) \right) \right] e^{\frac{iH_0 t}{\hbar}}$$

$$= e^{-\frac{iH_0 t}{\hbar}} \left[\mathcal{T} \left(\mathbb{1} - \frac{i}{\hbar} \int_{t_0}^t V_I(t') dt' \right) \right] e^{\frac{iH_0 t}{\hbar}}$$

$${}_s \langle e, \psi | \hat{U}_s |g, 0\rangle_s = e^{-iE_\psi t / \hbar} e^{iE_g t_0 / \hbar} \langle e, \psi | \mathcal{T} \left(\mathbb{1} - \frac{i}{\hbar} \int_{t_0}^t V_I(t') dt' \right) |g, 0\rangle_s$$

$$\therefore P_{|g,0\rangle \rightarrow |e, \psi\rangle} = \left| \langle e, \psi | \mathcal{T} \left(\mathbb{1} - \frac{i}{\hbar} \int_{t_0}^t V_I(t') dt' \right) |g, 0\rangle_s \right|^2$$

$$= \left| \langle e, \psi |g, 0\rangle - \frac{i}{\hbar} \int_{t_0}^t \langle e, \psi | V_I(t') |g, 0\rangle dt' \right|^2$$

Since $\langle e, \psi | g, 0 \rangle = 0$

$$P_{|g,0\rangle \rightarrow |e,\psi\rangle} = \frac{1}{\hbar^2} \left| \int dt' \langle e, \psi | V_{\text{int}}(t') | g, 0 \rangle \right|^2$$

For our case $\hat{V} = \hat{H}_{\text{int}} = \hat{m} \hat{\phi}$

$$\hat{V}_{\pm}(t) = e^{\pm i\hat{H}_0 t/\hbar} \hat{m} \hat{\phi} e^{\mp i\hat{H}_0 t/\hbar}$$

$$\begin{aligned} & \langle e, \psi | e^{-i\hat{H}_0 t/\hbar} \hat{m}(0) \hat{\phi}(t=0) e^{-i\hat{H}_0 t/\hbar} | g, 0 \rangle \\ &= \langle e, \psi | e^{-i\frac{H_0^{\text{atom}} t}{\hbar}} \hat{m}(0) e^{-i\frac{H_0^{\text{atom}} t}{\hbar}} e^{i\frac{H_0^{\text{atom}} t}{\hbar}} \hat{\phi}(t=0) e^{-i\frac{H_0^{\text{atom}} t}{\hbar}} | g, 0 \rangle \\ &= \langle e | e^{-i\frac{H_0^{\text{atom}} t}{\hbar}} \hat{m}(0) e^{-i\frac{H_0^{\text{atom}} t}{\hbar}} | g \rangle \langle \psi | \hat{\phi}(a, t) | 0 \rangle \\ &= \langle e | \hat{m} | g \rangle e^{i(\frac{E_g - E_e}{\hbar})t} \langle \psi | \hat{\phi}(a(t)) | 0 \rangle \end{aligned}$$

Thus

$$P_{|g,0\rangle \rightarrow |e,\psi\rangle} = \frac{1}{\hbar^2} \int dt' \langle e | \hat{m} | g \rangle e^{i\frac{\Delta E t'}{\hbar}} \langle \psi | \hat{\phi}(t') | 0 \rangle \times \int dt'' \langle g | \hat{m} | e \rangle e^{-i\frac{\Delta E t''}{\hbar}} \langle 0 | \hat{\phi}(t'') | \psi \rangle$$

$$= \frac{1}{\hbar^2} |\langle g | \hat{m} | e \rangle|^2 \iint dt dt' e^{i\frac{\Delta E (t-t')}{\hbar}} \langle 0 | \hat{\phi}(t') | \psi \rangle \langle \psi | \hat{\phi}(t) | 0 \rangle$$

If we are interested in what happens to the atom only

$$P_{|g\rangle \rightarrow |e\rangle} = \sum_{\psi \in \text{Fock basis}} P_{|g,0\rangle \rightarrow |e,\psi\rangle} = \frac{1}{\hbar^2} |\langle g | \hat{m} | e \rangle|^2 \iint dt dt' e^{-i\frac{\Delta E (t-t')}{\hbar}} \langle 0 | \hat{\phi}(t') \sum_{\psi} |\psi\rangle \langle \psi | \hat{\phi}(t) | 0 \rangle$$

$$= \frac{1}{\hbar^2} |\langle g | \hat{m} | e \rangle|^2 \iint dt dt' e^{-i\frac{\Delta E (t-t')}{\hbar}} \langle 0 | \hat{\phi}(t) \hat{\phi}(t') | 0 \rangle$$

$$P_{|g,0\rangle \rightarrow |e_i;2_k\rangle} = |\langle e_i;1_k | \psi_I(t) \rangle|^2$$

$$P_{|e_i;2_k\rangle} = |\langle e_i;2_k | \psi_I(t) \rangle|^2$$

$$P_{|g,0\rangle \rightarrow |e_i;1_{k_1},1_{k_2}\rangle} = |\langle e_i;1_{k_1};1_{k_2} | \psi_I(t) \rangle|^2$$

In general

$$\begin{aligned} P_{|g,0\rangle \rightarrow |e,\psi\rangle} &= |{}_I \langle e_i \psi | \psi_I(t) \rangle a|^2 = |{}_I \langle \psi | \hat{U}_I | \psi_1(0) \rangle|^2 \\ &= |{}_I \langle e_i \psi | \underbrace{e^{\frac{-iH_0 t}{\hbar}} \hat{U}_I e^{\frac{iH_0 t_0}{\hbar}}}_{\hat{U}_s} | g, 0 \rangle|^2 \end{aligned}$$

$$\hat{U}_g = e^{\frac{-iH_0 t}{\hbar}} \hat{U}_I e^{\frac{iH_0 t_0}{\hbar}}$$

$$e^{\frac{-iH_0 t}{\hbar}} \left[\gamma \left(e^{\frac{-i}{\hbar} \int_{t_0}^t V_I(t') dt'} \right) \right] e^{\frac{iH_0 t}{\hbar}}$$

$$e^{\frac{-iH_0 t}{\hbar}} \left[\gamma \left(1 - \frac{i}{\hbar} \int_{t_0}^t V_I(t') dt \right) \right] e^{\frac{iH_0 t}{\hbar}}$$

$$P_{|g,0\rangle \rightarrow |e_i;2_k\rangle} = |\langle e_i;1_k | \psi_I(t) \rangle|^2$$

$$P_{|e_i;2_k\rangle} = |\langle e_i;2_k | \psi_I(t) \rangle|^2$$

$$P_{|g,0\rangle \rightarrow |e_i;1_{k_1},1_{k_2}\rangle} = |\langle e_i;1_{k_1};1_{k_2} | \psi_I(t) \rangle|^2$$

∴

$$\begin{aligned} P_{|g,0\rangle \rightarrow |e,\psi\rangle} &= |\langle e \psi | \left[\gamma \left(1 - \frac{i}{\hbar} \int_{t_0}^t V_I(t') dt \right) \right] | g, 0 \rangle|^2 \\ &= \left| \langle e_i \psi | g, 0 \rangle - \frac{i}{\hbar} \int_{t_0}^t \langle e_i \psi | V_I(t') | g, 0 \rangle dt' \right|^2 \end{aligned}$$

Since

$$\langle e, \psi | g, 0 \rangle = 0$$

$$P_{|g,0\rangle \rightarrow |e,\psi\rangle} = \frac{1}{\hbar^2} \left| \int dt' \langle e, \psi | V_I(t') | g, 0 \rangle \right|^2$$

For our case, $V = \hat{H}_{int} = \hat{m} \hat{\phi}$

$$\hat{V}_I(t') = e^{\frac{i\hat{H}_0 t'}{\hbar}} \hat{m} \hat{\phi} e^{-\frac{i\hat{H}_0 t'}{\hbar}}$$

$$\langle e, \psi | e^{\frac{i\hat{H}_0 t'}{\hbar}} \hat{m}(0) \hat{\phi}(t'=0) e^{-\frac{i\hat{H}_0 t'}{\hbar}} | g, 0 \rangle$$

$$\langle e, \psi | e^{\frac{i\hat{H}_0^{atom} t'}{\hbar}} \hat{m}(0) e^{-\frac{i\hat{H}_0^{atom} t'}{\hbar}} e^{\frac{i\hat{H}_0' t'}{\hbar}} \hat{\phi}(t'=0) e^{-\frac{i\hat{H}_0' t'}{\hbar}} | g, 0 \rangle$$

$$\langle e | e^{\frac{i\hat{H}_0^{atom} t'}{\hbar}} \hat{m}(0) e^{-\frac{i\hat{H}_0^{atom} t'}{\hbar}} | g \rangle \langle \psi | \hat{\phi}(x, t') | 0 \rangle$$

$$\langle e | \hat{m} | g \rangle e^{\frac{E_e - E_g}{\hbar}} \langle \psi | \hat{\phi}(x(t')) | 0 \rangle$$

Thus

$$P_{|g,0\rangle \rightarrow |e,\psi\rangle} = \frac{1}{\hbar^2} \int dt' \langle e | \hat{m} | g \rangle e^{\frac{\Delta E}{\hbar}} \langle \psi | \hat{\phi}(t') | 0 \rangle \times \int dt' \langle g | \hat{m} | e \rangle e^{-\frac{\Delta E}{\hbar}} \langle 0 | \hat{\phi}(t') | \psi \rangle$$

$$\frac{1}{\hbar^2} \langle g | \hat{m} | e \rangle^2 \iint dt dt' e^{\frac{\Delta E(t-t')}{\hbar}} \langle 0 | \hat{\phi}(t') | \psi \rangle \langle \psi | \hat{\phi}(t') | 0 \rangle$$

If we are interested in what happens to the atom only

$$P_{|g\rangle \rightarrow |e\rangle} = \sum_{\psi \in \text{Fock basis}} P_{|g,0\rangle \rightarrow |e,\psi\rangle} \frac{1}{\hbar^2} \langle g | \hat{m} | e \rangle^2 \iint dt dt' e^{\frac{-\Delta E(t-t')}{\hbar}} \langle 0 | \hat{\phi}(t') \sum_{\psi} |\psi\rangle \langle \psi | \hat{\phi}(t') | 0 \rangle$$

$$\frac{1}{\hbar^2} \langle g | \hat{m} | e \rangle^2 \iint dt dt' e^{\frac{-\Delta E(t-t')}{\hbar}} \langle 0 | \hat{\phi}(t) \hat{\phi}(t') | 0 \rangle$$

In general, I can write down what is the likelihood of initial state which was $g(0)$ going to some state in which atom is excited while the field is in one of the possible excited or ground state. I am writing as ψ . ψ can be a mimic for either vacuum or single excitation or double excitation or triple excitation, whatever you want to put it here. And as I saw, as we saw that this will be obtained from mod square of this projection. Now, remember this projection is in the interaction picture. So, I have to evolve this

$\psi_i(t)$ is in the interaction picture. So, I will take the interaction unitary and evolve the initial $\psi | 0$. So, $\psi | 0$ times acted upon by U_I gives $U_I \psi_i(t)$. That is the definition of $\psi_i(t)$. And then I try to write down this is at time t while this is at time t_0 , $\psi_i(t)$ now has become at time t_0 . Initial $\psi_i(t_0)$ is obtainable from $e^{-iH_0 t_0/\hbar}$ acting on ψ Schrodinger at time t_0 . In the Schrodinger picture, the initial state was $g | 0$. Both the systems were in their ground state. $e^{-iH_0 t_0/\hbar}$ has to act/because this is the definition of $\psi_i(t_0)$. So, these two terms are together $\psi_i(t_0)$. Similarly, the two terms over here which are at time t now, this is the final time. These two terms are definition of ψ at time t in the interaction picture. So, you see in between I have a $e^{-iH_0 t/\hbar}$ and $e^{iH_0 t/\hbar}$ and the interaction unitary is squeezed between them. Remember that was your exercise, this is the Schrodinger unitary in between. Schrodinger unitary is nothing but the interaction unitary squeezed between the free Hamiltonian exponential. Now let us use the definition of what was the interaction unitary. Interaction unitary as we recall was time ordered exponential of V_I . Let us go to the previous slide and see it with clarity once more that this was the interaction unitary. Time ordered of the exponential of V_I where V_I is just the perturbation Hamiltonian squeezed between the unitary of the free Hamiltonian. This V_I which has appeared over here which is the interaction picture potential. Now, let us see once this exponential has appeared over here and we have decided that the interaction potential is feeble which is not very strong. I can see that this exponential has a character e^{-x} which is small with a small x the interaction Hamiltonian is not very strong. I can tailor expand this exponential. So, this is $1 - x$ kind of term there could also be $x^2 + x$ cube and those kind of term which I am ignoring because remember when we were discussing linear coupling or bilinear coupling I had already thrown higher order field operators ϕ^2 or a^2 So, therefore no point containing or having higher order exponential terms here because in the full Hamiltonian also I have thrown away the higher order terms in the field and interaction Hamiltonian contains a field. So, its higher power will contain higher powers of fields which we do not want. So, therefore I can just tailor expand this exponential into $1 -$ this integration of this quantity that is all. And this would be t_0^2 between, this would be t_0^2 between exponential. So, this should be t_0 here. So, this should be t_0 here. So now let us open it up and try to see what do I get. I have to take this Schrodinger unitary. See, the transition probability is the Schrodinger unitary us, squeeze between the Schrodinger state $g(0)$ and $E\psi$ from left and right. So I have to take this us, squeeze it between state $g(0)$ initial state and the projected final state. And mod square of that will give me probability. So let us first squeeze it between these two states of interest and then we will do the mod square. So, when I squeezed it between $E\psi$ and $g(0)$ from left and right, this $E\psi$ will go and sit on the left side, while this $g(0)$ will go and sit on the right side here. Now, you will see there are two exponentials $e^{iH_0 t/\hbar}$ and $e^{-iH_0 t/\hbar}$ and $e^{iH_0 t/\hbar}$. These operators will hit this states $E\psi$ on the left and $g(0)$ on the left. When the operator, this exponential hits the left coming bra state which is $E\psi$, I will get the total excited state energy of the whole system E and ψ put together. This is the free Hamiltonian, $H_{\text{atom}} + H_{\text{field}}$, no interaction. So, therefore, I will both of them here E and ψ , E is the excited state of atom and ψ is one of the excited state of the So, overall this is eigenstate of this operator H_0 . So, therefore, I will get total energy of the whole state which I am calling $E\psi$. $E\psi$ is not just the state energy eigenvalue of the field, it is the total energy of the whole system $E\psi$. Similarly, from the right hand side when this $e^{iH_0 t/\hbar}$ acts on $g(0)$. that will give me $e^{iH_0 t/\hbar}$ to the power total energy in the phase total energy of the full system. So, I will get $e^{iH_0 t/\hbar}$, these two are just complex numbers now pure phase. So, this can be clumped out while these are eigenstates acting, these are the operators acting on their eigenstates, the state will come back meaning $e^{iH_0 t/\hbar}$ acting on $E\psi$ will give me this phase factor and the bra back, $E\psi$ bra back. And similarly the right hand side $e^{iH_0 t/\hbar}$ acting on the $g(0)$ will give me this phase factor times the $g(0)$ back. So, therefore now I have this whole time ordered expansion term squeezed between $E\psi$ and $g(0)$ and outside I have a pure phase, a couple of pure phases actually, one from left side, one from right side, but pure phase after all. Now, as we discussed the probability is the

mod square of such a term. So, when I take the mod square, these pure phases which have accumulated outside, they get under the mod square to give me 1. So, that means they do not play any active role. All I am left with is the time ordered expansion squeezed between the states of interest and mod square of that. So, let us open it term by term. First term is identity, time ordering of identity is just identity. So, first term will just be overlap between the two states of interest, initial state $g(0)$ and my projection of likelihood of going to $E\psi$. So, the state of interest $E\psi$. The second term is the first integral of $V_I(t')$ from t_0 to t . Now, this is an integral and single integral and time order of single integral does not do anything. Time ordering matters only from second integral onwards. So, therefore, time ordered of the first two terms of the expansions are thus terms themselves. So, I do not have to worry about the time ordering up to lowest order perturbation theory or leading order perturbation theory. So, I have two terms, once the overlap between $E\psi$ and $g(0)$ and then I \hbar times this integral where the interaction Hamiltonian or interaction perturbation term in the interaction picture is squeezed between the two states of interest and mod square of the whole term. Now, since this $E\psi$ and $g(0)$, they are eigenstates of H_0 , total free Hamiltonian of atom and field put together, they are orthogonal to each other. So that means their overlap is essentially 0. So that means I am left with only the second terms mod square and mod square of the second term is 1 over \hbar^2 because I or $-I$ rather under mod square will come out as 1 and \hbar will come out under mod square as \hbar^2 and then this integral mod square of that. So, at this point we have to just recollect what was the definition of v_i . The definition of v_i was the perturbation Hamiltonian squeezed between the unitaries of the free Hamiltonian. Our perturbation Hamiltonian is $\hat{m} \hat{\phi}$. Recall in the Unruh – de Witt double detectors system I had the atomic operator \hat{m} and field operator $\hat{\phi}$. So, this was my V . Its interaction version will be the operator squeezed between the unitaries of the free Hamiltonian H_0 . H_0 is composite of two fields, two Hamiltonians of atom and field put together. Now, recall that the V_I if I have to write in the interaction picture. In the interaction picture the two exponentials come on the left and right hand side with the same time parameter t' . In the unitary of Schrodinger and interaction the two different exponentials come at two different times on left and right. This was just to be careful about. So, I know what is my $v_i(t')$, this is just the operator interaction coupling squeezed between two unitaries and this thing has to be evaluated with the matrix element has to be evaluated between $E\psi$ and $g(0)$. So, I take the whole thing, this is the whole v_i for me and squeeze it between $E\psi$ and $g(0)$. Now again, there is a couple of steps of algebra you should be alert about and see how does it get you through. As we discussed, this H_0 is the free Hamiltonian of both the system, atom as well as the field. So, I split it up, $H_{\text{atom}} + H_{\text{field}}$, free part of them. And since H_{atom} and H_{field} are just two separable Hilbert spaces, they commute among themselves, so exponentials can be separated. So, I have a $e^{iH_{\text{atom}}t'/\hbar}$ of atom only and $e^{iH_{\text{field}}t'/\hbar}$ of the field only. Here I have the field only. So, similarly on the right hand side $e^{-iH_{\text{field}}t'/\hbar}$, again this H_0 can be broken up into H_{atom} and H_{field} with a $-$ sign at t' . So, I have a atom with a $-$ sign here and the field which is $-$ sign here. So, I have just broken things apart. So, I have just broken things apart. Since they are commuting operators belonging to two different operators, I collect all the atomic operators on one side and all the field operators on one side. So, you can just convince yourself that the whole thing over here can be written as operator \hat{m} squeezed between two unitaries of atomic free Hamiltonian. And similarly field operator is squeezed between the unitaries of free field operator. They will just give you timed evolved field operators and timed evolved monopole operator of the atom, because this is u, u^\dagger on the operators. But let us not write it in that way, I just keep the atomic part as a unitary times the operator times u^\dagger . While the field I write it in terms of the time evolved field. This squeezing of field operator between $e^{iH_{\text{field}}t'/\hbar}$ and $e^{-iH_{\text{field}}t'/\hbar}$ will just give me the p_i at a future time t' . This is just the definition of the time evolution of the operator. Here also it is time of evolution of the monopole operator m , but I am not writing it as explicitly because that helps in the next step because you can see here again the same game will happen. This will give me a pure phase times the state back and while on the right hand side also you can see this term also will give me a pure phase and the state back. So, I will have a pure

phase e to the power I excited state energy coming from the left – ground state energy coming from the right divided by \hbar times t and the operator \hat{m} which is here that will now get squeezed between e and g . So these exponential operators convert into pure complex numbers and the state go through them and hit the m . So you see the whole first half of this term is just this exponential of energy gap divided by \hbar times t and the monopole operators matrix element or transition element it is called sometimes. While on the right hand side I have the field which is between the initial state which was ground and possibly the final state which is ψ , that is the projected state which I am looking at. So, now I have the full expression for the squeezed v between $E\psi$ and $g(0)$. What I have to do? I have to do the mod square, I have to first do the integration and then do the mod square. Now you see here the t , actually this should be t' because everywhere Hamiltonian was coming with t' . So, you see the phase has come with t' , but this t' has to undergo integration. This is not a constant t' , it is changing t' . So, therefore, all the terms which are coming in the integration do not have the same phase. They have different, different phases at different, different time. Therefore, I cannot pull it as a common factor and under mod of² it will become 1. That is not going to happen unlike before. So this is a running phase. This changes at different different times. So let us better way will be take this thing, takes its complex conjugate because I have to first integrate it, multiply its conjugate, complex conjugate to get the mod square. So that's what I do. I first write down the term which I have obtained, integrate it with label $\partial t'$ and then write down the complex conjugate of that. The complex conjugate of the whole term is also an integration. Only thing that this emg will become gme this time because \hat{m} is a Hermitian operator and the complex conjugate of this thing will just flip the role opera and ket. The exponential will become a – sign over here and again the flipping between ψ and 0 because the scalar field also is a Hermitian operator. Now I am settled. Now I have the full probability of transition. So outside I will get $|a|^2$ because they are complex conjugates of each other, I will get $|a|^2$. So outside I will get because they are complex conjugates of each other, I will get $|a|^2$. Inside I will be left with $\partial t/\partial t'$ integration and you see now there is a relative time difference which appears in the phase. Because these phases were running phases at each integral different different time. Two such terms have to be multiplied, so therefore exponentials will have a relative phase with a relative time difference of running times. What I have done in the next step, I have called $t' - t$ and $t'' - t'$. This is just for simplification of notations. All of them are running variable, you could call it x and y , here also x and y , you will see both the terms are equivalent to each other. Only thing you have to be careful about that, now this is a complex number, a real field or Hermitian field squeezed between two different states can give me a complex number. So, these are not operators anymore. They are just matrix elements or complex numbers. So, I can flip them. So, I first write $\phi(t)$ double E' between 0 and ψ and then $\psi \phi(t')$ between, then $\psi \phi(t')$ at 0. I have flipped this complex number because complex number can be flipped, operators cannot. And you remember t'' was being called t' in the next step and t' was being called as a t in the next step. So, you see the exponential which was $t' - t''$ can be written as $t - t'$. And the operator which is coming first is $\phi(t'')$ which has become $\phi(t')$. And later I had written as $\phi(t')$ which has become $\phi(t)$. So, this is somewhat of a neatly looking form. Now, suppose we are interested only in knowing what is the probability of excitation of atom irrespective of whatever happens to the field. That means I am not interested about where the field goes, let it go to any of the fork basis, I do not care. That means I should ask for all the possible processes, add up their probabilities where atom gets excited. That means 0 to E , but field goes to 0 to 0 or 0 to I_k or 0 to 2_k , 0 to I_k , I_k not and all such possibility of ψ that means ψ of the all focuses I am going to add over, all the probabilities in which the first part gets excited, second goes wherever it likes. So, all the final states I sum over. If I do that, if I ask this limited question that I do not care what happens to the field, I just care about what happens to the atom, you will see that this summation of the probability, this ψ which is being summed over will go and sit here you see there is outer product $\psi-\psi$ coming in the previous state. So this summation over ψ will go and sit over the outer product of $\psi-\psi$. Remember ψ was a Fock basis element, one of the Fock basis element whichever you like. And now I am summing over all Fock basis elements. That means this is

the outer product of all basis elements and this constitutes an identity in the field space. So therefore, I am left with, if I just want to ask this limited question, what is the probability of atoms excitation? Then that limited question gets answered by this exponential, the integral, double integral, this exponential. See here also in the exponential, I have done a funny way that I am writing $t - t'$, there is a $-$ of $t' - t$. This is just algebraic manipulation, nothing serious. But in the interior the two, the operators which are sitting inside is ϕ at t' identity, then ϕ at t squeezed between $0, 0$, vacuum of the field, where the field is started with. Where it lands up I do not care, but I know where it started with, it started with the vacuum. And then this identity does not do any active role, so $\phi(t')$ identity $\phi(t)$ is equivalent to $\phi(t')$ $\phi(t)$. Now, I again relabel. So, t' appears first in the exponential with a $-$ sign and $\phi(t')$ appears here. t appears in the argument of exponential at the later stage and the t here appears second which means I can just flip, call it x and y again, t' and t can be flipped, flipped in the sense of I can relabel whatever was t' , I can call it t and whatever was t , I can call it t' . This is just integration variable change. Under that, you can see that, under that you can see that this just becomes $e^{-i(t-t')}$, because I am going to call t' as t and t as t' . ϕ at t' becomes $\phi(t)$ and ϕ at t becomes $\phi(t')$. So, you see this is the correlator of ϕ at time t and ϕ at time t' which has appeared. When I summed over all the possible states of fork basis where the field can land up, I end up getting the two point correlator of the field at time t and time t' and this t and t' s are running from initial time t not to final time t . The probability of transition collects the information of the two-point correlator correlation function and does this some sort of a double Fourier transform. It looks like a Fourier transform by structure. We will see what it is in the next class. We will see what it is in the next class. Two-point correlator at time t and time t' . at the locations of the atom. So atom is at time t at some location, at some other time t' it has moved to some other location. And the field values there, how much they are correlated, that tells us what is the transition probability. So that is why we are calculating the field correlators in the previous classes. Now we see that the field correlators decide the transition probability between different states and therefore their useful quantity is not useless. See we started in vacuum state. Field expectation in vacuum as we know now is zero. $\sqrt{\text{There is no active photon expectation value wise in the field. But vacuum has a quantum fluctuation. That quantum fluctuation is captured by this two point correlator in the vacuum state. Had it been a classical state, If there was no classical field, there will be no two-point correlator as well, that would be 0 as well and no probability of transition. But this time, despite its expectation being 0 in the vacuum state, the correlator is non-zero. That is a quantum feature and that correlators double Fourier transform of this kind will tell me what is the probability of the atoms excitation. So, therefore, we will need to compute this two-point correlator at t and D plane which we have done already. So, I will just use that result in the next class and show you what is the probability of transitions of atoms at excitations, de-excitations and other things of atom through quantum fluctuations. This is not a classical field anymore, this is quantum field and quantum fields two-point correlator function is deciding us, for us what is the transition probability. So, in the next class, we will see what is the transition probabilities with the usage of this correlation functions. And now we will see the quantumness of fields have started mattering when I couple quantum field to quantum matter. So, I stop here for today. In the next class, we will see the transition probabilities.}}$