Foundation of Quantum Theory: Relativistic Approach

Matter-Field interaction 1.1

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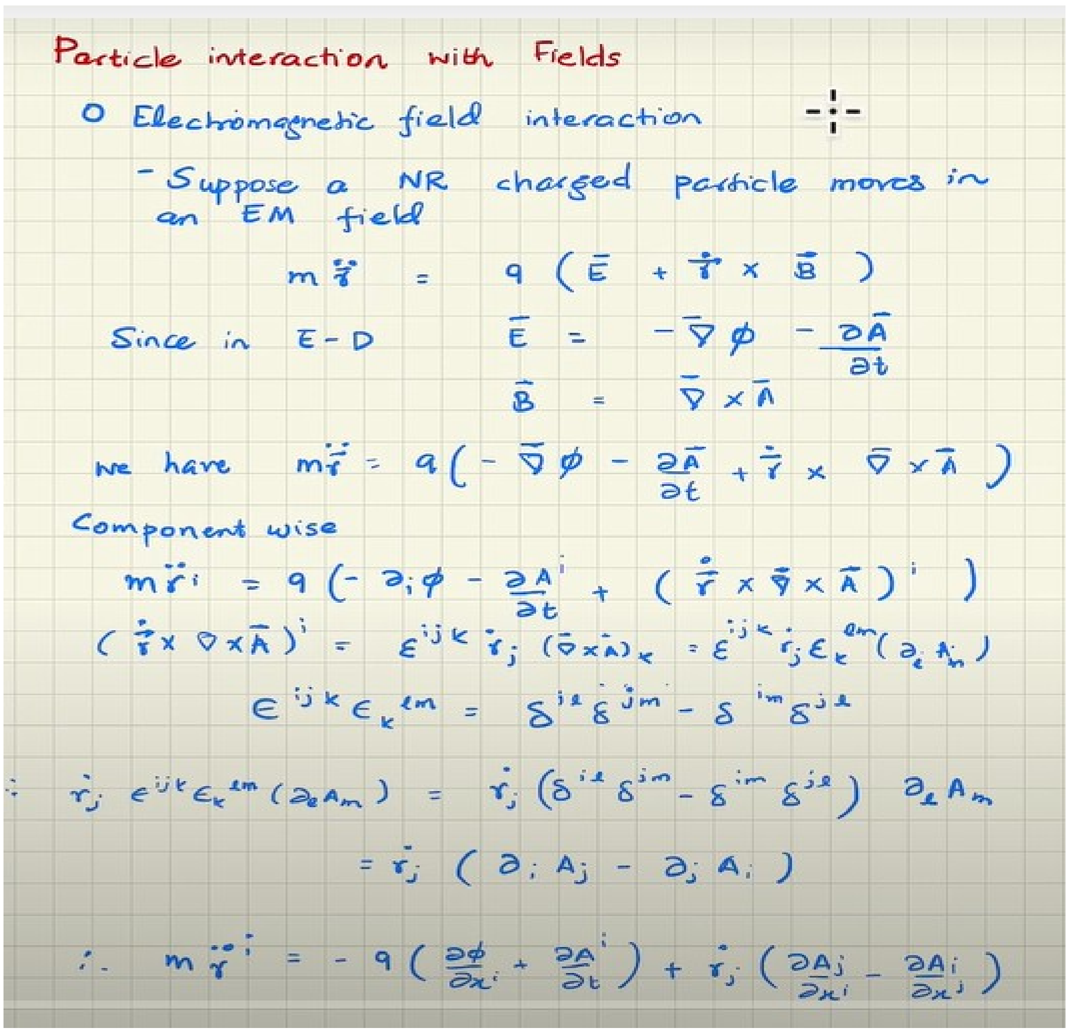
Atoms in Background Quantum Fields

Lecture- 33

So, having learned that how to quantize fields and how to obtain relevant quantities out of the quantum fields, now we move on to start coupling atoms with quantum fields which are present in nature and we will see how their interaction leads to new phenomena in terms of light matter interaction or quantum



field matter interaction.

Particle interaction with fields

bullet● Electromagnetic field interactions

– Suppose a NT cjathed pativlr movrd in the field

$$m\ddot{\vec{r}} = q(\vec{E} + \dot{\vec{r}} \times \vec{B})$$

Since in E.D. $\vec{E} = (-\nabla\phi - \frac{\partial \vec{A}}{\partial t})$

$$\vec{B} = \nabla \times A$$

We thus have $m\ddot{\vec{r}}^{i} = m(-\nabla\phi - \frac{\partial \vec{A}}{\partial t} + \dot{\vec{r}} \times \nabla \times \vec{A})$ Component wise

$$m\ddot{\vec{r}}^{i} = m(-\partial_i \phi - \frac{\partial \vec{A}^{i}}{\partial t} + (\dot{\vec{r}} \times \nabla \times \vec{A}))$$

$$(\dot{\vec{r}} \times \nabla \times \vec{A})^{i} = \epsilon^{ijk} \dot{r}_j (\nabla \times \vec{A})_k - \epsilon^{ijk} r_j \epsilon_k^{lm} (\partial_l A_m)$$

$\therefore$

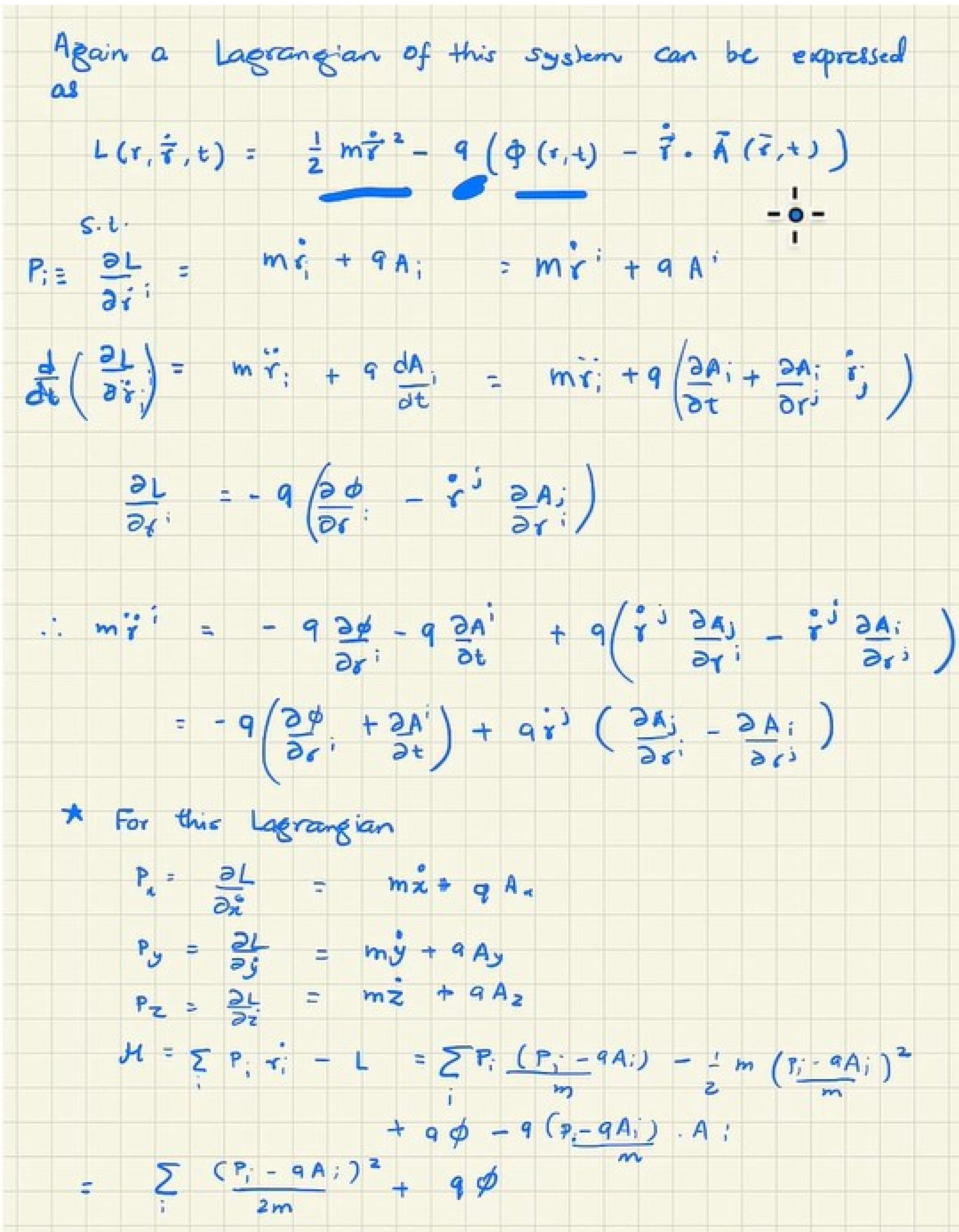
$$\dot{r}_j \epsilon^{r}_{j} \epsilon^{lm} (\partial_j A_j - \partial_j A_i) = \dot{r}_j (\delta^{ik} \delta^{\Im} - \delta^{\Im} \delta^{il}) \partial_l A_m = \dot{r}_j (\partial A_j - \partial A_i) \partial_l$$

$$\therefore m\ddot{\vec{r}}^{i} = -q(\frac{\partial \phi}{\partial x^{i}} - \frac{\partial A^{i}}{\partial x^{t}}) + \dot{r}_j (\frac{\partial A^{i}}{\partial x^{i}} - \frac{\partial A^{j}}{\partial x^{j}})$$

So, having learned that how to quantize fields and how to obtain relevant quantities out of the quantum fields, now we move on to start coupling atoms with quantum fields which are present in nature and we will see how their interaction leads to new phenomena in terms of light matter interaction or quantum field matter interaction. So the toy example, I will start with interaction of atoms, quantum mechanical systems with electromagnetic fields present in the nature. In the background, there is a field which is electromagnetic field, which now will be treated not classically but quantum mechanically as well. So both the systems are quantum, quantum system, quantum atom and quantum fields talking to each other. Let us first write down the equations of motions and relevant Lagrangians which will give rise to relevant Hamiltonian which would be useful for studying their interaction. In terms of electromagnetic field interaction, we can think of a charged particle which is talking to the fields and we have a equation of motion of the kind which we know from Newtonian or electromagnetic interaction is that the acceleration generated at least for non-relativistic particles is proportional to the forces acting on the charged particle. One component of the force is the electric field force which talks to the charge of the particle and gives rise to this qe kind of term. And then there is another set of term which is qv cross B where v is here written as $\dot{r}_i$. So, this kind of terms as we have seen are usually present in the loading force description which we have considered previously as well. Now, in terms of electromagnetic potentials, electromagnetic four vector potential, I can write down the electric field as a gradient of a potential – $\partial \vec{A}/\partial t$, the vector potentials temporal derivative. Remember in electrostatic this term would not have been around. In the electrostatic, the $\partial \vec{A}/\partial t$ term would not have been around and only the gradient term would have been there. But since we are considering electrodynamic or electromagnetic field theory, the dynamics in electric field has to be set up which is generated through the $\partial \vec{A}/\partial t$ term which is present over here. All right, the magnetic field remains as the previous version which is the $\nabla \times A$ of a vector potential which is this. Now, we can see that we have over all the equations of motion written in terms of all the vector potentials and the scalar potential as this, where I have a gradient term then there is a electric field term which is also made up of $\partial \vec{A}/\partial t$ and then there is a $v \times \vec{B}$ term which I have written as $\dot{r}_i$ term over here and the magnetic field term which is a curl equation $\nabla \times \vec{A}$ now this is a vector equation a vector acceleration is coupled to or is written in terms of vector forces now we can have a different comp we can have different components of these in terms of three acceleration components which I am going to call $\dot{r}_i$, $m\ddot{r}_i$ stands for 1, 2, 3, the three components of acceleration. They will be obtained from the x component for example I is equal to 1 which is x component of the acceleration will be obtainable from the x component of the gradient of the scalar potential – the x, the temporal derivative of the x component of the vector potential $\partial A_x/\partial t$ + the x component of the whole $qv \times B$ term. So, $v \times B$ is a scalar This is a vector product, cross product which in terms of the vector potential becomes $\dot{r}_i$ which is $v \times \vec{\nabla} \times \vec{A}$. And the whole vectors X component should be selected in order to communicate it to the X component of the acceleration. This is just a notation wise issue. There is a compactor way of writing this which is in terms of Levy-Civita tensor which we have previously discussed as well. So if I have to obtain the x component of the cross product of two vectors, suppose this is vector A and this is vector B. If two vectors have to be taken across product, I can write it in terms of Levy's Civita beta tensor ε_{ijk}. And if I have to get the ith component over here, I will put that ith component as the first index of the Levy's Civita beta symbol and the second or third component j and k will appear with the first vector in the cross product and then the second vector in the cross product this is the first vector which is $\dot{r}_i$ its index will become j and the third vector is $\vec{\nabla} \times A$ its index will become k and this is this k which gives rise to this $\varepsilon_{klm}{}^{\delta lm}$

Okay so now we can see we have two Levy's Civita tensors coming about, ε_{ijk} coming from the first cross product which we have written and then ε_{klm} coming from the second cross product of this. Let me clean it up so that we can see it from clarity. $\varepsilon_{ijk}\ \varepsilon_{klm}$ is equivalent to δ^i , the first index here should be the first index which is not summed over there l and the second index should be j and m. So, I should be l and j should be m – I should be m and j should be l, the reverse version. The first index which is not summed over becomes the last index which is not summed over in the second term and vice versa. So, this identity is handy. Because using this identity, we can write down the overall term which is appearing as $\dot{r}_j\varepsilon_{ijk}\ \varepsilon_{klm}\ \delta_{il}\delta l_m$ as we can subsume the two Levy's Civita beta tensors in terms of product of chronicle delta. $\delta_{il}\ \delta_{jm} - \delta_{im}\ \delta_{jl}$ and you see these will give as to two terms. First term this will just replace l with i and j with m with j so I will have a $\delta_i{}^{aj}$ and the second chronicle delta product will do the reverse the first index of the derivative will be replaced with j from this δ_j and the last index m will be replaced by I from this δ_{im} so therefore I have a δ_{jj} so now we have all the things in our hand we have the For ith component of the acceleration is equal to – q times these two terms where this first gradient with respect to ith coordinate, then the temporal derivative of the ith component of a vector potential and the ith component of the cross product between v and $\vec{\nabla} \times A$ which is this term. So, we have all the components of the, all the components of the fourth term which we have written previously and this is the compact notation of writing various component of the process. So, let us go forward and try to ask which Lagrangian gives rise to this kind of force term, force equation. This will be handy because we want to go to the Hamiltonian of the system and therefore quantize the system in the canonical approach. So, I just again like before propose a Lagrangian which gives rise to this equation of motion. Again, you can ask from where I can write, I do write this Lagrangian. In this course, we are not discussing how to write consistent Lagrangians corresponding to interactions, but anyway you can see it later on that all the symmetries which we have required so far which is consistency with Lorentz transformations will allow for a moderate, a Lagrangian of a moderate kind which would be useful and the simplest Lagrangian in that setting will be this Lagrangian which we are just discussing as of now. So, let us first do it the analysis and then we will see its covariant form in the relativistic picture. So, the Lagrangian which I am proposing is just the kinetic term which is the usual – the potential term is of a particular nature where I have a q times the potential $q\phi$ and then there is $\dot{A}$ product between the velocity and the vector potential $q\dot{r}_i a$. This is the dot product between the velocity and the vector potential.

First we obtain the conjugate momenta corresponding to the various components of the velocities, the various components of the configuration space variable which is r, r like our q. So, $\partial L/\partial \dot{q}$ there are three *q's*,r_1 ,r_2, r_3, *x, y,* z. So, with respect to i[th] position vector, ith component of the position vector, the momentum will be defined as pi. And you take the Lagrangian and obtain the partial derivative with respect to $\dot{r}_i$. is appearing in two terms.



Again a Lagrangian of this system can be expressed as

$$L(r,\dot{\vec{r}},t) = \frac{1}{2}m\dot{\vec{r}}^2 - q\left(\phi(r,t) - \dot{\vec{r}}\cdot\vec{A}(\vec{r},t)\right)$$

s.t.

$$P_i \equiv \frac{\partial L}{\partial \dot{r}^i} = m\dot{r}_i + qA_i = m\dot{r}^i + qA^i$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}_i}\right) = m\ddot{r}_i + q\frac{dA_i}{dt} = m\ddot{r}_i + q\left(\frac{\partial A_i}{\partial t} + \frac{\partial A_i}{\partial r^j}\dot{r}_j\right)$$

$$\frac{\partial L}{\partial r^i} = -q\left(\frac{\partial \phi}{\partial r^i} - \dot{r}^j\frac{\partial A_j}{\partial r^i}\right)$$

$$\therefore\ m\ddot{r}^i = -q\frac{\partial \phi}{\partial r^i} - q\frac{\partial A^i}{\partial t} + q\left(\dot{r}^j\frac{\partial A_j}{\partial r^i} - \dot{r}^j\frac{\partial A_i}{\partial r^j}\right)$$

$$= -q\left(\frac{\partial \phi}{\partial r^i} + \frac{\partial A^i}{\partial t}\right) + q\dot{r}^j\left(\frac{\partial A_j}{\partial r^i} - \frac{\partial A_i}{\partial r^j}\right)$$

★ For this Lagrangian

$$P_x = \frac{\partial L}{\partial \dot{x}} = m\dot{x} + qA_x$$

$$P_y = \frac{\partial L}{\partial \dot{y}} = m\dot{y} + qA_y$$

$$P_z = \frac{\partial L}{\partial \dot{z}} = m\dot{z} + qA_z$$

$$\mathcal{H} = \sum_i P_i\dot{r}_i - L = \sum_i P_i\frac{(P_i - qA_i)}{m} - \frac{1}{2}m\left(\frac{P_i - qA_i}{m}\right)^2 + q\phi - q\frac{(P_i - qA_i)}{m}\cdot A_i$$

$$= \sum_i \frac{(P_i - qA_i)^2}{2m} + q\phi$$

Again a Langrangain of this system can be expressed as

$$L(r,\dot{\vec{r}},t) = \frac{1}{2}m\dot{\vec{r}}^2 - q(\phi(r,t) - \dot{\vec{r}}\cdot\vec{A}(\vec{r},t))$$

$$p_i \equiv \frac{\partial L}{\partial \dot{r}^i} = m\dot{\vec{r}}^i + qA^i$$

$$\frac{\partial}{\partial t}\frac{\partial L}{\partial \dot{\vec{r}}_i} = m\ddot{r}_i + q\left(\frac{\partial A_i}{\partial t}\right) = m\ddot{r}_i + q\left(\frac{\partial A_i}{\partial t} + \frac{\partial A_i}{\partial t} r^j\right)$$

$$\frac{\partial L}{\partial r^i} = -q\left(\frac{\partial \phi}{\partial r} - \dot{r}^j \frac{\partial A}{\partial r^i}\right)$$

$\therefore$

$$m\ddot{r}^i = -q\frac{\partial \phi}{\partial r^i} - q\frac{\partial A^i}{\partial t} + q\left(\dot{r}^j \frac{\partial x_j}{\partial r^j} - \dot{\vec{r}}^j \frac{\partial A_i}{\partial t}\right)$$

$$= \quad -q\left(\frac{\partial \phi}{\partial r^i} + \frac{\partial A^i}{\partial t}\right) + q\dot{r}^j\left(\frac{\partial x_j}{\partial r^j} - \frac{\partial A_i}{\partial t}\right)$$

⋆ For the Langrangian

$$p_x = \frac{\partial L}{\partial \dot{x}} = m\dot{x} + qA_x$$

$$p_y = \frac{\partial L}{\partial \dot{y}} = m\dot{y} + qA_y$$

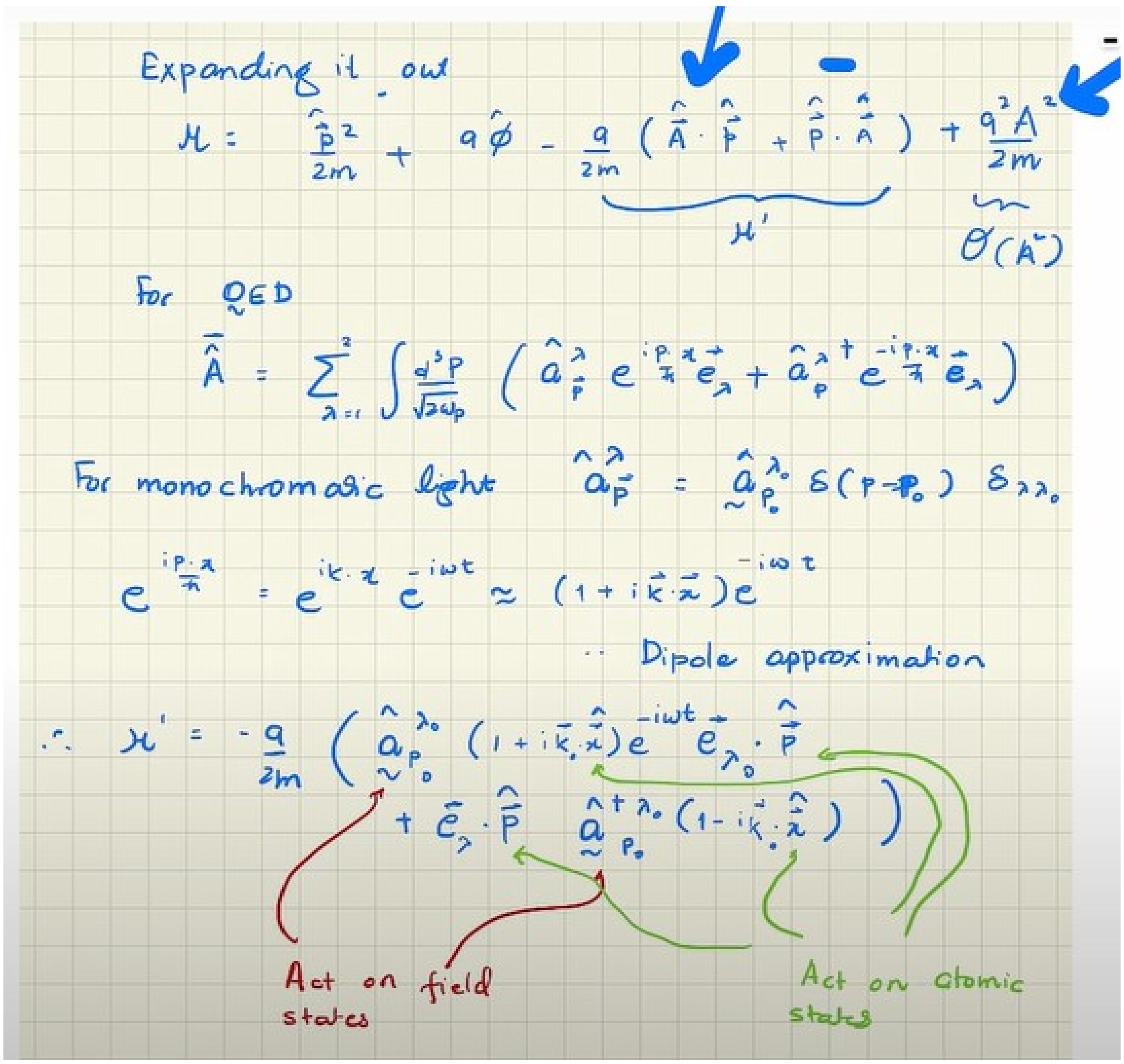
$$p_x = \frac{\partial L}{\partial \dot{z}} = m\dot{z} + qA_z$$

$$H = \sum_i p_i \dot{r}_i - L = \sum_i \frac{p_i(p_i - qA_i)}{m} - \frac{1}{2}\left(\frac{p_i - qA_i}{m}\right)^2 + q\phi - q\frac{p_i - qA_i}{m}A_i$$

$$= \quad \sum_i \frac{(p_i - qA_i)^2}{2\text{m}} + q\phi$$

First it is appearing in the kinetic term over here. It is made up of $\dot{r}_1^2 + \dot{r}_2^2 + \dot{r}_3^2$. So, $\dot{r}_i$ is hiding here as well as in this term where also I have a $\dot{r}_q$ times $A_x + \dot{r}_2 A_y + \dot{r}_3 A_z$ with a – sign after all. So, you can see that overall if I do the partial derivative of the Lagrangian with respect to $\dot{r}_i$, then I will get m$\dot{r}_i$ from the kinetic term and + qA_i There is a – sign here and there is a – sign afterwards alongside outside as well. There are overall there will be A. So, I will have a $m\dot{r}_i + qA_i$ that would be the conjugate momenta conjugate to r_i. Now, the equation of motion will be obtained from Euler Lagrangian equation which is $\partial/\partial t$ of the momenta. So, $\partial/\partial t(\partial L/\partial \dot{r}_i)$. Take the time derivative of this equation over here. The first term will be just the acceleration term $m\ddot{r}_i$ and then the total derivative of q times A_i which becomes q times $\partial A_i/\partial t$. This is the total derivative. Therefore, the total derivative can be written in terms of partial derivative with respect to t and then the chain rule, the partial derivative of A_i with respect to r_j and then dr_j/dt which is written as $\dot{r}_j$. So, this would be the first term in the Euler Lagrangian equation, left hand side term. Right hand side term in the Euler Lagrangian equation is $\partial L/\partial r_i$ which again I will search for where is the occurrence of r_i in the Lagrangian. r_i appears in the potential over here, in the scalar potential over here and in the vector potential over here. So, there are two terms which contain the information of position, one is the scalar potential and one is the vector potential. So, both these terms will undergo the derivative with respect to r_i. So, first term I will get $\partial\phi/\partial r_i$ and the second term I will get just $\dot{r}_i\partial A/\partial r_i$. I have written the product between velocity vector and the a vector as $\dot{r}_j A_j$ and j is summed over and the partial derivative with respect to i, r_i is just going to hit the A_j part of it. Remember this j and upper j and lower j are appearing in the diagonal and they are summed over. So, now both the terms of the Euler Lagrangian equation is obtainable.

Now we can equate this to the equation over here $\partial L/\partial r_i$ and overall what I can do, I can take the extra term from the left hand side which is the $q\partial A_i/\partial t + \partial A_i/\partial r_j\ \dot{r}_j$ to the right hand side. So, I bring it on the right hand side. So, I see there are certain kind of structure which appears in the Lagrangian equation of motion. Right hand side already had a $\partial\phi/\partial r_i$ term and A – $q\partial A_i/\partial t$ terms comes from the left hand side. So, this term over here comes to the right hand side and becomes a – $q\partial A_i/\partial t,$ which is fine on the right hand side already I had $a + q$ – here + qr_j And $\partial A_j/\partial r_i$. So, $\dot{r}_j\partial A_j/\partial r_i$ which was already there on the right hand side. Now the remaining last term which is present in the left hand side would be just transported on the right hand side to become – of $q\dot{r}_j\partial A_i/\partial r_i$. This is the last term which has appeared. Now, compare the equation of motion we just wrote above. This is the same equation of motion which we discussed for a charged particles motion. So, you see we have obtained a Lagrangian which gives rise to our usual familiar electromagnetic field interaction of a charged particle.



Expanding it out,

$$H = \frac{\hat{p}^2}{2\text{m}} + q\hat{\phi} - \frac{1}{2\text{m}}\underbrace{(\hat{\vec{A}}\cdot\hat{\vec{P}} + \hat{\vec{P}}\cdot\hat{\vec{A}})}_{H'} + \underbrace{\frac{q^2A^2}{2\text{m}}}_{\Theta(\vec{A})}$$

For QED

$$\hat{\vec{A}} = \sum_{\lambda=1}^{2} \int \frac{d^3 \vec{p}}{\sqrt{2\omega_{\vec{p}}}} (\hat{a}_{\vec{p}}^{\lambda} e^{ip_{\vec{k}} x} \vec{e}_{\lambda} + \hat{a}_{\vec{p}}^{\lambda \dagger} e^{-ip_{\vec{k}} x} \vec{e}_{\lambda})$$

For monochromatic light hat

$$a_{\vec{p}}^{\lambda} = a_{\vec{p}_0}^{\lambda_0} \delta(p - p_0) \delta_{\lambda \lambda_0}$$

$$\therefore H' = \frac{-q}{2\mathrm{m}} (a_{\vec{p}_0}^{\lambda_0} (1 + i \vec{k}_0 \cdot \vec{x}) e^{-\omega t} \vec{e}_{\lambda_0} \cdot \hat{\vec{p}} + \vec{e}_{\lambda_0} \cdot \hat{\vec{p}} a_{\vec{p}_0}^{\dagger \lambda_0} (1 - i \vec{k}_0 \cdot \vec{x}))$$

So, this Lagrangian will be useful as we discussed because we can write down the Hamiltonian of the system from this Lagrangian. Again, how do I do that? I have all the three momenta Px, Py, Pz, which is $\partial L/\partial \dot{x}$, $\partial L/\partial \dot{y}$, $\partial L/\partial \dot{z}$. It is just elaborate form of $\partial L/\partial \dot{r}_i$ which I had written previously. We know it will be of the structure $m\dot{r}_i + qA_i$ that means $m\dot{x} + qA_x$, $m\dot{y} + qA_y$ + mj dot + qA_z. This would be the three momenta. And the Hamiltonian which can be obtainable from this momenta would be summation over p_i and $\dot{r}_i - l$, px $\dot{x}$ + py $\dot{y}$ + pz $\dot{z} - l$ and at the end of the day all the dots have to be converted into momenta. So, this is what we do, p_i I have converted all the dots which is $\dot{x}$, $\dot{x}$ is nothing but px – qA_x/m that is $\dot{x}$. So, p$\dot{x}$ = px· $\dot{x}$ will be $px \cdot px - qA_x$/m. Similarly $py \cdot \dot{y}$ will be $py \cdot py - qA_y/m$. And $pz \cdot \dot{z}$ will be $pz \cdot pz - qA_j /m$. Collectively I am writing as $p_i \cdot p_i - qA_i/m$ and summation over i. This is the representation of this term and the Lagrangian also had $\dot{A}$ in it. Remember Lagrangian had two places where dots were appearing, one was in the kinetic term and one was in the potential term. That is what I do, all the dots I replace with $p_i - qA_i$/m, this becomes the first term over here becomes the^2 of this because this was $\dot{r}_i^2$. And the second term over here was just $\dot{r}_i A_i$ So, this $\dot{r}_i$ is just $q \cdot p_i - qA_i$/m$\cdot A_i$ Okay, let me clean it up once we have understood where are the terms coming from. Now, you can see the Hamiltonian which we were writing has a very neat structure. Actually, you can collect all the You can collect all the terms which are $p_i - qA_i$ common and you will see that ultimately this whole expansion is nothing but $(p_i - qA_i)^2$ divided by twice m and sum over I + $q\phi$. So, this $q\phi$ which is appearing as a solitary term independent of p remains like it is and all the other p_i dependent terms make up for a perfect^2 make up for a perfect^2 kind of term which is available here. So, this is how the Hamiltonian of a non-relativistic system talking to electromagnetic field is written up as written about. Now, what we can do? We can just open the Hamiltonian up the whole square term which is appearing over here and then try to see if I quantize both the quantum particle that means I will quantize the p_i which is belonging to the quantum particle, position x_i, y_i, z_i which belongs to the quantum particle and the electromagnetic field that means the potentials A_i and ϕ both of them belong to the quantum field. If I quantize all of them what happens to the whole Hamiltonian. So let us open the whole term. I have a First term of the Hamiltonian is $(p_i - qA_i)^2/2m$. Open it up, you will get a $p^2/2m + q^2A^2$ by $2m$ which is the two^2 terms and then there is cross term p_i – twice of $p_i\ qA_i$ divided by $2m$. But that is the classical If they are quantum operators, two times of term will come out, one with $\dot{A}\ p$ and other is $p\dot{A}$. Unless these two things commute, we do not know how to write them as a common term, so I will keep it like that, $\dot{A}\ p + p\dot{A}$. As we had discussed previously as well, in quantum mechanical systems, let us not jumble the order, let us keep the order in which they appear. So, this^2 term which we traditionally remember from the classical expression of $(x + y)^2$ is $x^2 + 2xy + y^2$ is actually $x^2 + xy + yx + y^2$. So, therefore, this xy + yx in our context would be written as $\dot{A}\ p + p\dot{A}$ and these two things may not necessarily be the same thing. So, therefore, I will keep it as they are. At this stage, if I say that the electromagnetic field which is set up in the background is very weak, that means potentials are not very large, then the a^2 term which is appearing as the last term over here can be thought to be very, very weak. If I enforce that a is very weak, the vector potential and the scalar potential themselves are very weak, that means their^{2}d versions. So therefore, I should be legitimately be eligible to drop off this term from the full Hamiltonian saying that it will just cause insignificant changes into our description.

Leading order changes from the quantumness of electric field and the quantum particle both, electromagnetic field and quantum particle both is captured by this term q by $2m\ \dot{A}\ p + p\dot{A}$ where a and p both are operators. So, this I am going to define as a leading order perturbation Hamiltonian. This would be the second order perturbation Hamiltonian, but right now we are just contained with a leading order perturbation theory. So, I would not worry without with much of a cause concerned because this indeed would cause some trouble, but not at the leading order. It will be very subdominant kind of correction this can bring about. And next I know how to write the quantum operators p for quantum particles. Its representation is $i\hbar\partial/\partial x$ in position space. And I know how to write the operators a over here in quantum electrodynamics which we have just done. We know how to write down the vector potential. Vector potential are written in terms of operators which come with not only the knowledge of the momentum, but also with the knowledge of the polarization vector a lambda. And there is a vectorial character to the field. A field is a vector field, so therefore there is a basis E lambda which accompanies the field description along. So, this would be the full description up to leading order of a quantum particle talking to electromagnetic field. Now, for a special case we can think of electromagnetic field which is monochromatic in light. That means I am artificially going to impose a condition. This is not very natural condition, but let us just for sake of discussion, if I impose that This a lambda p operator is just present only for a particular momenta p_0. Suppose there is no field modes excited or anything at any other momenta apart from p_0, I can write down this a lambda p to be made up only with one momenta p_0. So, therefore, in this integration, if I write down this a lambda p as a $\lambda_0 p_0$, a new operator, a tilde operator, a lambda tilde, see there is a tilde in the denominator, in the, below the a symbol, so that they become dimensionally different and delta function takes care of the game that it is only present at one momenta and this delta function, Kronecker delta just identifies which direction it is polarized along. So, this is a very artificial kind of arrangement I am just discussing about that only in a particular direction the vector potential is pointing. That means electric field and magnetic fields direction, I know which direction we are talking about. And I want to say that I am talking about a monochromatic light. That means only one wavelength is present. That this integration will be replaced by just one number. In this whole integration only one term survives with momenta p_0 and with polarization λ_0. And in this case, if that is the case, then the term which is accompanying it along $e^{ip\dot{x}}$, which can be broken in terms of space and time part, $e^{ik\dot{x}}$ and $e^{-i\omega t}$, which is just decomposition of this 4 dimensional inner product. And then what I can impose suppose my momenta which I had the only momenta which is present in the field is very big wavelength that means k is very small. If k is very small then e^{ikx} can be approximated as $1 + ik\dot{x}$. This expansion of exponential function gives rise to $(1+ ik\dot{x} + ik\dot{x})^2/2$, but it higher order terms will become higher powers of k and k is supposed to be very small. So, I can truncate the series up to leading order which is called the dipole approximation. Actually, the zero-thotadipole approximation is just dropping $k\dot{x}$ all together. But if you wanted to have some information of wavelength into the game, I will just replace this $e^{ik\dot{x}}$ with $1 + ik\dot{x}$. And k belongs to the same wavelength where the integration has survived. So, it will become k dot, k0 $\dot{x}$. So, therefore, $e^{ip\dot{x}}$ was appearing which will become $1 + ik_0\dot{x}$ and $e^{-i\omega 0t}$. So, ω also is at the same frequency. And then e lambda not 1 polarization survives dotted with the p, p dot product of p is coming from here. So, the first term which is appearing over here becomes this whole operator in the dipole approximation. The first term, the first whole term is the $\dot{A}p$ term. Similarly, if I write down the $p\dot{A}$ term, this time the dot product the p will appear on the left as it is present over here and the a will appear on the right hand side and it I have to write in terms of this thing I have to write in terms of the a and $a^{\dagger}$s again. So, ultimately I would write down this $a + a^{\dagger}$, $a+ a^{\dagger}$ term which is appearing over here. So, this whole term which I have written as a $\lambda_0 p_0$ is actually the statement of the whole a with a dipole approximation. Similarly, when I have written a p_0 dagger that is the whole integration with respect to with our delta function in between which converts the integration over here into just sum of two terms. All those two terms I am collectively writing over here. So, there is one more term in principle which is made up of a λ_0, a $\lambda_0 p_0$ and $1 + ikxe^{i\omega t}$. So, I have just written it figuratively as of now. I am not going to do computations with this. I am just telling you if I just open it up, do the dipole kind of approximation, I get various kinds of terms in the Hamiltonian. I have not written all the terms because in the realistic setting which we are going to discuss later on, I will cleanly write all the Hamiltonian term. But you can see an emergent structure which is present in the Hamiltonian over here. You see all the terms have a character that There are certain operators, there is a operator A which is belongs to the quantum field and then operator p which belongs to the atomic set. So, there are mixing of operators, some operators acting on the field states and some operator which act on the atomic states. So, I have a two party system now, one state which is the state of the atom I have to write down the state of the atom and then there is another part which is the state of the field itself.

So, this is two party quantum system and the operators of both of them appear as a product term in the Hamiltonian. That is the generic character in terms of a light matter interaction Hamiltonian where you will see most of the times operators of this kind, interaction operators of this kind which do appear in the game and you will see that these interactions Hamiltonian would be useful in computing various quantities. We should just wrap up with the discussion of relativistic version of that so that we are just comfortable with dealing with relativistic particles if needed. Most of the times we will be discussing only with non-relativistic particles. Nevertheless, let us go ahead and discuss the structure in terms of relativistic particle. So, in the relativistic setting, the action has to be written in terms of a Lorentz invariant object as we have discussed previously. And one of the most simplest Lorentz invariant object for particles can be written in the following terms. I write down the first term in terms of the total path covered by the particle, momenta mc in some sense, some units of momenta times the path cover $\partial\tau$.

$$S = -mc\int d\tau - q\int d\tau\, A_\mu \frac{dx^\mu}{d\tau}$$

$$= -mc\int \sqrt{-\eta_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}}\, d\tau - q\int d\tau\, A_\mu \frac{dx^\mu}{d\tau}$$

$$= -mc\int \sqrt{c^2\left(\frac{dt}{d\tau}\right)^2 - \left(\frac{d\vec{x}}{d\tau}\right)^2}\, d\tau - q\int d\tau\, A_\mu \frac{dx^\mu}{d\tau}$$

$$= -mc^2\int dt\sqrt{1-\frac{\dot{x}^2}{c^2}} - q\int dt\, A_\mu \frac{dx^\mu}{dt}\frac{dt}{d\tau}$$

$$\mathcal{L} = -mc^2\sqrt{1-\frac{\dot{x}^2}{c^2}} - qA_\mu \frac{dx^\mu}{dt}$$

$$\frac{\partial\mathcal{L}}{\partial\dot{x}^i} = \frac{m\dot{x}^i}{\sqrt{1-\frac{\dot{x}^2}{c^2}}} - qA_i$$

$$\frac{d}{dt}\left(\frac{d\mathcal{L}}{\partial x^i}\right) = \frac{m\ddot{x}^i}{\sqrt{1-\frac{\dot{x}^2}{c^2}}} - \frac{\frac{m}{2}\dot{x}^i\left(\frac{-2\dot{x}^i}{c^2}\right)\ddot{x}^i}{\left(1-\frac{\dot{x}^2}{c^2}\right)^{3/2}} - q\frac{dA_i}{dt}$$

$$= \frac{m\ddot{x}^i\left(1-\frac{\dot{x}^2}{c^2}\right) + m\frac{\dot{x}^2}{c^2}\ddot{x}^i}{\left(1-\frac{\dot{x}^2}{c^2}\right)^{3/2}} - q\frac{\partial A_i}{\partial t} - q\frac{\partial A_i}{\partial x^j}\dot{x}^j$$

$$= \frac{m\ddot{x}^i}{\left(1-\frac{\dot{x}^2}{c^2}\right)^{3/2}} - q\frac{\partial A_i}{\partial t} - q\frac{\partial A_i}{\partial x^j}\dot{x}^j$$

$$\frac{\partial\mathcal{L}}{\partial x^i} = -q\frac{\partial A_\alpha}{\partial x^i}\dot{x}^\alpha$$

$$\therefore \frac{m\ddot{x}^i}{\left(1-\frac{\dot{x}^2}{c^2}\right)^{3/2}} = -q\left(\frac{\partial A_0}{\partial x^i} + \frac{\partial A_j}{\partial x^i}\dot{x}^j\right) + q\frac{\partial A_i}{\partial t} + q\frac{\partial A_i}{\partial x^j}\dot{x}^j$$

$$S=\int d\tau-\int d\tau A_\mu \frac{\partial X^\lambda}{d\tau}$$

$$= \quad -mc\int\sqrt{-\eta_{\mu\nu}\frac{dX^M}{d\tau}\frac{dX^\mu}{d\tau}\frac{dX^\nu}{d\tau}}d\tau-q\int d\tau A_\mu\frac{dX^\mu}{d\tau}$$

$$= \quad -mc\int\sqrt{c^2(dt/d\tau)^2-(\frac{d\vec{X}}{dc})^2}dc-q\int d\tau A_\mu\frac{dX^\mu}{d\tau}$$

$$= \quad -mc^2\int dt\sqrt{1-\frac{\dot{x}^2}{c^2}}-q\int dt A_\mu\frac{dX^\mu}{dt}\frac{dt}{d\tau}$$

$$L=-mc^2\sqrt{1-\frac{\dot{x}^2}{c^2}}-qA_\mu\frac{dx^m}{dt}$$

$$\frac{d}{dt}(\frac{\partial L}{\partial x^i})=\frac{-m\ddot{x}^i}{\sqrt{1-\dot{x}^2/c^2}}-\frac{\frac{m}{2}\ddot{x}^i(\frac{-2\mathrm{x}^i}{c^2})}{(\sqrt{1-\dot{x}^2/c^2})^{3/2}}-q\frac{dA_i}{dt}$$

$$= \quad \frac{m\ddot{x}^i}{(\sqrt{1-\dot{x}^2/c^2})^{3/2}}-q(\frac{\partial A_i}{\partial t}-\frac{\partial A_i}{\partial x^j}\dot{x}^j)$$

$$\frac{\partial L}{\partial x^1}=-q\frac{\partial A_i}{\partial x^i}\dot{x}^\alpha$$

$$\therefore\frac{m\ddot{x}^i}{(\sqrt{1-\dot{x}^2/c^2})^{3/2}}=-q(\frac{\partial A_0}{\partial x^1}+\frac{\partial A_i}{\partial x^j}\dot{x}^j)+q\frac{\partial A_i}{\partial t}+q\frac{\partial A_i}{\partial x^j}x^j$$

Remember $\partial\tau$ is, $\partial\tau^2$ is $-\eta_{\mu\nu}\,\partial x_\mu\,\partial x_\mu$. So, this $\partial\tau$ is the *root* of $-\eta_{\mu\nu}\,\partial x_\mu\,\partial x_\mu$ which is the path length. Then there is a term which is motion of a charged particle q in presence of a_μ and there is a coupling between the a_μ and the velocity of the particle $\partial x_\mu/dt$. If I just forget about the second term for a moment, The first term is just the path length information, the dynamical quantity is path length, *m* and *c* are fundamental constants in some sense, not fundamental but constants. So, only thing which changes along with the path is the path length. So, if I just forget about the extra term, then if suppose I am talking about a free particle, not talking to any electromagnetic field, then the second term will not be there and the first term has to be extremized in order to obtain the equation of motion. So, therefore, what will get extremized is the path length. So, particles, free particles will move on the extremal path, the shortest path, the geodesics. So, this is the geodesic motion path. The second term which I have now put in is just the force due to charged particle moving with velocity $\partial x_\mu/\partial\tau$ in a vector potential A mu. Remember, this is just a natural generalization. I want to have a coupling between the velocity 4 vector and the a_μ vector potential. This is the simplest scalar I can write out of that. $a_\mu\partial x\mathrm{b}/\partial\tau$, this is the local velocity at a time tau let us say and a_μ is the vector 4 potential at that location. So, this is a scalar overall and I integrate it to accumulate all the scalars along its path. So, now in preserves of electromagnetic force, electromagnetic field, the charged particle does not move on a geodesic, it has to be corrected by this much of a force and so on and so forth. So, this is the Lorentz invariant interaction. That is how you write down any Lagrangian of the system which is consistent with the Lorentz symmetries. Now, next what I do, I again write up the $\partial\tau$ in terms of $\sqrt{(-\eta_{\mu\nu}dx_\mu dx_\nu)}$. I divide and multiply the whole term by $\partial\tau$ and $\partial\tau$. So, first this $\partial\tau$ has to be written as $\eta_{\mu\nu}dx_\mu dx_\nu$ Then I multiply and divide by $\partial\tau$. So, one $\partial\tau$ I leave outside and another one by $\partial\tau$ I sneak in into the root term. And into the root it becomes $\partial x_\mu/\partial\tau$ and $\partial x_\nu/\partial\tau$. So, therefore this become velocity 4 vectors. The second term I just leave it alone. I do not do anything to it. See what happens. In the interior of the root , this contraction is there. $_\mu$and nu are supposed to take all possible values. But remember, $\eta_{\mu\nu}$ in Cartesian coordinate system can be written as a matrix whose only diagonal entries are non-zero. – 1, 1, 1, 1. Other thing, everything else is 0. So, I will have only four terms surviving. The first term of that would be η_{00} which is – 1. So, overall – of η_{00} is + 1 and then $\partial x_0\ \partial x_0/\partial\tau$ which is $c^2\ \partial t^2/\partial\tau^2$. So, that is just the η_{00} term. The remaining three diagonal terms are the velocity, spatial velocity vector square $\partial x/\partial\tau$ whole square. So, the whole term, the$\sqrt{}$ term can be written in terms of these. Again as before I have left the interaction between the particle and the electromagnetic field as untouched. Now what I do? In the next step, I pull out this $c^2\ \partial t/\partial\tau$ *whole square*. When I pull it out, I will get a $c\cdot\partial t/\partial\tau$. One extra c I supply here and c^2 will be there and $\partial t/\partial\tau$, I cancel that $\partial\tau$ which is appearing here. So, overall the integration measure becomes ∂t. Since there is a $\partial t/\partial\tau$ and already there was a $\partial\tau$ outside, those two ∂taus cancel each other. The same thing I can do on the second term, which was untouched till now. The derivative which is $d/\partial\tau$ can be written as $d/\partial t\cdot\partial t/\partial\tau$ as a chain rule. And again the same game, this $\partial\tau$ here and this $\partial\tau$ appearing over here can be cancelled out. So ultimately I have a Lagrangian which is – mc^2 integration of $dt\sqrt{(1 - v^2/c^2)}$ and $- q\int dta_\mu\partial x_\mu$. This is the action and the Lagrangian would be just $- mc^2\ \sqrt{(1 - v^2/c^2)}$ $- q\int \underline{d}t\ a_\mu\partial x_\mu/\partial t$. Now note one fact that in this game Here I have obtained the Newtonian velocity not the Lorentz covariance velocity. Lorentz covariance velocity would have been $\partial x_\mu/\partial\tau$. This time I have obtained $\partial x_\mu/\partial t$ because of the chain rule I replaced the derivative with respect to tau into derivative with respect to t. Similarly here $\dot{x}$ here When I pulled out the $c^2\ \partial t^2/\partial\tau^2$, the $\partial\tau,\ \partial\tau$ got cancelled and inside also I had a $(\partial x/\partial t)^2/c^2$. So, this $\dot{x}$ is also Newtonian velocity. So, now I have a Lagrangian which is Newtonian velocity, written in terms of Newtonian velocity, $\dot{x}$ here is a Newtonian velocity magnitude and $\dot{x}_\mu/\partial t$ is $\partial\dot{x}_\mu$, $\partial x_\mu/\partial t$ is the Newtonian 4 velocity. $\partial x_0/\partial t$ will be 1, $\partial x_i/\partial t$ will be the three components of the spatial velocities. So, if once this Lagrangian is given to me, I will proceed ahead and I will write down the Euler Lagrangian equation for motion. First by taking the Lagrangian partial derivative with respect to $\dot{x}_i$, $\dot{x}_i$ is sitting at two places. This v^2/c^2 otherwise $\dot{x}^2/c^2$ is made up of $\dot{x}_1^2 + \dot{x}_2^2 + \dot{x}_3^{\ 2}/c^2$. So, $\dot{x}_i^2$ is sitting, $\dot{x}_i$ is sitting here as well as in this interaction term $\partial x_\mu/\partial t$ is summed over for all mus. So, when I do the partial derivative with respect to $\dot{x}_i$, this$\sqrt{}$ term will undergo derivative which will give me the first term and the second term will survive only when $\dot{x}_i$ will see the corresponding term with A_i multiplication, so qA_i will be the second person to survive. What I have to do further for Euler Lagrangian equation, I have to do a time derivative of this $\partial l/\partial\dot{x}_i$. Do this, this is just a simple algebra you can do, you have to just see where is the time dependency in this term, $\dot{x}_i$, $\dot{x}^2$ and A_i all these three functions are time dependent, m, c and q *a*re not time dependent. So, overall this term will undergo time derivative, the $\sqrt{}$ will undergo a time derivative and A_i will undergo the time derivative and you will get three types of terms. This is again a plain and simple algebra you should be able to do and you can just write down the total derivative of $\partial A/\partial t$ in terms of partial derivative + partial derivative with respect to position times $\partial\dot{x}_j/\partial t$. So, this is just the simple expansion of the partial derivative of partial derivative of A_i with respect to p. A total derivative of A_i with respect to p can be written in terms of two terms. So overall this would be the left hand side of the Euler Lagrange equation. The right hand side is just $\partial L/\partial\dot{x}_i$. I will search in the Lagrangian where is the position

dependency. First term in the Lagrangian has only velocity dependency, no position dependency. Only the second term through a_μ position dependency comes about $a_\mu \partial x_\mu/\partial t$ or a $\alpha\dot{x}$ α $a_\mu \dot{x}_\mu$ whatever way you write. Then the partial derivative will just hit the a because only special derivative position can hit only the a and I will get this term.
What I have to do? I have to just equate the left hand side and the right hand side and you will see I have this structure. Where I have brought in the two terms of the left hand side. The last two terms of the left hand side are also thrown onto the, the last two terms of the left hand side are also thrown into the right hand side. So now you see the terms which are appearing on the right hand side have a peculiar structure in them. You can see I have a $\partial A_i/\partial t$ here and $\partial A_0/\partial x_i$ over here. These two terms I am going to combine and write it like this over here. The remaining terms which comes with xj dot, here I have x_j and last term has $\dot{x}_j$. They do, the two terms also I combine and I get these two terms. And now you can realize that the two terms, one term which is independent of $\dot{x}_i$ and another term which is dependent on $\dot{x}_i$ or $\dot{x}_j$. They have a structure of F_{i0} and F_{ij} respectively. This is just the ijth component of the field strength tensor of electromagnetic theory and this is just F_{i0} component of the $F_{\mu\nu}$. This gets multiplied with 1 which is u_0, remember u_0 was $\partial t/\partial t$, this is Newtonian velocity, so that is why this is 1 and this is xj dot which is a u_j So, ultimately you see the two terms are $F_{i0}u_0 + qF_{ij}u_{ij}$ collectively it is Q_{fI} αu_α where α is being summed over 0 and 1, 2, 3. So, this is the relativistic sort of force law for a relativistic particle which is moving along in presence of electromagnetic field. You see this is almost like the non-relativistic term apart from an extra $(1 - v^2/c^2 \text{ r})^{3/2}$ term over here. So, you see this $(1 - \dot{x}^2/c^2)^{3/2}$ actually is an extra, but that is about it. You can just, if I ignore that term I am just in the non-relativistic limit whose equations of motion we had just seen. So, now we can move forward and write down effectively the Hamiltonian of the system.

$$\frac{m\ddot{x}^i}{\left(1-\frac{\dot{x}^2}{c^2}\right)^{3/2}} = +q\left(\frac{\partial A_0}{\partial x^i} - \frac{\partial A_i}{\partial t}\right) + q\left(\frac{\partial A_i}{\partial x^j} - \frac{\partial A_j}{\partial x^i}\right)\dot{x}^j$$

$$= +q F_{i0} u^0 + q F_{ij} u^j$$

$$= +q F_{i\alpha} u^\alpha$$

From $P_i = \frac{m\dot{x}^i}{\sqrt{1-\frac{\dot{x}^2}{c^2}}} - qA_i$

$$(P_i + qA_i) = \frac{m\dot{x}^i}{\sqrt{1-\frac{\dot{x}^2}{c^2}}}$$

$$(\bar{P} + q\bar{A})^2 = \frac{m^2\dot{x}^2}{1-\frac{\dot{x}^2}{c^2}} \Rightarrow \frac{(\vec{P}+q\vec{A})^2}{m^2c^2} = \frac{\frac{\dot{x}^2}{c^2}}{1-\dot{x}^2/c^2}$$

Componendo dividendo $\frac{(\vec{P}+q\vec{A})^2 + m^2c^2}{m^2c^2} = \left(\frac{1}{\sqrt{1-\dot{x}^2/c^2}}\right)^2$

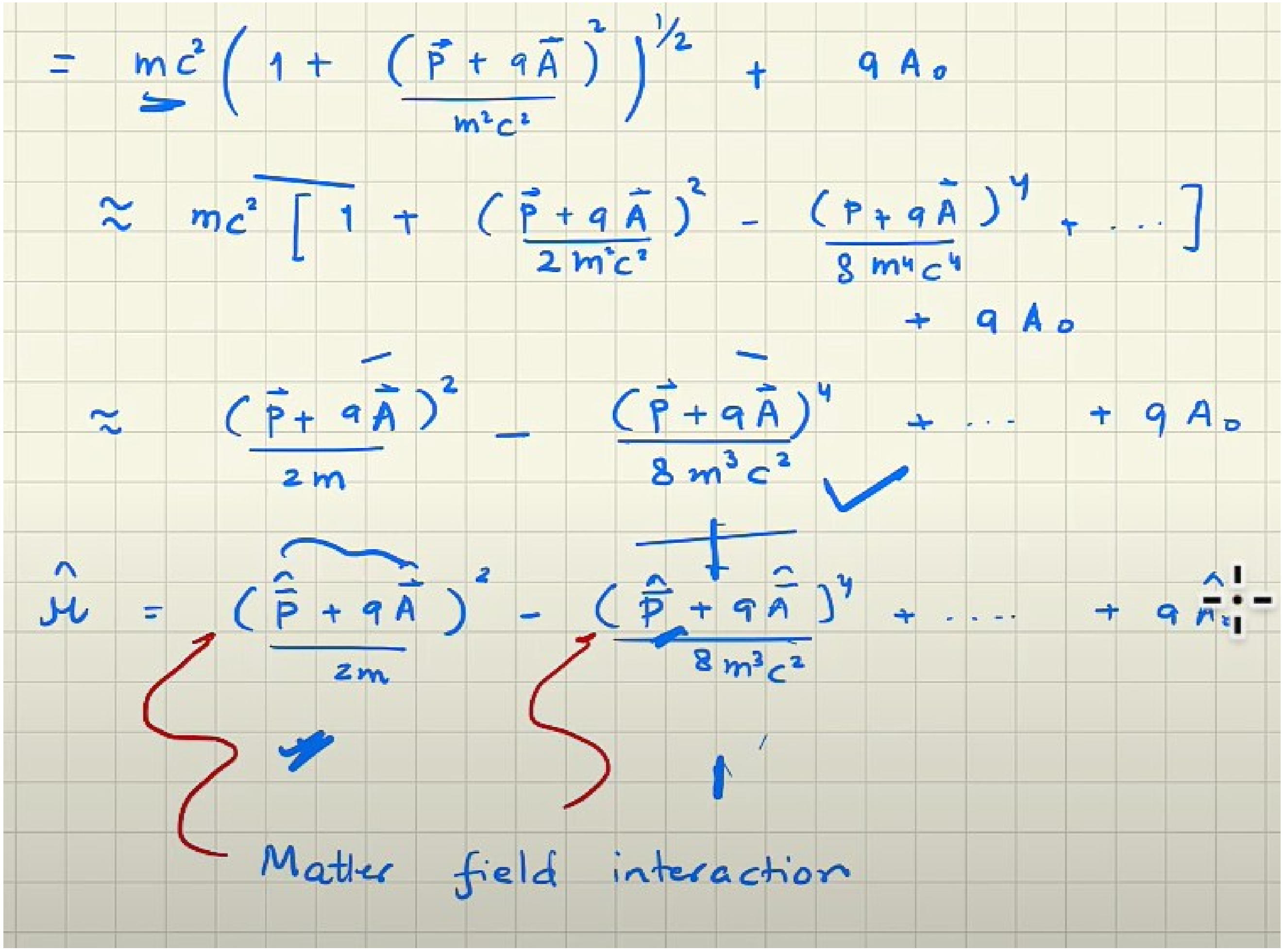
$$H = \sum_i P_i \dot{x}_i - \mathcal{L}$$

$$= \sum_i\left(\frac{m\dot{x}_i^2}{\sqrt{1-\frac{\dot{x}^2}{c^2}}} - qA_i\dot{x}^i\right)$$

$$+ mc^2\sqrt{1-\frac{\dot{x}^2}{c^2}} + qA_\mu\dot{x}^\mu$$

$$= \frac{m\dot{x}^2 + mc^2\left(1-\frac{\dot{x}^2}{c^2}\right)}{\sqrt{1-\frac{\dot{x}^2}{c^2}}} + qA_0$$

$$= \frac{mc^2}{\sqrt{1-\frac{\dot{x}^2}{c^2}}} + qA_0 = mc^2\left(\frac{(\vec{P}+q\vec{A})^2 + m^2c^2}{m^2c^2}\right)^{1/2} + qA_0$$



$$\frac{m\ddot{x}^i}{(\sqrt{1-\dot{x}^2/c^2})^{3/2}} = +q\left(\frac{\partial A_0}{\partial x^i} - \frac{\partial A_i}{\partial t}\right) + q\left(\frac{\partial A_i}{\partial x^j} - \frac{\partial A_i}{\partial t}\right)\dot{x}^j$$

$$= \quad +qF_{io}u^0 + qF_{ij}u^j$$

$$= \quad F_\alpha u^\alpha$$

From $\quad p_i = \dfrac{m\dot{x}^i}{\sqrt{1-\dfrac{\dot{x}^2}{c^2}}} - qA_i$

$$(P_i + qA_i)^2 = \frac{m^2\dot{x}^2}{1-\frac{\dot{x}^2}{c^2}} \Rightarrow \frac{(\vec{p}+q\vec{A})^2}{m^2c^2} = \frac{\frac{\dot{x}^2}{c^2}}{1-\frac{\dot{x}^2}{c^2}}$$

Componendo Dividendo

$$\frac{(\vec{p}+q\vec{A})^2 + m^2c^2}{m^2c^2} = \left(\frac{1}{\sqrt{1-\dot{x}^2/c^2}}\right)^2$$

$$H=\sum_i p_i x_i - L$$

$$= \sum_i (m\frac{\dot{x}_i^2}{(\sqrt{1-\dot{x}^2/c^2})} - qA_i\dot{x}^i) + mc^2(\sqrt{1-\dot{x}^2/c^2}) + qA_\mu\dot{x}^\mu$$

$$\frac{m\dot{x}^2+mc^2(1-\dot{x}^2/c^2)}{(\sqrt{1-\dot{x}^2/c^2})}+qA_0$$

$$= \frac{mc^2}{(\sqrt{1-\dot{x}^2/c^2})}+qA_0$$

$$= mc^2(1+\frac{\vec{p}+q\vec{A}^2}{m^2c^2})+qA_0$$

$$\approx mc^2\left[1+\frac{\vec{p}+q\vec{A}^2}{2m^2c^2}-\frac{\vec{p}+q\vec{A}^4}{8m^4c^4}+...\right]$$

$$\approx \frac{(\vec{p}+q\vec{A})^2}{2m}-\frac{(\vec{p}+q\vec{A})^4}{8m^3c^2}+...+qA_0...$$

$$\hat{H}=\frac{(\vec{p}+q\vec{A})^2}{2m}-\frac{(\vec{p}+q\vec{A})^4}{8m^3c^2}+....+.\,qA_0$$

which are Matter field interaction.

The conjugate momenta corresponding to x_i is already available with us from the $\partial L/\partial \dot{x}_i$ term which is this. What I am going to do?I am going to convert all the $\dot{x}_i$s which are appearing in the Lagrangian in terms of their momenta. I just take this qA_i dot term on the left hand side. So, I will be left with $m\dot{x}_i$/square root term. What I do?This is ith component on the both side. I multiply one more $p_i + qA_i$ on the left hand side. As the result, I will be multiplying $m\dot{x}_i$ the√ quantity on the right hand side and then I sum over all the i's. So, left hand side will become the vector dot product between $p + qA$ which is the magnitude square of *the* $p+q\vec{A}$ and the right hand side will become $m^2\,\dot{x}^2$ divided by $1 - \dot{x}^2/c^2$ which is fine. So, I can take this m^2c^2 as well. So, you can see that I can divide by c^2 multiply by c^2 in the numerator and take m^2c^2 on the left hand side so I will be having p + qA vector mod square $/m^2c^2$ is equal to $v^2c^2\ /(1-v^2/c^2)$ and now we can play simple componendo dividendo kind of term I add the numerator denominator on the top and leave the new denominator in the bottom and this is the structure I will get. I have written $\surd(1-v^2/c^2)$, the whole square. This is just the funny way of writing $1-v^2/c^2$. Why I am doing so, that will become clear in a mi nute. Now, let us look at the Hamiltonian. The Hamiltonian is $p_i(\dot{x}_i)$ – Lagrangian. p_i, I already know is $m\dot{x}_i$ divided by this root factor – qA_i. So, Hamiltonian will be $p_i\dot{x}_i$. So, both the terms here and here will get multiplied with $\dot{x}_i$. These are the two terms and this is the Lagrangian subtracted out now you see again open it up this is again simple algebra you should do it as a homework exercise you open it up you will see three terms a I x I dot gets cancelled in the four terms here so I have $A_\mu\dot{x}_\mu$ this $\dot{A}_0\dot{x}_0$, $\dot{A}_1\dot{x}_1$, $A_2\dot{x}_2$, $A_3\dot{x}_2$, three of them get cancelled from the spatial portion coming from the first term here. I am left only with q term. And the remaining terms combine neatly into just this object mc^2 divided by $1-\dot{x}^2c^2$. Let me erase it up so that things become very visible to us. So, So, the remaining root terms can be combined together out of potential terms, vector potential terms. Many of the terms got cancelled, only one term qA_0 will survive. qA_0 with $\dot{x}_0$, $\dot{x}_0$ in the Newtonian version is 1. And while the root terms can be combined together as you can see and ultimately I will get mc^2 divided by $\sqrt{1-v^2/c^2}$ You can identify this term as the e is equal to mc^2 expression where the gamma factor is an extra. So, this is how the typical e is equal to mc^2 is written. The *m* effective is *m* rest mass, proper mass divided by this loading factor. This is a free particles energy. In presence of electromagnetic fields interaction, what electromagnetic field does is to ascribe an extra energy part into the Hamiltonian which is charged particle talking to the scalar potential A_0.

And now you will realize why I had written this thing as a funny way because $\sqrt{1-\dot{x}^2/c^2}$ is description is needed for writing the Hamiltonian momentum. So, root of the whole quantity on the left is the quantity which is appearing over here. So, you see my Hamiltonian is $m\,c^2$ this Lorentz factor is just $(p + qA)^2 + m^2c^2/m^2c^2$ to the power 1/2. So, this is the Hamiltonian of a relativistic system, relativistic particle. What you can see the same term 1/2, $\sqrt{1/2}$ term can be written as $1 + p + qA/m^2c^2$. This denominated m^2c^2 cancel this m^2c^2 to get you 1 and the second term is $p + (qA/mc)^2$ and overall 1/2 factor and qA_0 term remains as it is. We have seen it before. Previously, we had seen if no electromagnetic field was present, A_0 was not there, a vector a was not there, then it was $1 + p^2/m^2c^2$ 1/2. This is what the kinetic energy kind of expansion we had done before. This time, the role of *p* is played by $p + qA$. So, you see for small momenta, if rest mass is higher compared to the momenta, then I can tailor expand this thing as we had done before as well. Previously, I had obtained rest mass energy $+ p^2/2m - p^4/8m^3c^2$. This time also I am going to get those terms, but *p* replaced by $p + q$. So, you see it has very much similar kind of non-relativistic + expansion, meaning non-relativistic term is just the first term, relativistic correction is this and higher and higher powers will come through, through the expansion of these terms. Only change is that *p* is replaced by $p + qA$. Now you see the structure in relativistic setting as well that if I open it these terms out I will get various terms p^2 and a^2 from the first term but there is a $p\dot{A}$ and $\dot{A}p$ terms also. Now you see the structure in relativistic setting as well that if I open it these terms out I will get various terms p^2 and a^2 from the first term but there is a $p\dot{A}$ and $\dot{A}p$ terms also. Remember again these are individual operators when I go to quantum theory. So I would not write $p\dot{A} + \dot{A}p$ as twice $p\dot{A}$ or twice $\dot{A}p$. I will just keep them along $p\dot{A} + \dot{A}p$. In the second term again, I have $a\ (p + qA)^4$. So, I will have a p^4 term and A^4 term and then there will be various kind of term p^3 and A then *pA*, *Ap* meaning all the factorizations of *p* and *A* together they should add up to four powers. Three powers of *p* and one power of A, two power of *p*, two powers of A, one power of *p* and three powers of A are will be the cross terms and those these can they can be written in terms of as many forms as quantum mechanics allows them meaning I would not club them as one term I would get *p* to the power something and a to the power something or *p* a *p* a a those kind of kind of structure but you see in most of the systems which we have discussed and seen the coupling terms have this structure that one operator belongs to the quantum mechanical set *p* operator for instance here and one another operator which is coming in the cross product belongs to the quantum field set which is the A operator here. So, ultimately in light matter interaction I would have a generic structure like that, that there will be in the effective Hamiltonian or the perturbation Hamiltonian will contain only structure such that two operators one from field side and one from the quantum mechanical side will be talking to each other, they will be cross product with each other. And those operators will hit the state, join the state of the field and the matter. I have now two party systems, two party quantum system, quantum atom and quantum field. They are joint operators, the coupling between them is through one operator from atom, one operator from field talking to each other having cross product. And this perturbation Hamiltonian will evolve the state correspondingly as we have seen. A more representative, illustrative strategy of dealing with that would be done in terms of undue dv detector which in the next class we will see where most simplistic but a very general setup would be considered where one operator from atomic sector and one operator from field sector will be put together as a coupling Hamiltonian and then we will take states and evolve with that perturbation Hamiltonian to see where the states land up ultimately. So this is what we would do in the next class. I stop over here for today.