Foundation of Quantum Theory: Relativistic Approach

Thermal Fields 1.1

Prof . Kinjalk Lochan

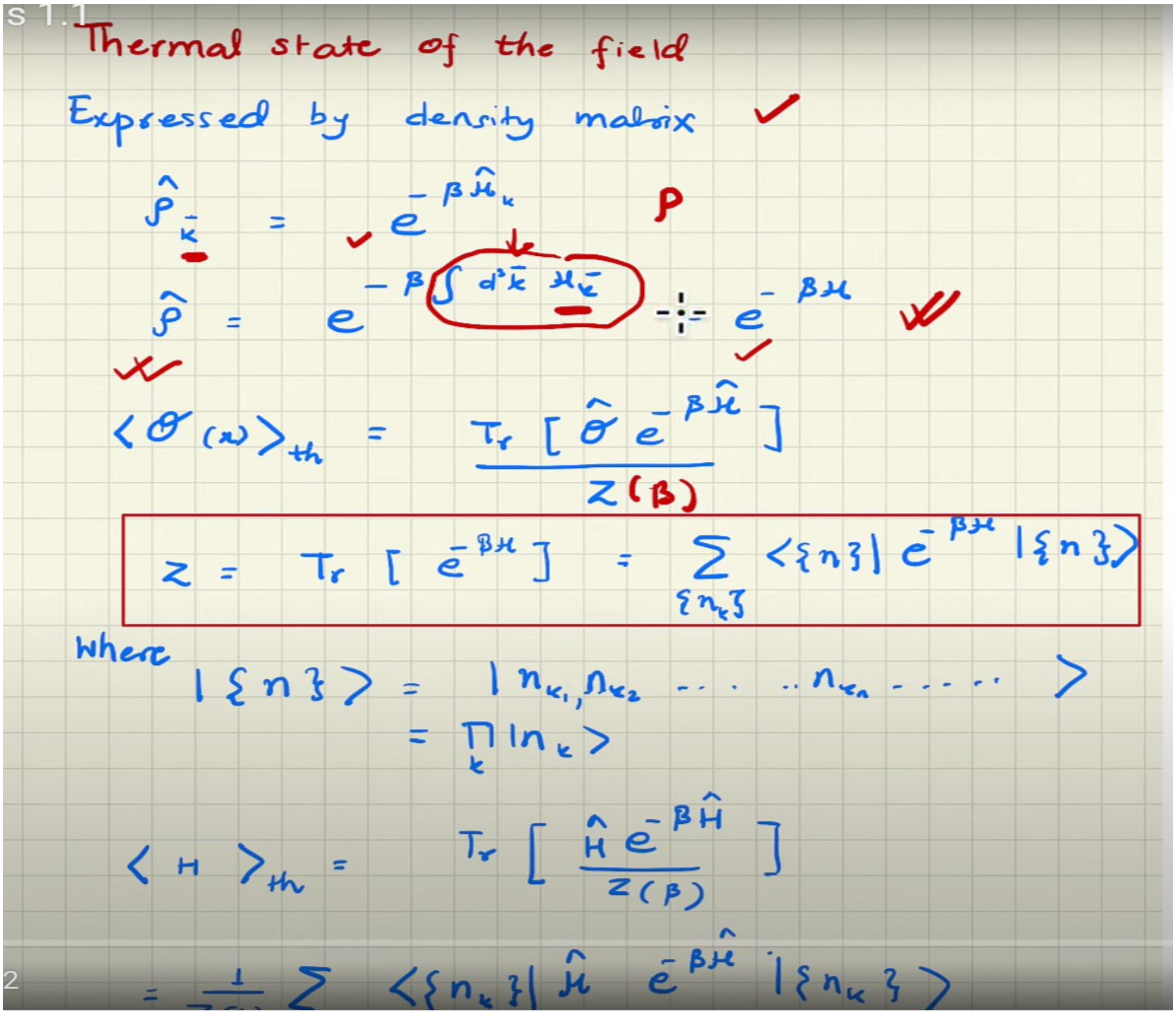
Department of Physical Sciences

IISER Mohali

Scalar field thermal state

Lecture- 31

So, today we will be discussing about the aspects of thermal state of a quantum field.



$$\hat{H} |\{n_k\}\rangle = \int d^3\bar{k} \; n_k \; \omega_k \; |\{n_k\}\rangle$$

$$e^{-\beta\hat{H}} |\{n_k\}\rangle = e^{-\beta \int d^3\vec{k} \; n_k \omega_k} |\{n_k\}\rangle$$

Thermal state of the field

Expressed by density matrix

$$\vec{\rho}_{\vec{k}} = e^{-\beta \hat{H}_k}$$

$$\vec{\rho} = e^{-d^3\vec{k} H_{\vec{k}}} e^{-\beta H}$$

$$\langle \theta(x) \rangle_{th} = \frac{Tr[\hat{\theta} e^{-\beta H}]}{Z(\beta)}$$

$$Z = Tr[e^{-\beta H}] = \sum_{\{n_k\}} \langle \{n\} | e^{-\beta H} | \{n\} \rangle$$

Where $|\{n\}\rangle = |n_{k_1}, n_{k_2} \ldots \ldots n_{k_n}\rangle$

$$= \prod_k |n_k\rangle$$

$$\langle H \rangle_{th} = \frac{Tr[\hat{H} e^{-\beta H}]}{Z(\beta)}$$

$$= \frac{1}{Z(\beta)} \sum_{\{n_k\}} \langle \{n_k\} | e^{-\beta H} | \{n_k\} \rangle$$

$$\hat{H} |\{n_k\}\rangle = \int d^3\vec{k} \, n_k \omega_k |\{n_k\}\rangle$$

$$e^{-\beta \hat{H}} |\{n_k\}\rangle = e^{-\beta \int d^3\vec{k} n_k \omega_k} |\{n_k\}\rangle$$

Remember, we started our discussion in this subsection regarding three important classes of states. One was the vacuum state which we have discussed about. In the previous discussion session, we covered the aspects of coherent state. And now we are going to discuss thermal state of the field which would be one of the more reasonable kind of state which we will be dealing with most of the time in the laboratory settings. In order to make concrete predictions, it will be good to have an idea how fields which very frequently could be described by a thermally populated field should behave in terms of expectations and correlations. So, now as we know from quantum mechanics that the thermal state of any quantum system cannot be described by a pure state, it has to be a mixed state. So, therefore, it has to be described by a density matrix. So, that would be true for any quantum system, therefore this is true for our field as well and as we know our field is constituting of individual oscillators in Fourier domain. So, therefore, I can think of that each Fourier mode k which is like an oscillator has a thermal density matrix description if the whole field is in the thermal state. So, the k^{th} mode oscillator state is being described by this density matrix which is $e^{-\beta\hbar k}$ which is which is the thermal density matrix of the system. The full field theoretic computations of this would be the row which is effectively collection of all the oscillators put together. So, it could be $e^{-\beta\hbar_{k1}}$, $e^{-\beta\hbar_{k2}}$, $e^{-\beta\hbar_{k3}}$. Each $\hbar k_1$, $\hbar k_2$, $\hbar k_3$ is the individual Fourier Hamiltonian which we have discussed about which is made up of $\hat{a}_k$ and $\hat{a}_k^\dagger$. Okay but we also know all aks and $\hat{a}^\dagger_k$s they talk to each other only for the same $\hat{a}_k$ meaning $\hat{a}_k$and $\hat{a}^\dagger_k$ commutation is non-zero only for the same case if I talk about different $\hat{a}_k$s and different $\hat{a}^\dagger_k$s $\hat{a}_k$ with $\hat{a}^\dagger_k$ or $\hat{a}_k$ with $\hat{a}_k$ commutation is always zero so therefore I can club this product into summation over the argument in the exponential such that the Hamiltonian operator appearing over here is commuting with each other. So, ultimately the full density matrix of the whole system is integration over d^3k and hk each individual Fourier modes Hamiltonian itself. Collectively the whole thing which is appearing in the exponent is nothing but the total Hamiltonian of the system and therefore, there we have a consistent definition of the full thermal density matrix of the whole quantum field $e^{-\beta H}$. which is general to any quantum systems or any even classical. So, density matrix effectively can describe a classical ensemble of particles as well. So, in that sense any generic system which is thermalized can be described by a density matrix of this kind of thing, this kind of expression. Now, we also know Given a density matrix, how to compute things out of it, that is to know what are the expectation values, what are the probabilities of finding some things and that is obtained by taking the operator in consideration. Its thermal expectation will be the trace of the operator times the density matrix divided by the normalization of the density matrix. Since, this is density matrix which we are writing in a thermal base, this sometimes the downstairs which is coming about should be a function of temperature as well $Z(\beta)$. Sometime this is given a name path integral as well in the quantum field theoretic discussion session, but this is just the partition function of the system if you are tailored to think along that line. Notwithstanding what you want to call it, the definition of Z is nothing but the trace of $e^{-\beta H}$. That is, if I project this operator in all the possible good quantum basis of the whole quantum field, then sum over all those basis elements, the trace would be that. Now, we know quantum field is made up of vacuum state, let us say, where all Fourier modes are in their respective vacuum. or let us say one of them is in excited state. So, let us say 1k would be just all other momenta is grounded only one of the momenta let us say k_0 that is excited singly. Similarly, there can be twice excitation in the same k_0 which is equivalent to every other thing in vacuum and twice excitation in k_0. or I can have a one k_0 and another three particles at k_0. That means only around this pair I have particles and all other things are in vacuum. So, in principle different excitations in different k modes constitute a full basis for the description of a field. Field could be either found with some probability in vacuum or in one particle excited state, which one particle excited state, two particles excited state, which two particle excited state and so on. So ultimately a good complete basis of the description of a field is this collection N . This set N tells me how many particles are excited in which momentum mode. And this is a basis element. There are infinite dimensional space because infinite many possibilities are there for bosonic systems. Each mode can be populated from 0 to infinity. So, therefore, there are infinitely number of basis element and each basis element can be covered by one collection of n_k. Okay, so therefore, when I am supposed to do this trace of $e^{-\beta H}$, at the end of the day, I should be doing summation over all possible distributions of n_k varying from 0 to infinity in all possible case. and then put that along the density matrix, squeeze the density matrix along that and that is giving me the partition function of the path integral of this. We have not discussed the covariant approach of the field, so I should refrain from calling it path integral in Euclidean basis, but anyways there is a colloquial name people come up about with this. So, now as we discussed this ket collection of n is ultimately telling me how many particles in mode k1, how many particles in mode k_2 and this can be different. It can be 0, this can be 5, this can be 7, any arbitrary collection. This is one such element. So, ultimately that one such element can be put as a product over*k*and k. This is just a representation of that. Now, this partition function is known. and how to obtain any operator's expectation is known through the trace of operator times the density matrix divided by the partition function. So, let us go ahead and compute the expectation value of the Hamiltonian itself.

So, in this process the Hamiltonian will write along with the density matrix divided by Z and the trace of the whole thing would be taken. Again as we discussed trace means put the basis elements on both sides and sum over all the possibilities of the basis element. This sum is happening for different possibilities of writing this n_k. So, now let us see there are two kind of terms appearing, one term is over here and one term is over here. So, if I compute this object, and I compute this object, then the overlap of these two things gives me the whole expectation of Hamiltonian in this density matrix. So, let us do it side by side. So, first I want to compute this object. This object can be obtained from the dagger of Hamiltonian operator on n_k. If I take the whole thing and do the dagger of this, since Hamiltonian is a Hermitian operator, it will come back to itself. The ket will become the bra and I will get a h. Therefore, I obtain this, take the dagger of this, whatever I get would be my first element appearing over here. So, let us try to do that. So, I compute Hamiltonian's actions on n_k. Remember Hamiltonian, in the normalized Hamiltonian we will be talking about, we have thrown away the vacuum part which was divergent.The Hamiltonian is written as d^3k, ω_k and n_k. So, that is what I have supplied over here. This operator number operator hits this and then we get what it is so first this n_k operator which was $\hat{a}^\dagger_k\hat{a}_k$ this is true for a particular*k*So this operator will search in this get, this get meaning this whole distribution, this whole distribution over here is the full state set n_k. So when Hamiltonian is operating that means $\hat{a}^\dagger_k\hat{a}_k$ number operator will hit this get of collection of N. it will search for its own*k*value somewhere along the line n_k will be here and how many particles are excited in that*k*mode will come out so that is the n_k and ω_k is already there from the definition of hamiltonian so number operator goes and plucks out the excited number of particles in that particular mode and then you integrate over all the modes so this is as simple as that So action of Hamiltonian on this gives you back. This is an eigenstate of the Hamiltonian with this eigenvalue. If this is eigenstate of the Hamiltonian, we are familiar by now that exponentials of Hamiltonian will also be operator whose eigenstates are n_k. So, you can find it to work it out that if this is eigenstate of $H,$ it is also eigenstate of $e^{-\beta H}$. So, wherever This sum is happening for different possibilities of woperator is appearing, this summation d^3k integration $d^3kn_k\omega_k$ will appear and the state will be back to me. So, these two terms appear in my description of the expectation of S. Now, I know what is this, I know what is this, what I have to do, take the overlap of these two outcomes. So, we take the overlap divide by this partition function to get the whole expectation values. So, I have from the left side coming as this things dagger. So, this is real number, this is real number, this integration is all real number. So, this remains intact. Only thing happens this n_k will convert itself into ket of collection of n_k.

Okay and this will go ahead and hit this operator over here this operator whose eigen value equation is this so this will go ahead and hit this everything on the right hand side of the second term is also now number no operator so this n_k will go all across to this and hit it over there n_k n_k overlap will come about so ultimately you will get this n_k n_k overlap here and all the remaining eigen values coming intact like they are Here again I have decided to write down this $e^{-\beta}$ integration of d^3k and $k\omega_k$ as product of$ke^{-\beta}$ and $k\omega_k$. Because from there itself this product form appeared. The product form was converted into integration in the exponential remember here. When I wrote down the total Hamiltonian, that total Hamiltonian of this thing which had appeared on the top was just the single Hamiltonian put together too many times, the product became this integration. So, that is what I have rewritten again. This is same as $e^{-\beta}$ integration $d^3k\ n_k\ \omega_k$. Okay, so I have just rewritten it conveniently. All right, so if that is the case, then there is a $Z(\beta)$ which is in the downstairs. Remember $Z(\beta)$ was what? $Z(\beta)$ was $e^{-\beta H}$ squeezed between n_k, n_k. So I will use this term, squeeze it between n_ks. So again, I will get the same thing back and n_k, n_k overlap. So z therefore is just $e^{-\beta\omega}\ n_k\omega_k$ and product over$k$the way I have written and then n_k this nn overlap. So whatever nn overlap which is appearing in the top gets exactly cancelled from nn overlap appearing in the bottom and in the bottom I am left with this finite piece only. So the normalizations are taken care of From now on, I am just writing $Z(\beta)$ as this quantity. Full $Z(\beta)$ is this quantity product form times this object. But this object in the denominator is also present, in the numerator is also present. So, I will not write it just for cancellation.

$$\therefore \langle H \rangle = \frac{1}{Z(\beta)} \sum_{\{n_k\}} \int d^3\vec{k}\ n_k\ \omega_k \prod_k e^{-\beta\omega_k n_k}$$

$$= \frac{1}{Z(\beta)} \int d^3\vec{k}\ \omega_k \sum_{n_{k_1}=0}^{\infty} \sum_{n_{k_2}=0}^{\infty} \cdots \sum_{n_{k_n}=0}^{\infty} n_k\ e^{-\beta\omega_k n_k}$$

$$= \frac{1}{Z(\beta)} \int d^3\vec{k} \left(\omega_k \sum_{n_k} n_k e^{-\beta\omega_k n_k}\right)\left(\prod_{k'\neq k} \sum_{n_{k'}=0}^{\infty} e^{-\beta\omega_{k'} n_{k'}}\right)$$

Now,

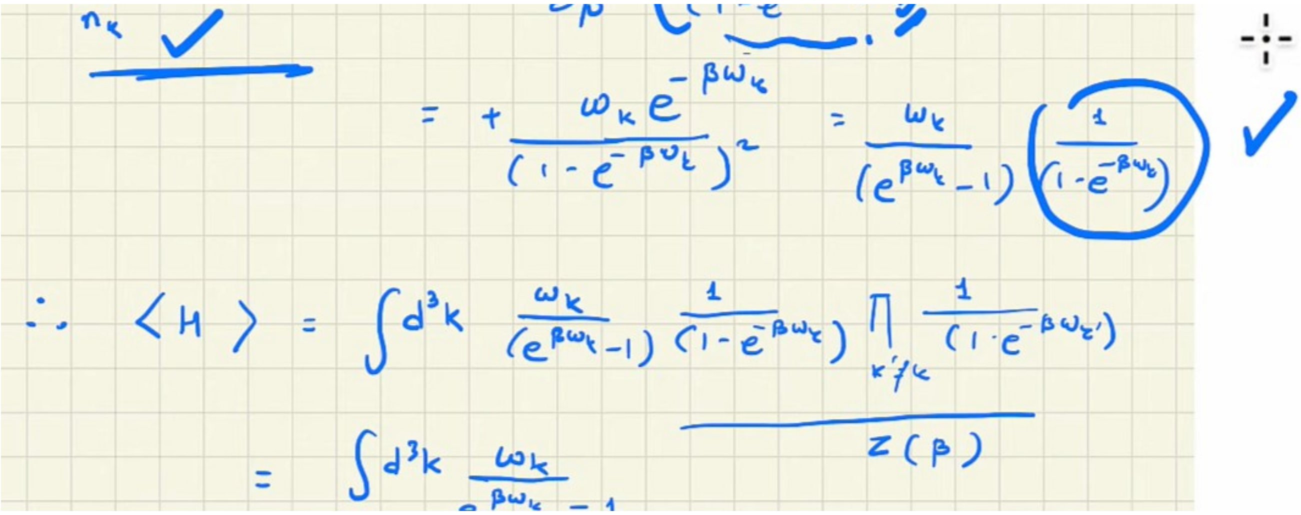
$$\prod_{k'\neq k} \sum_{n_{k'}} e^{-\beta\omega_{k'} n_{k'}} = \prod_{k'=k} \frac{1}{1-e^{-\beta\omega_{k'}}}$$

Thus,

$$Z(\beta) = \prod_k \frac{1}{(1-e^{-\beta\omega_k})}$$

and

$$\sum_{n_k} n_k\omega_k\ e^{-\beta\omega_k n_k} = -\frac{\partial}{\partial\beta} \frac{1}{(1-e^{-\beta\omega_k})}$$



$$\therefore \langle H \rangle = \frac{1}{Z(\beta)} \int d^3\vec{k}\, n_k \omega_k \prod_k e^{-\beta\omega_k n_k}$$

$$= \frac{1}{Z(\beta)} \int d^3\vec{k}\, \omega_k \sum_{n_{k_1}}^{\infty} \sum_{n_{k_2}}^{\infty} \sum_{n_{k_3}}^{\infty} n_k e^{-\beta\omega_k n_k}$$

$$= \frac{1}{Z}(\beta) \int d^3\vec{k}\, \omega_k \Big(\sum_{n_k} e^{-\beta\omega_k n_k}\Big)\Big(\prod_{k'\neq k} \sum_{n_k'=0} e^{-\beta\omega_k n_k'}\Big)$$

Now,

$$\prod_{k\neq k'} \sum_{n_k'=0} e^{-\beta\omega_k n_k} = \prod_{k'\neq k} \frac{1}{1-e^{-\beta\omega_k}}$$

$$Thus\ Z(\beta) = \prod_k \frac{1}{1-e^{-\beta\omega_k}}$$

and $\sum_{n_k} n_k \omega_k e^{-\beta\omega_k n_k} = \frac{-\partial}{\partial\beta} \frac{1}{1-e^{-\beta\omega_k}}$

$$= \frac{+\omega_k e^{-\beta\omega_k}}{(1-e^{-\beta\omega_k})^2} = \Big(\frac{\omega_k}{e^{\beta\omega_k}-1}\Big)\Big(\frac{1}{1-e^{-\beta\omega_k}}\Big)$$

$$\therefore \langle H \rangle = \int d^3x \Big(\frac{\omega_k}{e^{\beta\omega_k}-1}\Big)\Big(\frac{1}{1-e^{-\beta\omega_k}}\Big) \prod_k \frac{1}{1-e^{-\beta\omega_k}}$$

So, for the time being, I am going to use the definition of *Z(β)* only up to this. You can think of the way that it is normalized state. So, this might be 1. But not quite because we have seen that single particle excited state is not a normalizable state, two particle excited state is not a normalizable state in quantum field theory. So, this are not made up from normalizable states. So, this is not one in principle to speak with. But whatever appears as this in the numerator which is potentially divergent term, there is a equally divergent term in the denominator, they cancel each other exactly and we are left with a finite piece. That finite piece I am writing over here. Okay, now thereafter this is just plain manipulations of things which are appearing over here. Okay, now thereafter this is just plain manipulations of things which are appearing over here. Remember what are we supposed to do?

We have to sum trace we have to do. Trace means summing over all the possibilities. So that means I have to sum over all the bracket of n_k, possible set of n_k and here one of the n_k is appearing and all other k's are appearing as a multiplication over here. So, there are various n_k's which are appearing and there is a single n_k which is appearing over here in substance. So, you can call it k' if you wish such that to distinguish it up against this single n_k which is appearing over here. Okay, but no matter of that either I can break this integral*k*and this summation over n_k. So, this will be summation, this product means $e^{-\beta\omega k_1}$ n_{k1}+times $e{-}\beta\omega_k 2n_{k2}$ and similarly all the exponentials. And then there is a summation of all possibilities of $n_k 1$ being from 0 to infinity n_{k2} running from 0 to infinity, n_{k3} running from 0 to infinity and so on. And this whole collection space of possibility is collectively written over this over here. So what we can do?

We can write the whole thing as this term.

Here this has gone like this. One of n_k was surviving which has appeared over here, the ω_k I am writing over here. And this series sum I am writing in the expanded form like that. That is all.

Now see what is happening. This is a particular mode *k* and that particular mode k has to change according to the integration. But for the moment, fix this that there is first a single mode k and then you see all nks are appearing. So I should write down this is $e^{-\beta}$. I have missed out here. So let me correct it this way. that here all other $e^{-\beta\omega_k}$ and *k*' are also appearing as a product. Let us say *k*not summation. So, let me write it cleanly once more in the usual color as well so that it becomes part of the node. So, it is e product over k' which is not equal to*k*and $e^{-\beta\omega_k'}$ and k'. So, what does it do?

All this*k*which we had decided to call it like that, are appeared here one of the term which is just*k*is equal to*k*I have plucked out I have written over here so the whole product over here is just this in which one term I have just isolated and brought it to the fold of $n_k\omega_k$ all other things are left over here okay so this is how I have written so let me clean it up so that it is visible more carefully so Now, we have to just sit tight and see what happens out of this algebra. So, we go to the next step, the single *k* which is appearing as n_k here and $e^{-\beta n\omega k}$, that single*k*has to be summed over for different possibilities of n_k. The series summation which was broken into parts will also contain a particular n_k which I am talking about, this *k*. So that summation I have separated and all other things whose individual summation as is supposed to be present is being done like this. So, this times this is equivalent to this and the remaining term over here is equivalent to this, this times this is here and the remaining summation times this is appearing over here. So, this is just the reorganization of the thing which we have written, nothing profound about it. Now you see what kind of structure has appeared. There are so many summations of exponential of something kind, all k' but for one. Only one *k* is not put in this product. That *k* has appeared here and it has got multiplied with an extra n_k and ω_k actually. But apart from this *k*, all other *k*'s are written as exponential and exponential's argument is changing in integer steps. This we have done so many times in statistical mechanics if you have done a course on. So, this can readily be written individually. This is the GP, geometric progression. The first term starting from 0 will be 1, then $e^{-\beta\omega,}$ then $e{-}2\ \beta\omega$ and all the way up to infinity. And that summation can be written as 1 over 1−the common difference $e^{-\beta\omega_k}$. So, this is true for all ks not equal to k. This is this. So, this product I know becomes this. But about downstairs, the *Z(β)* is appearing.*Z(β)* we know is just $e^{-\beta\omega_k n_k}$. product over *k* and then all the possible n_ks. So, this is exactly the same kind of computation, but this time it involves even this k which we had separated out. So, this $Z(\beta)$ is exactly the same thing With the factor that$k$appearing over here involves all the values. It does not leave out anything. This part here leaves out one mode. Which mode? That is being summed over. I have to first do k is equal to k_1. Put this thing. Separate out k1. Put it here. And evaluate that.+the second term will be d^3k integral resuggesting. The k will change the value. This function will change the value to k_2. So k_2 will be plucked out. the product form will have everything apart from k_2 and so on. But in the partition function which is appearing downstairs, nothing is left out. The whole $e^{-\beta H_k}$ is undergoing the squeezing along with n_k, n_k from both sides. So, ultimately you can convince yourself the partition function has all the $1- e^{-\beta\omega_k}$. Here I have all the $1- e^{-\beta\omega_k}$ up to a single k which was plugged out which is appearing over here because an extra n_k came along with that. But in partition function I do not have that, I only have to obtain the exponential simultaneous expectation value which will exactly give me this. So, the two things we have computed, we have computed this quantity, we have computed the partition function. So, we should compute whatever is appearing over here as well, which is here I have written. You can quickly see this is nothing but $\partial/\partial\beta$ of $\partial/\partial\beta$ of this $n_k e^{-\beta\omega_k n_k}$ That means if I take the derivative with respect to $-\beta$, I will get down $n_k\omega_k$ and it will exactly become this quantity. And this summation I know how to compute. We have computed three times now, here, here and once more. So, you will get this. So, that means I have to do a derivative of the whole quantity here with respect to $\partial/\partial\beta$ with a –sign. You do that, you will get $\omega_k e^{-\beta\omega_k}$ in the numerator and $(1-e^{-\beta\omega_k})^2$. This is appearing in the downstairs, whole square of that and the exponential–exponential derivative go on the top which is this, which can be separated neatly as this factor and 1 over $e^{-\beta\omega}$, which you can convince yourself this is equal to that. Now, you see what happens. In the upstairs, this term was providing structures like this and only one$k$was missing in this. That$k$missing piece is supplied by this summation as a factor. So, overall this along with this become exactly equal to $Z(\beta)$ and they cancel each other. So, $Z(\beta)$ in the downstairs full contribution from the second term and one of the term coming from the first term exactly cancel out each other. So, I am left with only the reminiscent piece which is over here. So you see, this d^3ks what I have written.$k$not equal to$k$is already supplied. Put together everything, this is $Z(\beta)$ and they cancel each other. So I am left with only the first quantity over here. So Hamiltonian expectation is $d^3k\omega_k e^{-\beta\omega_k}$ So, this is the useful number, this is the useful result which we have. It looks like it is summation over $\hbar\omega_k$, $\hbar$ is there, I am writing in the unit of $\hbar$ is equal to 1. So, energy of oscillators, energy of the oscillators, but with number expectation $1/(e^{\beta k}- 1)$. So, this looks like Hamiltonian operator was supposed to be number operator times ω_k, number expectation operator times ω_k integration over this way. So, therefore, it looks like that each individual mode is populated with these many particles at temperature β and frequency ω_k.

Field expectation

$$\langle \phi(x) \rangle_\beta = \frac{1}{Z(\beta)} \sum_{\{n_k\}} \langle \{n_k\} | \phi(x) e^{-\beta H} | \{n_k\} \rangle$$

$$\hat{\phi}(x) | \{n_k\} \rangle = \frac{1}{(2\pi)^{3/2}} \int \frac{d^3\vec{k}}{\sqrt{2\omega_k}} \left(\hat{a}_k e^{ik\cdot x} + \hat{a}_k^{\dagger} e^{-ik\cdot x} \right) | \{n_k\} \rangle$$

$$= \frac{1}{(2\pi)^{3/2}} \Big[\int \frac{d^3k}{\sqrt{2\omega_k}} e^{ik\cdot x} \sqrt{n_k} \, | \{n_1 \dots n_k - 1 \cdot n_{k_n} \dots \} \rangle$$

$$+ \int \frac{d^3k}{\sqrt{2\omega_k}} e^{-ik\cdot x} \sqrt{n_k + 1} \, | \{n_1, \dots n_k + 1, \dots n_{k_n} \dots \} \rangle \Big]$$

$$e^{-\beta H} | \{n_k\} \rangle = e^{-\beta \int d^3\vec{k} \, n_k \omega_k} | \{n_k\} \rangle$$

$$\langle \phi(x) \rangle_\beta = \frac{1}{Z(\beta)} \frac{1}{(2\pi)^{3/2}} \int \frac{d^3\vec{k}}{\sqrt{2\omega_k}} \Big[e^{-ik\cdot x} \sum_{\{n_k\}} \sqrt{n_k} \, e^{-\beta \int d^3\vec{k} \, n_k \omega_k} \times$$

$$\langle \{n_1 \dots n_k - 1 \dots n_{k_n} \dots \} | \{n_1, \dots n_k \dots n_{k_n} \dots \} \rangle$$

$$+ e^{ik\cdot x} \sum_{\{n_k\}} e^{-\beta \int d^3\vec{k} \, n_k \omega_k} \sqrt{n_k + 1} \times$$

$$\langle \{n_{k_1} \dots n_k + 1, \dots n_{k_n}, \dots \} | \{n_{k_1}, \dots n_k, \dots n_{k_n} \dots \} \rangle$$

$$\langle\phi(x)\rangle_\beta=\frac{1}{Z(\beta)}\sum_{\{n_k\}}\langle\{n_k\}|\phi(x)e^{-\beta x}|\{n_k\}\rangle$$

$$\hat{\phi}(x)|\{n_k\}\rangle$$

$$=\quad\frac{1}{(2\pi)^{3/2}}\int\frac{d^3\vec{k}}{\sqrt{2\omega_{\vec{k}}}}(\hat{a}_k e^{ik\,x}+\hat{a}_k^{\dagger}e^{-ik\cdot x})|\{n_k\}\rangle$$

$$=\quad\frac{1}{(2\pi)^{3/2}}\left[\int\frac{d^3\vec{k}}{\sqrt{2\omega_{\vec{k}}}}\sqrt{n_k}e^{ikx}|\{n_1,n_{2,}\ldots n_{k-1},n_{k\ldots}\}\rangle+\int\frac{d^3\vec{k}}{\sqrt{2\omega_{\vec{k}}}}\sqrt{n_k+1}\,e^{-ikx}|\{n_1,n_{2,}\ldots n_{k+1},n_{k\ldots}\}\rangle\right]$$

$$e^{-\beta H}|\{n_k\}\rangle=e^{-\beta\int d^3\vec{k}\,n_k\omega_k}|\{n_k\}\rangle$$

$$\langle\phi(x)\rangle_\beta=\frac{1}{Z(\beta)}\frac{1}{(2\pi)^3/2}\int\frac{d^3\vec{k}}{\sqrt{2\omega_{\vec{k}}}}\left[e^{-ikx}\sum_{\{n_k\}}e^{-\beta\int d^3\vec{k}\,n_k\omega_k}\right]\times\langle\{n_{1,}\ldots,n_{k-1},\ldots n_{kn},\ldots\}|n_{1,}\ldots,n_{kn},\ldots n_{k+1},\ldots\rangle$$
$$+e^{ikx}\sum_{\{n_k\}}e^{-i\beta\int d^3\vec{k}\,n_k\,\omega_k}\sqrt{n_k+1}\times\langle\{n_{k_1},\ldots,n_{k+1},\ldots n_{kn},\ldots\}|n_{1,}\ldots,n_k,\ldots,n_{kn},\ldots\rangle$$

So having obtained the expectation value of the Hamiltonian, let us now go forward and see what is the field expectation value in the thermal state itself. Recall in vacuum state, field expectation value turned out to be zero. In coherent state, it became a function of α_k, the coherent state parameter. Coherent state parameter $\alpha_k e^{i\omega kt}/\sqrt{2\omega}$. The Fourier transform of this quantity told me what should be the field. expectation so the parameter of the state decided what is the expectation is this true for thermal state as well the parameter of interest for thermal state is the temperature β β if you remember I have not defined it but I am hoping that you are familiar with the β, β is $1/k_BT$ so this temperature sets your β and that βH opefully will set the expectation value of the field as well for hamiltonian it indeed has set up the expectation value of the Hamiltonian which is this much, whether this will set the expectation value of the field or as well that we should compute.

Again by the definition of expectation of the field, we should be calculating the trace of ϕ times rho. That means I should put this quantity ϕ times rho, rho being $e^{-\beta H}$ in the diagonal basis or the eigen basis of the system, complete basis of the system and divide it by the partition function $Z(\beta)$. That is the standard definition we keep using. So this is the operator method and we will write down the field in its mode and the $\hat{a}_k$ and $\hat{a}_k$ digraph like we have been doing all along. So let us write down first what is this. Why are we computing ϕ acting on n_k? The same thing. I would want this part to be evaluated first, this part to be evaluated first, take the overlap, compute the sum, that is the game which we played for Hamiltonian. Previously, the Hamiltonian was sitting over here and this time ϕ is sitting here. So, let us compute what is ϕ acting on n_k. And then I will take the dagger of the whole equation. If ϕ is a real scalar field, that means it is Hermitian conjugate is the same as itself. So, therefore, when I take the dagger, it will get back to itself. And n_k ket will become n_k bra like before. So, let us do this. I take the expansion of the field in terms of mode function and the ladder operator which is this. The operators are only $\hat{a}_k$ and $\hat{a}^{\dagger}_k.\hat{a}^{\dagger}_k$ will hit this state and make it search for its own k value across this whole decomposition. Remember, it means n_{k1}, n_{k2}. So there will be some n_k here, then other n_{kn} and so on. So, this $\hat{a}^{\dagger}_k$ will search for its label, it will find n_k particle in it, operation of that will make $\sqrt{n_k}+1$ coming out and the n_k becomes $n_{k+1.}$ So, $\hat{a}^{\dagger}_k$ will raise the particle content in mode k by 1. Similarly, $\hat{a}_k$ operator will hit this state, search for its own k and decrease the number of expectation, number of excitation in that state by 1, n_k-1 and $\sqrt{n_k}$ will come out as the factor. This is the usual ladder operator's business. So, therefore, $\phi(k)$ acting $\phi(x)$ acting on n_k is integration of this quantity times this $e^{ik\dot{x}}$ will come along with $\sqrt{n_k}$ and n_k-1 in the kth mode. And the second term will come out with $e^{-ik\dot{x}}$ here and this will come

along with $\sqrt{n_k}+1$ with n_k+1 particle in the k^{th} mode. So, I have obtained what is this $e^{-\beta H}|n_k>$, we have already obtained in the previous computation of Hamiltonian expectation, we had already computed this quantity here, $e^{-\beta H}$. Again let us clean it up, so that we can see it with clarity. The last line of this page is the computation which we are after, so I will just repeat copy this and put it over here. So $e^{-\beta H_k}$ is known to me. What is left? Now let us take the overlap between these two. If I do this overlap, I will take the dagger of the whole thing. If I take the dagger of the whole thing, this will become e^{-ikx}, this will become e^{+ikx}, $\sqrt{n_k}$ will remain $\sqrt{n_k n_k}$, this will remain $\sqrt{n_k}+1$ and only thing changing will be this will become bra operators. So, ultimately you can see if I take the dagger of the first box and contract it or take inner product with respect to the second box, what I will get all the stars of this left hand side which will appear with e^{-ikx} $\sqrt{n_k}$ and this will have a overlap between this n_k which is being written as n_{k1}, n_{k2}, n_{k3} and so on and bra of this which is n_{k1}, n_{k2} and only in the k^{th} mode I have a n_{k-1} particle.. Similarly, for the second term this exponential will become e^{+ikx} and $e^{-\beta}$ this thing will come out. $\sqrt{n_k}+1$ will remain $\sqrt{n_k}+1$ this will become a bra and that will hit this. So, I have a n_k , n_{k+1} this time in the bra and $n_k n$ and right hand side it is just this n_k. Now you see I have a structure two kind of terms are appearing in the expectations one is one term is coming with the overlap of set n_k with almost the set n_k only the one particular mode has one less particle. Similarly, the second term is right hand side the ket of the system is set n_k and the bra of the system is almost the set n_k but one of the mode has one extra particle. So we need to compute these two things, supply over here and then get done with the summation. However, we can see this can be written as n_{k1} ket, n_{k2} ket. Similarly and n_kn set and this will keep going on and similarly here it is n_{k1} bra n_{k2} bra in between somewhere I have an one extra particle in the kth mode n_k+one and then all other thing So, this n_{k1} will undergo contraction with n_{k1}, this n_{k2} will undergo contraction with n_{k2}, everything will go along only thing n_k here which is appearing as n_k particle and from left and right n_{k+1} particle, that overlap will happen.
And similarly for the first term n_{k-1} and n_k overlap will happen. And similarly for the first term n_{k-1} and n_k overlap will happen. One particular mode inner product like this will appear, all other modes will have the same ket and bra. And this is fairly easy exercise to prove that if I start with a ket n_k and take its overlap with respect to n_{k-1} or n_{k+1}, both the cases, the overlap is 0. So, that means this is 0 as well as this first term is 0. Therefore, nothing survives in this computation and the field expectation in the thermal state becomes identically 0. This is an exercise which is very simple exercise you should be able to prove even for harmonic oscillators different eigenstates of Hamiltonian and orthogonal as simple as that. So, therefore, you see like vacuum state the expectation of the field is 0 in this state as well. But we know this is not of empty state because it is filled with particles and how many? these many particles number expectation we have just written these many particles in k^{th} mode. The distribution is such that ultimately when get to obtain the field expectation it sums up to 0. But that does not constitute it of empty field. In the next discussion session we will start looking at how much of a correlation it has. Vacuum state also was mean 0 or expectation 0 object with but it had a non-zero correlations thermal fluctuation non-zero fluctuation vacuum fluctuations what about thermal fluctuations are they identical to vacuum fluctuations or they have more robust structure that we will discuss in the next clip.