Foundation of Quantum Theory: Relativistic Approach

Quantum Field Coherent State

Prof . Kinjalk Lochan

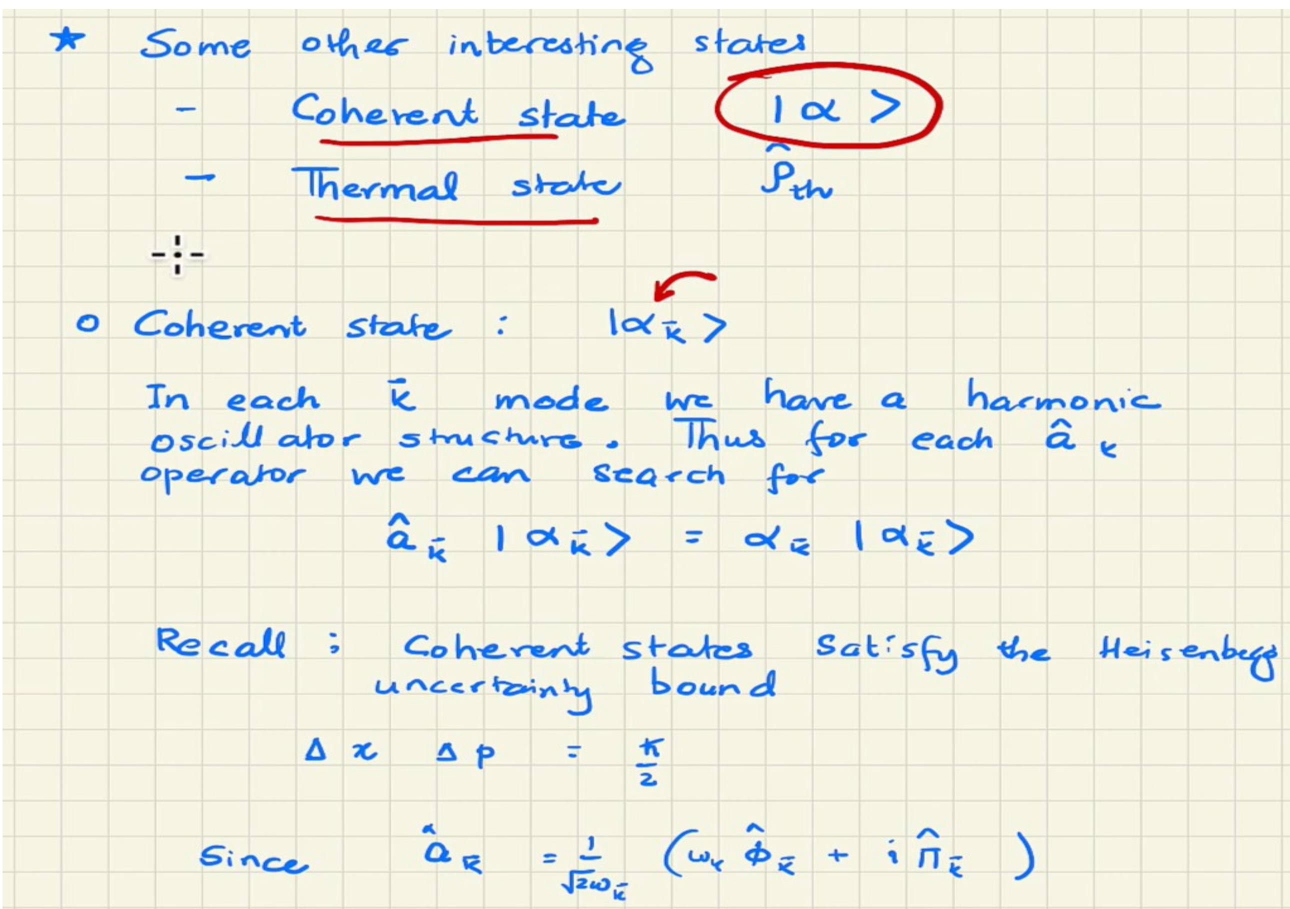
Department of Physical Sciences

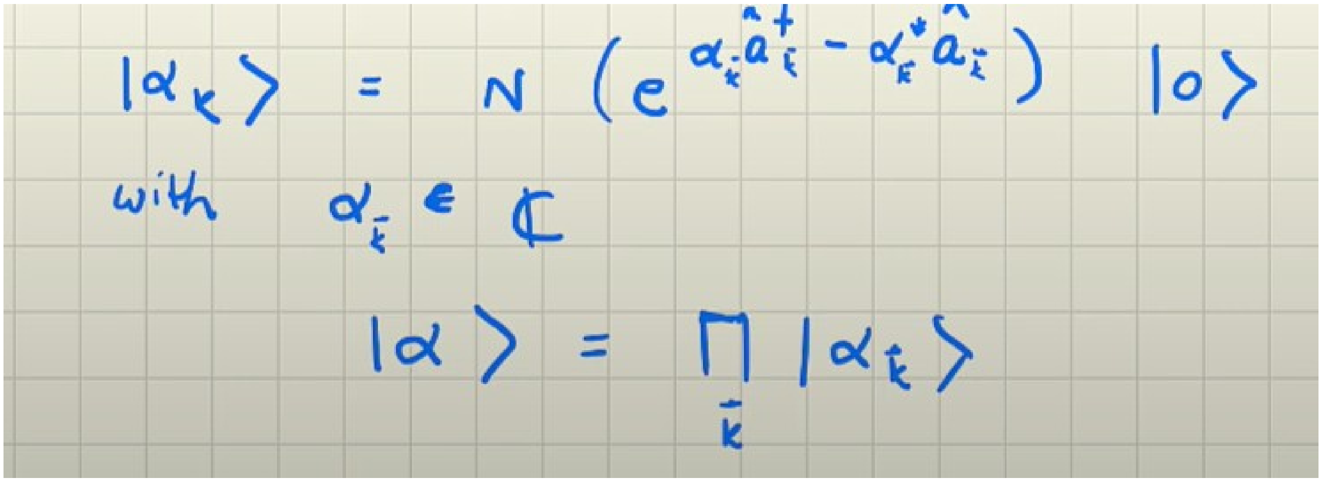
IISER Mohali

Scalar field coherent state

Lecture- 30

In today's discussion session, we will start discussing some other interesting states of quantum fields and which would be handy when we discuss more interaction with atoms with different kind of classical or quantum fields. So in previous classes, we have learnt about the structure of the vacuum of the field, vacuum state, how vacuum state is not completely empty. It comes up with correlations.





Hat α_ bar k = \ lline Hat α_ bar k \ rangle = α_ bar k = \ lline α_ bar k \ rangle
Recall : Coherent states satsify the Heisenberg uncertainty bound

$$\Delta x\, \Delta p = \frac{\hbar}{2}$$

$$\hat{a}_{\vec{k}} = \frac{1}{\sqrt{\omega_k}}\left(\omega_k \hat{\phi}_k + i\,\hat{\Pi}_k\right)$$

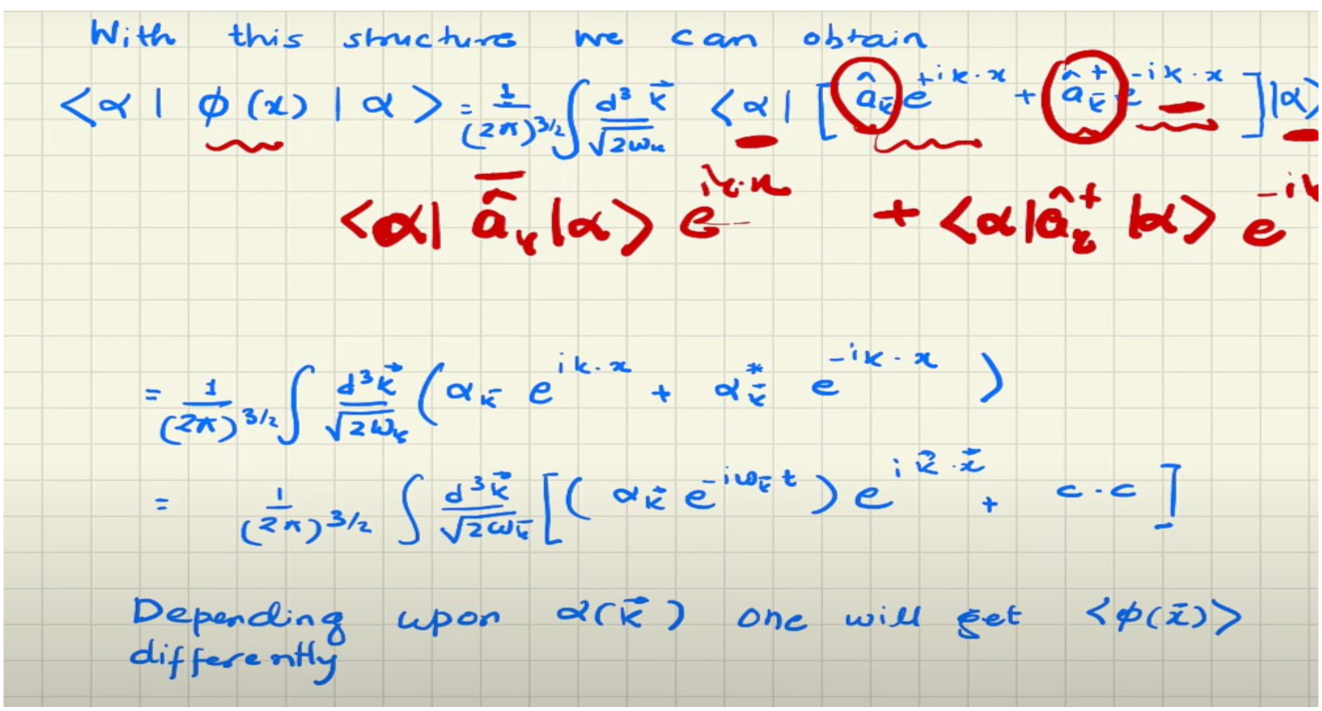
$$|\hat{\alpha}_k\rangle = N\left(e^{\alpha_{\vec{k}}\hat{\alpha}^{\dagger}_{\vec{k}} - \alpha^{*}\hat{\alpha}_{\vec{k}}}\right)$$

$$\alpha_{\vec{k}} \in \mathbb{C}$$

$$|\alpha\rangle = \prod_{\vec{k}} |\alpha_{\vec{k}}\rangle$$

And those correlations have very nice structure of invariant distance, determining the amount of correlation the particles could have.Now, we will see whether this structure remains true for non-vacuum states or not and in particular non-vacuum state of a particular point which is very close to a classical description goes by the name of the coherent state. There are thermal states as well which we will discuss in the next class, but in today's class we will discuss about the coherent state.Now, before going to coherent structure what they are, if you are familiar with harmonic oscillator coherent structure, you should be able to sense the direction where we are headed to. So, since we know for each momentum mode k, our description of the quantum field is that of a harmonic oscillator. Therefore, I can have the definition of coherent state for that harmonic oscillator as well. So this tally very well with the discussions which you might have in the quantum mechanics course about the coherent state. The same kind of structure is present for the different k modes as well. Each one has its own coherent state. So we are going to list that state for particular mode k, its coherent state as α_k, the ket α_k. And this as we know is true because each different k is a separated harmonic oscillator structure. So, now we know from harmonic oscillator analysis, a coherent state α is supposed to be eigenstate of the ladder operator A, the annihilation operator. And we know that our description which we are demanding is there is set of oscillators for each k. So, therefore, in order to identify whose coherent state we are talking about, we put a label k. That means it is eigenstate of a_k. And that eigenstate we are saying comes up with the eigenvalue α_k. That is how we identify the state. The name α_k ket is identified with their eigenvalue of the annihilation operator. As we know annihilation operator is not Hermitian, a and $a^\dagger$ are not the same thing. So therefore, this is not necessarily a real number. This can be a complex number. And these states if you think if are not orthonormal states and as if you are familiar with the literature of coherent state, you must be knowing that it is over complete basis rather than the complete basis. What we further know that these coherent states satisfy the Heisenberg uncertainty bound that is it is respecting this $\delta(x)$ $\delta(p)$ is equal to h power/2. So, they are individual variances in position in case of harmonic oscillators do come up with the minimum distributions individually can come they can come up with so that h bar/2 is achieved. For instance vacuum is also a coherent state which also satisfies this of a particular parameter. What about these α_k's? These α_k's are not bounded operators eigen spectrum so therefore they are continuous variable and they are complex number. So, I can have a complex number right now unbounded from anything, it is just possible complex number and therefore, I can have a coherent state of a eigenstate with different α which are continuous complex numbers. In position space, the a operator, annihilation operator is of linear combination of x + ip in a particular way, the same kind of a strategy will tell you how the field operators combination should be generating ak. Recall that the role of x is played by ϕ and the role of p is played by ϕ. Therefore, the same kind of decomposition which gives me the relation between x and p should give me the relation between the creation annihilation operators between ϕ and Π. All right.

So, still now we are just moving ahead on the structure which is already available at the level of standard quantum mechanics. individual oscillator k. Also, again I am just quoting it from the quantum mechanical structure. If you want to write down the eigenstate of the creation annihilation operator as a, that is the coherent state, that state can be written as exponential of α_k times $a^\dagger$ and α star times k exponentiated with some normalization constant that acting on vacuum gives rise to this coherent state. Again, I am copying this from memory of quantum mechanics.You should go to quantum mechanics textbook if you are unfamiliar with this and try to see how the eigen states has this description. You can verify given this α_k indeed this is true. So, this is a good correct relation and how this comes about, there are many ways of obtaining this around. You can go to any standard textbook of quantum mechanics harmonic oscillator and try to ask what is the representation of a coherent state in terms of some operators hitting the D. For instance, single particle excitation state is $a^\dagger$. acting on g. Two particles excitation is double $a^\dagger$ with a factor 2 and so on. Actually, factor root 2 should be. So, if you keep hitting more and more $a^\dagger$, you will be getting more and more new states. Similarly, you can have a state where I have a twice this, this – this is also a possible state, right. Two particles excited with some weightage α and one particle with weightage β and I take a – sign. This is also a state, but this is written as in a particular way this state is double $a^\dagger$ acting on vacuum and this is a single $a\dagger$ acting on vacuum. So, this is not a common operator which is hitting this. So you can write so many different states by combining how many excitations you want in which way to appear in the different components of superposition. One particular such distribution is this exponential. Exponential is identity + this term + square of this term/2 and so on. All those operators put together give you a transcendental series of states, vacuum state + some coefficient time, excited state + some coefficient time, second excited state and so on and put together they constitute for you the coherent state. The exponential series that transcendental series summing up to exponential of this kind gives you the coherent state. The same thing I write for the field momenta mode case as well and we have this thing where α_k can take complex numbers. Now this is true for one oscillator set and we have repeatedly said that the full description of the state is the collection of state from all possible oscillators So, therefore the full coherent state will be coherent state in each of the momenta mode. With what value that will depend upon which α_k function you choose. Each k I have a different α value. So, I have a α_1 , α_2 , α_3 , α_4 and so on. This is the full coherent state of this. What is α_1 , α_2 , α_3 individually is fixed by this αk number. So, that is why a symbol k, subscript k was inserted. α is a function of k. This is a function of k. So, therefore, you would have a structure that coherent state of the full system is coherent state of individual things. And since vacuum is also a coherent state, one possible coherent state could be that α one α two then vacuum in three then vacuum in five and so on so different different values it is taking for example it could take value one this can take value two this is taking value zero α is equal to zero the vacuum state this can take value three again this can take value 0 again. So, ultimately if you plot as a function of k, you are getting 1, 2, 0, 2 again or 3 again and then 0 again. So, this kind of plot for α_k is coming about. So, in principle you can choose different different functions α_k and you will generate resulting α, the full coherent state of the field. Now, given this coherent state of the field, we want to know what are the properties, expectations in the coherent state and correlations in the coherent state, whether to share the same structure like vacuum or not. In principle, we know they are not supposed to satisfy the nice structures of vacuum in a particular way, in the sense that if I compute even for one particle excited state, suppose I have a field in which only kth momenta is excited, all others are in vacuum. Then also if you calculate ϕ_x ϕ_y, then we do not get this to be equal to the vacuum two-point function. So, it will be different from the vacuum two-point function. It is not equal to this. Actually it is equal to this vacuum thing + an additional thing, some function which depends upon which mode you have excited. So, it is a function of k and x and y. This had a nice structure that it was – $1/d$ times $1/4\pi^2$. That was the invariant distance which were appearing. This will be some arbitrary function which will not be necessarily equal to any invariant quantity. So, therefore, the coherent state also is not a vacuum state. So, it is very rare to very non-trivial to think whether they will constitute additional f which is containing some usual geometric identities or not. So, that is what we want to see. We want to compute for a particularly excited state like this and wish to know how different it is compared to the vacuum case. So, we go ahead and compute the expectation value of the field first.



$$\langle \alpha | \phi(x)\, \phi(y) | \alpha \rangle$$

$$= \frac{1}{(2\pi)^3} \int \frac{d^3\bar{k}}{\sqrt{2\omega_{\bar{k}}}} \int \frac{d^3\bar{k}'}{\sqrt{2\omega_{\bar{k}}'}} \langle \alpha | \left(\hat{a}_{\bar{k}} e^{ik\cdot x} + \hat{a}^{\dagger}_{\bar{k}} e^{-ik\cdot x} \right) \left(\hat{a}_{\bar{k}'} e^{ik'\cdot y} + \hat{a}^{\dagger}_{\bar{k}'} e^{-ik'\cdot y} \right) | \alpha \rangle$$

$$= \frac{1}{(2\pi)^3} \int \frac{d^3\bar{k}}{\sqrt{2\omega_k}} \int \frac{d^3\bar{k}'}{\sqrt{2\omega_{k'}}} \left[\begin{array}{l} \langle \alpha | \hat{a}_{\bar{k}} \hat{a}_{\bar{k}'} | \alpha \rangle e^{ik\cdot x + ik'\cdot y} + \\ \langle \alpha | \hat{a}_{\bar{k}} \hat{a}^{\dagger}_{\bar{k}'} | \alpha \rangle e^{ik\cdot x - ik'\cdot y} + \\ \langle \alpha | \hat{a}^{\dagger}_{\bar{k}} \hat{a}_{\bar{k}'} | \alpha \rangle e^{-ik\cdot x + ik'\cdot y} + \\ \langle \alpha | \hat{a}^{\dagger}_{\bar{k}} \hat{a}^{\dagger}_{\bar{k}'} | \alpha \rangle e^{-ik\cdot x - ik'\cdot y} \end{array} \right]$$

With his structure we can obtain

$$\langle \alpha | \phi(x) | \alpha \rangle = \frac{1}{(2\pi)^{3/2}} \int \frac{d^3\vec{k}}{\sqrt{2\omega_{\vec{k}}}} \langle \alpha | [\hat{\alpha}_{\vec{k}} e^{+ikx} + \hat{\alpha}^{\dagger}_{\vec{k}} e^{-ikx}] | \alpha \rangle$$

$$= \quad \frac{1}{(2\pi)^{3/2}} \int \frac{d^3\vec{k}}{\sqrt{2\omega_{\vec{k}}}} (\alpha_{\vec{k}} e^{+ikx} + \alpha^{\dagger}_{\vec{k}} e^{-ikx})$$

$$= \quad \frac{1}{(2\pi)^{3/2}} \int \frac{d^3\vec{k}}{\sqrt{2\omega_{\vec{k}}}} [\alpha_{\vec{k}} e^{-\omega_{\vec{k}} t} e^{+i\vec{k}\vec{x}} + c.c]$$

Depinding upon $\alpha(\vec{k})$ one wiill get $\langle \phi(x) \rangle$ differently.

$$\langle \alpha | \phi(x) \phi(y) | \alpha \rangle = \frac{1}{(2\pi)^3} \int \frac{d^3\vec{k}}{\sqrt{2\omega_{\vec{k}}}} \int \frac{d^3\vec{k}'}{\sqrt{2\omega_{\vec{k}}'}} \langle \alpha | (\hat{\alpha}_{\vec{k}} e^{+ikx} + \hat{\alpha}^{\dagger}_{\vec{k}} e^{-ikx}) (\hat{\alpha}_{\vec{k}} e^{+iky} + \hat{\alpha}^{\dagger}_{\vec{k}} e^{-iky}) | \alpha \rangle$$

$$= \quad \langle \alpha | \phi(x) \phi(y) | \alpha \rangle = \frac{1}{(2\pi)^3} \int \frac{d^3\vec{k}}{\sqrt{2\omega_{\vec{k}}}} \int \frac{d^3\vec{k}'}{\sqrt{2\omega_{\vec{k}}'}} \begin{bmatrix} \langle 0 | \hat{\alpha}_{\vec{k}} \hat{\alpha}_{\vec{k}}' | 0 \rangle e^{ikx + ik'y} + . \\ \langle 0 | \hat{\alpha}_{\vec{k}} \hat{\alpha}^{\dagger}_{\vec{k}}' | 0 \rangle e^{ikx - ik'y} + . \\ \langle 0 | \hat{\alpha}^{\dagger}_{\vec{k}} \hat{\alpha}_{\vec{k}}' | 0 \rangle e^{-ikx + ik'y} + . \\ \langle 0 | \hat{\alpha}^{\dagger}_{\vec{k}} \hat{\alpha}^{\dagger}_{\vec{k}}' | 0 \rangle e^{-ikx - ik'y} \end{bmatrix}$$

What we have to do that is the standard exercise by now we know how to compute expectations of field in various states. I will write down the decomposition of the operator ϕ in terms of a and $a^\dagger$ which is usual $d^3k/\sqrt{2\omega}$ integral with $(1/2\pi)^{3/2}$ factor outside. And inside I have plane wave times a_k and plane wave star with $a_k^\dagger$. That is the usual structure which we are familiar with and as we know compared to the vacuum case which the same exercise we had done. This time we have to just do the α state will be squeezing ak and $a_k^\dagger$ as well. So, the α state is squeezing this will come with $e^{ik\dot{x}}$ and this will come with e^{-ikx} that is all it will happen. This α and this α squeeze operated this and this all other things like our plane waves or integrals and other things. So, they are not operators. Only operator that gets squeezed is this. Recall, we are doing business only for scalar field for demonstration purpose. You can do the same thing for Spinar field or vector field as well. So, the operators which appear, squeezed operator which appear over here is a_k α and α $a_k^\dagger$. this operator gets this, this operator gets this. Recall the α is supposed to be product of all k's and α_k. So, therefore when I hit it with operator a_k, let us say let us call it k', this is integrated, this is product of all possible k'. So, when I hit with operator a_k, it will search for its own momenta. And there will be one particular α at the same momenta k. This will be hit upon by this ak. All other states will be ignored by this a_k operator. So, a_k acting upon α_1, α_2 , α_3 , *so on* α_k. Then α_n , all these things. This operator will only identify this state. All other states it will go across and get up to this α_k and hit it. When it hits it, since it is eigen state of a_k operator, I will get a α_k over here, $\alpha_k a_k$ back, α_k back. So, there the squeezed thing is actually now convert itself into eigen value equation. So, this becomes a number now from being an operator. So, this will come out and therefore, I will have answer as α_k and α_k, α_k in the product. Since we had normalized the state, if you recall, we had put a normalization function over here, that means the state is properly normalized. So, α_k α_k in our product is 1, I will be left with α_k over k. So, this becomes α_k. You can argue this operator here gives you α_k star times of bra α and therefore, bra α with ket α gives you 1 again. So, you will realize After this action of the whole thing, the squeezing things of a_k and $a_k^\dagger$ gives you α_k and α_k star with plane wave and plane wave star. So, now we have obtained expectation values as individually, we can just write it what these are in the open plane wave basis. So, if I write down the $ik\dot{x}$, I know there is a temporal part with a – sign which is $e^{-i\omega t}$ which I have separated. And the remaining thing which is just complex conjugate of this, a plane wave with a complex conjugate of this. So, overall there is this d^3k integral which is coming along, this d^3k integral which is coming along and there is a $e^{ik\dot{x}}$ which is here and some function which I can call α_k tilde, together it is α_k times $e^{-i\omega t}$. So this is some function overall. This looks like a Fourier transform of this function. And its complex conjugate added. So depending upon what function α_k you choose your business to start with. That means here I had a freedom to choose what α_k. Because these complex numbers just take any value from the complex space. So, I can choose any α_k to start my discussion with. That function multiplied with

$e^{-i\omega kt/\sqrt{\omega k}}$. The whole thing will undergo Fourier transform because everything depends on $\vec{k}$. ω_k depends on $\vec{k}$ and α_k is also a function of k. So, therefore, when I do the integration over d^3k, I get a Fourier transform of my choice α_k. So, depending upon what function of k you have chosen your α to be, you will get your expectation correspondingly. Expectation value will be the Fourier transform of your choice + its complex conjugation. So, therefore, anyone supplies you what is α_k, for example, if I give you it is $k^k\ k\ e^{-k}$, let us say, you will be able to do this Fourier transform at its complex conjugate and that will be your ϕ_x, okay. So, therefore, this depends on your choice of what α_k has gone. It is not a unique answer like a vacuum. Vacuum had a unique answer 0. This time it is non-zero. You can verify again if I take α to be 0, this will be 0. Complex conjugate of 0 is also 0. So, vacuum is a special coherent state where the choice of αs are 0 for all k. If any of the α_k is non-zero, then it is not vacuum state. So, now your choice of your function α as a function of k will tell you what is the expectation of field in that state. What about its correlator? You do the game again. you obtain the ϕ_x ϕ_y operator, squeeze it between the coherent state. So, then the field ϕ is here, ϕ at x is here, ϕ at y is here with their individual integrations. The field ϕ comes with the k integrations, field ϕ at y comes with k'

integrations. and you write it down again you will see there are four sets of operators when α_k goes with this α_k' you will get this which is squeezed between the coherent state. Then there is a possibility this α_k goes with this $\alpha_k^\dagger$ $k'^\dagger$ which is this squeezed between the coherent state. Pay attention that when these first two combine here the corresponding plane waves combine as a positive sign. When this α_k goes and hits this $a_k^\dagger$, I have a + $ik\dot{x}$ and – ik' y. So, the exponential function will change according to which of the operators you are trying to combine and what is their associated plane waves they are coming with. So, first two terms give rise to this, this and this. The third term will be this combining with this $\alpha_k^\dagger$ and α_k'. This time there will be – sign in the plane wave of xe^{-ikx} and + sign is ik'y. And lastly, this one combines with the last $a_k'^\dagger$ and you have exponential both with – sign. So, that is what the coherent state two-point correlator structure will emerge out to be and we will compute these things. Apparently, you can see that it is much different from the vacuum state. In vacuum state, many of these operators went away. Only one of the operators survived, which was the second number operator. Had it been 0, this would have annihilated it. Had it been 0, then the middle state would have survived. But all other states, you can argue, it would vanish. So, now you can see on face of it, it is different from the vacuum. And how much different it is from the vacuum can be computed further. So, you can find out what are the different operators appearing here. The four set of operators which are squeezed between α can also again be found from the information of a coherent state. Remember a_k acting on α will give me α_k back with α. The two statements are true, α_k hitting a α_k will give me α_k back with get α_k as well. But this is true and therefore the second line is also true for general α which is the full coherent state of the full vacuum to a full coherent state of the full field because this is α 1 α 2 α_k in between somewhere then progress to some other αs so therefore when a_k hits it it again goes and sees its own k momenta coherent state and k momenta coherent state and then apply this information that it will get back the state with extra α_k that α_k will come all the way out and remaining thing is still the full coherent state. So, you get this. So, this is this the a_k will again hit the first a_k' hitting this α The a_k' hitting this α has given me α_k' times α back. Then the second a_k will again hit this α and give me a_k hitting the second thing will hit this, a_k will hit this, I will get α_k' and α_k. So, the first thing will become α_k' α_k, the second thing has $a^\dagger$ in between. So, I have to do something to this, I do not know what is this. $a_k^\dagger$ hitting α_k. I do not know what it is. So, we have to do something about the middle term. The third term and the fourth term have this structure.This is also going to give me α_k' times α, while this is going to give me α_k star times ket the bra α. Similarly, this will give me α_k star bra α and this will give me α_k' star ket α. So, all those things where we have wrote down. This quantity is going to give me this. Similarly, the third quantity will give me 1 α_k star and α_k'. And the last quantity will give me α_k star and α_k star$'$ as well. What about the middle quantity which we had ignored? We can go back and use the commutation relation. We will flip again like for quantum state whatever we had done. We will flip this orders of a_k and $a_k^\dagger$ using their commutator which is delta function. So, this $a_k a_k^\dagger$ can be written as a $\delta(k - k')$ + a_k * $a_k a_k^\dagger a_k$. So, now this a_k will go and hit this. give you α_k and this will hit this other side and give you α_k' star α. So, there will be an extra $\delta(k - k')$ which will be earned by flipping this. So, all those four terms now we have answer to, the first term is this much, the second term has two pieces a delta under this term and the third and four pieces have each one of them. So, put it in the expressions which we wanted. In the previous slides, we have seen all the four expectations are coming with different exponential function. So, we take account of that and supply our non-zero values here, here, here and here.

$$\langle \alpha | \hat{a}_k^\dagger a_{k'} | \alpha \rangle = \alpha_{\vec{k}}^* \alpha_{\vec{k}'}$$

$$\langle \alpha | \hat{a}_k^\dagger \hat{a}_{k'}^\dagger | \alpha \rangle = \alpha_k^* \alpha_{k'}^*$$

Thus,

$$G_\alpha(x,y) = \frac{1}{(2\pi)^3} \int \frac{d^3\vec{k}}{\sqrt{2\omega_{\vec{k}}}} \int \frac{d^3\vec{k}'}{\sqrt{2\omega_{\vec{k}'}}} \Big[\alpha_{\vec{k}} \alpha_{\vec{k}'} e^{i(k\cdot x + k'\cdot y)} + \left(\delta(\vec{k}-\vec{k}') + \alpha_{\vec{k}'}^* \alpha_k\right) e^{i(k\cdot x - k'\cdot y)} + \alpha_{\vec{k}}^* \alpha_{\vec{k}'} e^{-ik\cdot x + ik'\cdot y} + \alpha_{\vec{k}}^* \alpha_{\vec{k}'}^* e^{-i(k\cdot x + k'\cdot y)} \Big]$$

$$= \frac{1}{(2\pi)^3} \int \frac{d^3\vec{k}}{\sqrt{2\omega_k}} e^{ik\cdot(x-y)} + \left[\left\{ \frac{1}{(2\pi)^{3/2}} \int \frac{d^3\vec{k}}{\sqrt{2\omega_k}} \left(\alpha_k e^{ik\cdot x} + \alpha_k^* e^{-ik\cdot x} \right) \right\} \left\{ \frac{1}{(2\pi)^{3/2}} \int \frac{d^3\vec{k}'}{\sqrt{2\omega_{k'}}} \left(\alpha_{\vec{k}'} e^{ik'\cdot y} + \alpha_{\vec{k}'}^* e^{-ik'\cdot y} \right) \right\} \right]$$

$$= G(x,y) + \langle \alpha | \phi(x) | \alpha \rangle \langle \alpha | \phi(y) | \alpha \rangle$$

Thus, for coherent states

$$\langle\alpha|\hat{\phi}(x)\hat{\phi}(y)|\alpha\rangle - \langle\alpha|\hat{\phi}(x)|\alpha\rangle\langle\alpha|\hat{\phi}(y)|\alpha\rangle = \langle 0|\hat{\phi}(x)\hat{\phi}(y)|0\rangle$$

- A classical like structure (upto an extra vacuum piece)

- Vacuum state is $\alpha = 0$

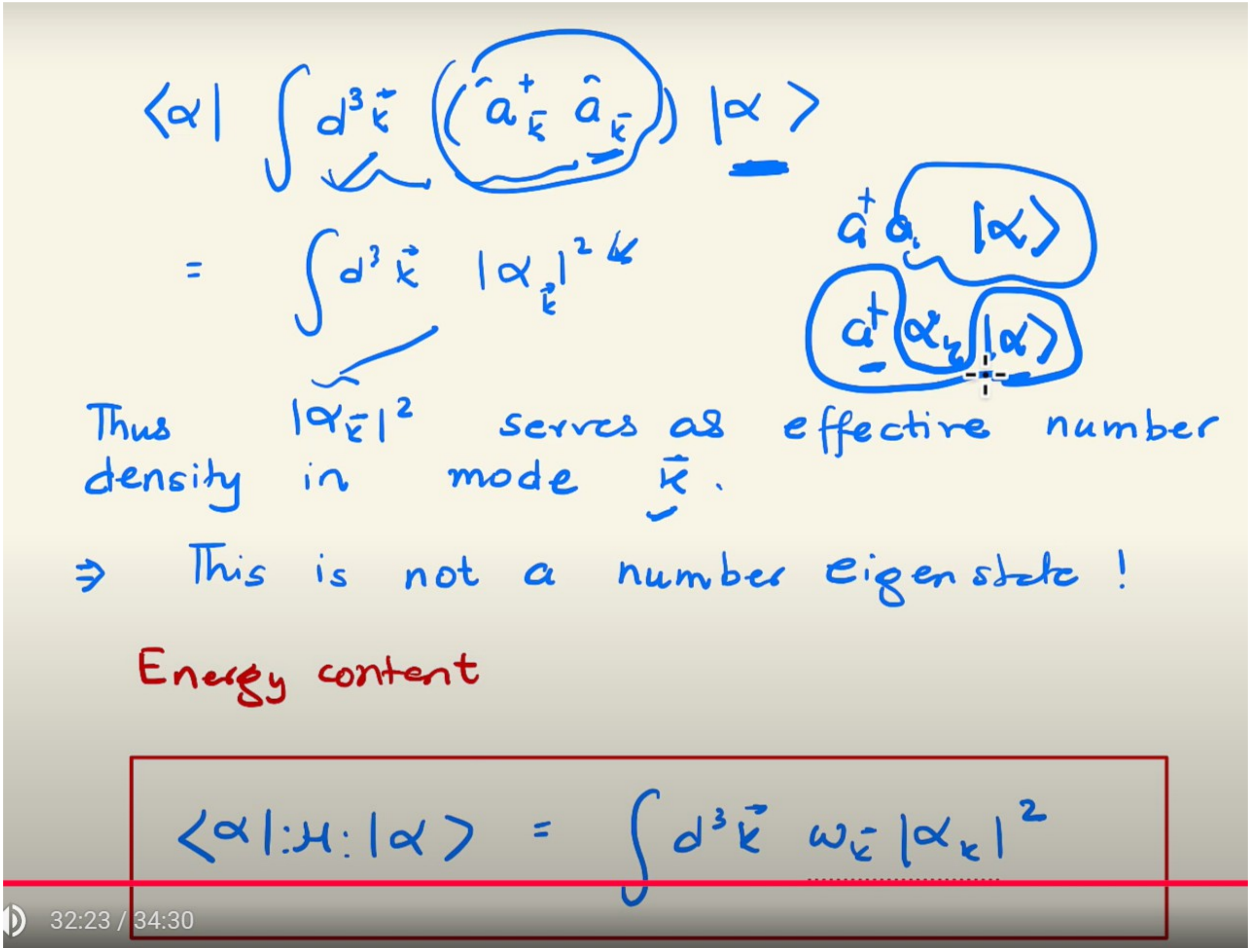
Since $\hat{\Pi}(x) = \frac{1}{(2\pi)^{3/2}} \int \frac{d^3\vec{k}}{\sqrt{2\omega_k}} (-i\omega_{\vec{k}}) \left(\hat{a}_k e^{ik\cdot x} - \hat{a}_k^{+} e^{-ik\cdot x}\right)$

One can verify that, even for $\hat{\Pi}(x)\hat{\Pi}(y)$

One can verify that, even for $\hat{\Pi}(x)\hat{\Pi}(y)$

$$\langle\alpha|\hat{\Pi}(x)\hat{\Pi}(y)|\alpha\rangle = \langle 0|\hat{\Pi}(x)\hat{\Pi}(y)|0\rangle + \langle\alpha|\hat{\Pi}(x)\hat{\Pi}(y)|\alpha\rangle$$

$$\langle\alpha|\hat{\Pi}(x)\hat{\phi}(y)|\alpha\rangle = \langle 0|\hat{\Pi}(x)\hat{\phi}(y)|0\rangle + \langle\alpha|\hat{\Pi}(x)|\alpha\rangle\langle\alpha|\hat{\phi}(y)|\alpha\rangle$$



Thus

$$\langle 0|\hat{a}_k^{\dagger}\hat{a}_k{}'|0\rangle = \alpha_{\vec{k}^*}\alpha_{\vec{k}}$$

$$\langle 0|\hat{a}_k^{\dagger}\hat{a}_k{}'|0\rangle = \alpha_{\vec{k}^*}\alpha_{\vec{k}}$$

Thus

$$\alpha_{\alpha}(x,y) = \frac{1}{(2\pi)^3}\int\frac{d^3\vec{k}}{\sqrt{2\omega_{\vec{k}}}}\int\frac{d^3\vec{k}'}{\sqrt{2\omega_{\vec{k}}'}}\begin{bmatrix}\alpha_{\vec{k}}\alpha_{\vec{k}}'e^{ikx+ik'y}+. \\ (\delta(\vec{k}-\vec{k}')+\alpha_{\vec{k}}^{\dagger}\alpha_{\vec{k}}')e^{ikx-ik'y}+. \\ \alpha_{\vec{k}}\alpha_{\vec{k}}'e^{-ikx+ik'y}+. \\ \alpha_{\vec{k}}\alpha_{\vec{k}}'e^{-ikx+ik'y}+.\end{bmatrix}$$

$=$

$$\frac{1}{(2\pi)^3}\int\frac{d^3\vec{k}}{\sqrt{2\omega_{\vec{k}}}}+\left[\begin{matrix}\frac{1}{(2\pi)^{3/2}}\int\frac{d^3\vec{k}}{\sqrt{2\omega_{\vec{k}}}}(\alpha_k e^{ikx}+\alpha_k^\star e^{-ikx})\\ \frac{1}{(2\pi)^{3/2}}\int\frac{d^3\vec{k}'}{\sqrt{2\omega_{\vec{k}}'}}(\alpha_k' e^{ik'y}+\alpha_k^\star{}' e^{-ik'y})\end{matrix}\right]$$

$$= \quad G(x,y)+\langle\alpha|\phi(x)|\alpha\rangle\langle\alpha|\phi(y)|\alpha\rangle$$

Thus, for coherent states

$$\langle\alpha|\phi(x)\phi(y)|\alpha\rangle\langle\alpha|\phi(y)|\alpha\rangle-\langle\alpha|\phi(x)|\alpha\rangle\langle\alpha|\phi(x)|\alpha\rangle\langle\alpha|\phi(x)|\alpha\rangle\langle\alpha|\phi(y)|\alpha\rangle=\langle\alpha|\phi(x)|\alpha\rangle\langle\alpha|\phi(x)\phi(y)|\alpha\rangle$$

- A classical like structure (upto an extra vacuum piece)

- vecuum state is α = 0
- Since $\hat{\Pi}(x)=\frac{1}{(2\pi)^{3/2}}\int\frac{d^3\vec{k}}{\sqrt{2\omega_{\vec{k}}}}(-i\omega_{\vec{k}})(\alpha_k e^{ikx}-\alpha_k^\star e^{-ikx})$
- One can verify that, even for $\hat{\Pi}(x)\hat{\Pi}(y)$

$$\langle\alpha|\hat{\Pi}(x)\hat{\Pi}(y)|\alpha\rangle=\langle 0|\hat{\Pi}(x)\hat{\phi}(y)|0\rangle+\langle\alpha|\hat{\Pi}(x)|\alpha\rangle\langle\alpha|\hat{\phi}(y)|\alpha\rangle$$

=

$$\langle\alpha|(\int d^3\vec{k}\,\hat{a}_k^\dagger a_k)|\alpha\rangle=\int d^3\vec{k}|\alpha_{\vec{k}}|^2$$

Thus $|\alpha_{\vec{k}}|^2$ serves as effective number density in mode $\vec{k}$.

⇒ This is not a number eigenstate!

Energy content

$$\langle\alpha|H|\alpha\rangle=\int d^3\vec{k}\,\omega_{\vec{k}}|\alpha_k|^2$$

Again you can cross back, cross check if αs are put to 0, only this delta function term survives, everything else washes out. So, now it is much more clear that it is different from the vacuum because there are surviving terms, non-zero surviving terms which are contributing into the 2.4 related. So, what I do? I take this delta function, this will convert one of the k' integrals into k integral and the usual vacuum structure. So, the vacuum term I separate out. This is the delta functions contribution. Then there are remaining four terms, this, this, this and that. And you can see they have very nice structure that they are individually made up from integral of this times integral of that. So, this is cross product or not cross because direct product of two individual integral. This integral and this integral put together generate all the four terms which are hiding here. And now we can identify this integral. This is nothing but if you remember expectation of the expectation of the field ϕ in the state α.So, you see the extra term which we have generated are nothing but the individual expectation at x and individual expectation at y in the coherent state. So, the two point function in the coherent state call it g α_{xy} is vacuum two-point function state and individual coherent state expectations. So, now we have quantified how much different it is from the vacuum state. The vacuum state would have just given me g this is the extra correction term due to presence of this α. So, therefore, we can write down that the two-point function – individual expectation at location x and location y, their difference is equal to the vacuum, the vacuum 2.4. In a classical system, if individual expectations are 0, individual numbers are 0 here and here, their product is also 0, classically. Quantum mechanically as we argued, if expectation is 0, expectation is 0 here, that does not mean the product of these two operators have also 0 expectation.

Now for coherent state, the difference between the product and the individual quantity is exactly the vacuum. So, it has almost a classical like a structure up to a vacuum. If I forget the vacuum piece, then this is what the classical thing would have done. The product is individual, the product expectation is individual expectations product. That is what a classical thing would have done. So, coherent state is as good as the classical system up to an extra vacuum piece. So, this 0 is the reminiscent quantum fluctuation of the vacuum. As we have repeatedly argued that vacuum state is just a particular realization of a coherent state that coherent state class is a class which includes vacuum in it as well. So, α is equal to 0 is the vacuum state. So, for α is equal to 0, this vanishes and therefore, this becomes a trivial identity. Now, this is for the field operator ϕ. What about its conjugate operator Π , Π if you recall is a derivative of ϕ, $\Pi\Pi$. Therefore, it will throw up a $-i\omega$ because exponentials involve $-i\omega t$. If I search for a time derivative, it will just give me exponential back with this factor. there will be a relative sign which will be coming out here, because the both the terms do come with a different signs in the i omega t. So, this is the Π operator. Now you can compute the expectation of Π, again you will get that in vacuum the Π's expectation is 0, in coherent state Π's expectation is Fourier transform of a $-i$ multiplied version of expectation of a_k which is you will get $\frac{-i\omega_k}{\sqrt{2\omega_k}}\alpha_k$, that is Fourier transform. This will appear over here and conjugate of that this time with a –, previously it was a + complex conjugate, this time it is a – complex conjugate because already we are talking about imaginary quantities. So, so far so good. So, that is exactly the similar kind of thing. And therefore, you will see that if I do the computation of 2.4 liter for conjugate field Π, everything else will fall through exactly like vacuum, exactly like the field ϕ description. Only thing if you go back, if I did the business for the, if I did the business for the correlation in Π Operator, what will just happen over here is in this discussion piece which we had over here, things will only change with extra $i\omega$ multiplications here and there and a sign difference. So, for example, if we are looking at Π and Π over here, a – sign should have come here, a – sign should have come here, a $-i\omega_k$ should have come from the first term and a $-i\omega_{k'}$ should have come from the second term.

And you can see all the operators expectations will still go through the same, extra a $-i\omega_k$, extra a $-i\omega_{k'}$ and a sign difference, first one +, –, – +. But operators in the computations which we are doing exactly remain the same kind of computations which we are doing. Even in the two, even in the single π expectation if I did here, π expectation of α. Only thing it might have happened as we discussed this will become a – and $-i\omega_k$ will extra come out if I did. So, you see if I computed the two-point correlator of ϕ, whatever we had done for Π almost goes through intact, only relative sign difference and extra couple of omegas come about. But at the end of the day exactly the same thing happens that even the two-point function of Π has the same structure as if that, sorry as if that it is, so I have missed α, it is a vacuum two-point correlator of Π^+ individual operator expectation of Π at location x and location y. So, in my notes if you are noting it down just correct there is a extra α, α in between. So, let me write it down. So, there is an α here and the α here. So, these are individual expectations. So, two point correlator is vacuum + individual expectation just like the field expectation did. And this is true for any combination of operators ϕ and Π as well also do the same thing that it is vacuum piece + an extra individual expectation. So, you see coherent state are almost doing the classical thing, classical things any two operators product is equal to the product of their individual expectation, all the time we are getting an extra vacuum term. So, you see coherent state are almost doing the classical thing, classical things any two operators product is equal to the product of their individual expectation, all the time we are getting an extra vacuum term. So, coherent state is therefore somewhat of a vacuum added classical state. It is almost doing classical business, but with an extra reminiscent quantum effect in terms of this vacuum correlators. You can compute what is the number expectation operator, number expectation is individual number operator in each mode and then sum over all possible modes. You can do this, you again work out the algebra as before, you will get the number operator is $|\alpha_k|^2$ integrated over all movement of modes. That means $|\alpha_k|^2$ here is like a number density in per mode. And then you integrate over the whole volume or k, you will get total number. However, vacuum state, so unlike vacuum state, α is not an eigen state of $a^\dagger a$. Because a will convert it back into a, this will give me α_k, a back, α state back. But $a^\dagger$ $a$cting on it does not remain into α state. α is eigen state of a operator only, $a^\dagger$ and a do not commute. So, therefore, it is not eigen state of $a^\dagger$ operator. So, therefore, it is not a number eigen state. Still, we can find out, still we can find out what is the effective number density in that. This is not eigen density, this is effective number state. And once we have effective description we can find out what is the normal order Hamiltonian as well which was remember apart from a divergent piece there was a number operator with extra ω_k multiplication. And once we have effective description we can find out what is the normal order Hamiltonian as well which was remember apart from a divergent piece there was a number operator with extra ω_k multiplication. So, you do that you will get that the effective number density times ω_k integrated over d^3k is going to tell you what is your total Hamiltonian above the vacuum divergent piece. So, this is the energy content. This is normal order thing is comparing its energy above the vacuum state. So, this is above the vacuum state. If you put all αs to 0, then that means you are talking about vacuum state. There is normal order Hamiltonian is 0, that means energy is exactly like the divergent piece which comes through vacuum. So, this is the broader structure of the coherent state which would be handy for us if when we discuss interaction of atoms with classical light. The classical light many of the times not most of the times can be mimicked by a coherent state up to a vacuum as we have seen. So, if you want to extract out hidden quantum features in classical light we can use this description of coherent light and try to see whatever the reminiscent quantum effects are there what they are supposed to do to a system that we will see when atom light interaction we discuss in more detail. So, for coherent state I stop the discussion, there is enough literature available on various other properties of coherent state, which we will not be going into that detail, but we have just constructed things and ideas which will be useful for us in our later part. So, I stop over here for this discussion session.