Quantum Fields expectation

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Scalar field vacuum structure

Lecture- 29

For today's discussion session, we will start learning about the expectation values of various field operators and their correlators in the given state of the field and we will start our journey with the most trivial kind of state which is our IQ. Till now we have learnt about the strategy of quantizing any kind of field and at the end of the day we learnt that the field itself becomes a set of collections of oscillator in momenta space and associated quantity with the field comes with various kinds of structure.

Quantum fields, states and their characteristics.

$$\varphi = \sum_q \int \frac{d^3 \vec{p}}{\sqrt{2 E_p}} (\hat{a}^q_{\vec{p}} e^{ipx} x_q + \hat{a}^{q\dagger}_{\vec{p}} e^{ipx} x_q)$$

$$[\hat{a}^q_{\vec{p}}, \hat{a}^{q\dagger}_{\vec{p}'}] = \delta(\vec{p} - \vec{p}') \delta^{qq'}$$

$$:H := \int d^3 \vec{p} \, \omega_{\vec{p}} \sum_q (\hat{a}^{q\dagger}_{\vec{p}} \hat{a}^q_{\vec{p}})$$

$$|0\rangle = \prod_{\vec{p}} \prod_q |0\rangle^q_{\vec{p}}$$

<u>Vacuum state</u> : Collection ground states of all oscillators

● Vacuum is normalized state.

$$._p^q\langle 0|0\rangle^q_p = 1$$

but excited state is not!

$$._p^q\langle 1|1\rangle^q_p = ._p^q\langle a^q_{\vec{p}} \big| a^{q\dagger}_{\vec{p}} \rangle^q_{p'}'$$

$$= \delta_{qq}' \delta(\vec{p} - \vec{p}') + a^\dagger a$$

For example, a scalar field will just be oscillator in momenta space and this chiq parameter which is appearing will become 1. There is nothing to associate it, it is just a scalar kind of thing. If it is a 10th, if it is a spinor kind of <u>field</u>, again I have oscillator structure with plane waves, but now I have a spinor part which gets multiplied over here. So, spinor and spinor, two different kinds of spinors will be coming along. Similarly, for vector degrees of freedom, I will have another set of oscillators with *pq*

and $pq^\dagger$ with this chiq now become the polarization directions. So, depending upon scalar or spinor or vectors, we will have associated quantities like ψ qs. So, this is the generic structure we will see in quantum fields. Now second thing is that all these oscillators operators they have the standard operator algebra that a and $a^\dagger$ have a commutation of identity it so happens that the momentum degrees of freedom are continuous so their identity representation is a direct delta function when the discrete degrees of freedom which will come either with the spin or the polarization will come with the chronicle delta for a scalar part this will not be present irrespective of whichever part we are talking about whichever kind of field we are talking about the Hamiltonian the normalized normal ordered Hamiltonian apart from the divergent term which always comes associated with this kind of fields. The overall above the ground state or above the vacuum Hamiltonian expectation value is obtainable from the rather the Hamiltonian operator is obtainable from the number operator in each mode and the frequency associated with that mode integrated over d^3 so this is the total oscillator energy for all sorts of oscillator which constitute the field point and as a corollary to that the ground state of the full field will be individual ground states of all possible oscillator states remember different oscillators come with a different level momentum p and let us say hyper spinner i'm tempted to call q for example spinor field it would be pins for vector fields it will be polarization so collectively I am calling it hyperspace. So this is in a nutshell the strategy of quantum field is there. Now that we have a vacuum state at our hand we can try to do some analysis of how the field will appear in vacuum and non-vacuum states gradually. So what is the property of the vacuum state that is the ground state of all the oscillators put together therefore it is ground state of a full field as well. And therefore this is a normalized state this is the state with which I start my business so this is a normalized state that means all oscillator state corresponding to label p and q they are normalized to unity so all this individual ground state of oscillator pq is normalized and as an artifact of that the whole vacuum is also normalized if I can compute zero zero the thing that is also one This state is a normalized state. However, if I start exciting states, exciting modes in different oscillator modes, then I get states which are not properly normalized. And then I take another oscillator which is excited at momenta p' and spin q' and try to compute the inner product between them. I will get first I can write down the one $p'q'$ state as vacuum of that oscillator and then acted upon on that by the raising operator.Similarly on the other side the bra of the same thing is appearing will have this structure. And we know that there is a commutation relation between a and $a^\dagger$ which is appearing over here which I can flip the order of a and $a^\dagger$ with a use of a delta function so $\delta(q$ -$q')$use this use this commutation relation here $\delta(q$ -$q')$ $\delta(p$ -$p')$ + $a^\dagger a$ would have appeared over here if I wrote like this. But $a^\dagger a$ is expectation in vacuum will vanish because vacuum is no number excited state, number operator is 0 or in other words the a annihilates the vacuum for each of qp pairs. So therefore, I see that only thing I will be left with this expectation of this thing and vacuum expectation of this operator in the vacuum state. And the vacuum state of this thing will not see any operator around, it is just identity operator, so they will be self normalized. So I will be just left with the piece which is the delta function. And as you can see, if I make the two oscillators excited state p is equal to p', q is equal to q', that will give me the normalization of the state. That becomes divergent because p going to p' is $\delta(0)$,$\delta(0)$ is infinite. So I have single excitation has created me a state which is not normalizable. This is not a surprise because as we know, single excitation creates only a plane wave. This is plane wave mode in position space. In oscillator space, momentum space, it is just first particle oscillator, one excited state oscillator. However, in physical space, you will see that it relates to opened up distribution in space, which is a plane wave. And we all know plane waves are not a normalizable state. So therefore, I realize this. So the state which we have generated despite being localized in momenta becomes opened up in position. This is the standard thing x and p since they are Fourier variables. Right now they are not operators because we are talking about quantum field theories in and p individually are not operators, but they are Fourier related variables, positions to momenta or Fourier momenta goes through a Fourier transform. And that property remains true as well that if one is rightly localized, that means I have a delta function in distribution of position, then I get a delta

function, I get opened up plane wave distribution in momenta and vice versa. So, that is what we are getting. We start with a definite momenta P, if I make it p and p', that means I know that state is p, the momenta of the state is definitely known to be p, there is no distribution about it. Therefore, the distribution profile in the momentum space is delta function and its representation in positional space is a plane wave. And in either case, you can see that it is not a normalized state. So, this is not a surprise. What we have to do? I have to smear a state with some smearing function. So, let us say some f of f of ω and then or call it p and then I take this $a^\dagger_p$ and this might be a normalized state of one excitation. But in a definite known moment if this becomes a delta function then it is not normalized. But in a definite known moment if this becomes a delta function then it is not normalized. If I make it $p - p$ naught, that means this integration will just create me a state at p naught. This is not a normalized state because of this delta appearing. But if it is not a delta but a regular function, then it is a normalized state. So, outcome is that normalized states of quantum field theory do not have a specific momenta. They have a distribution in momenta. And then there is a well-defined distribution, normalized distribution in position as well. that is what we would have expected of a wave packet in quantum mechanics as well the only thing we are now coming from quantum field theory domain so I would not make a quantum field theory made up only one excitation I should make my quantum theory quantum field out of it which are made up of many many many many modes and there is a weight factor to them such that the overall fields wave state is normalized okay so that is how this the strategy is to make states with excitations now we will go ahead and see what is the property of vacuum first of all to see before going to excited states so now let us try to compute what is the expectation of a field itself in the vacuum state this might be not very surprising for you because we have seen it before as well so first let us write down the field in terms of its bare operators the raising and the lowering operator so you see I have

written the field φ as the description which I had provided $d^3\vec{p}/\sqrt{2\omega_p}$ and then uh this these operators raising and lowering and the raising operator I am writing here a and $b^\dagger$ here most of the times the scalar or the vector fields we have seen a and $a\dagger$ appear only for spinal fields we saw that two different operators appear because we did not require the ψ field in spinor case to be real so I am writing a general structure if it is real then a and $b^\dagger$ b becomes $a\dagger$ meaning $b^\dagger$ is $a\dagger$ otherwise they are two different unrelated operators so irrespective of whether it is a or b the following is always true if I take the expectation of any spinor or vector or the scalar field in its vacuum at the end of the day what it is going to happen is that all other quantities like this plane wave like this ψ q like this integration they will be just riding along they will just remain like they are only operator in this game is this a and this $b^\dagger$ so only these operators undergo the expectation all other things remain intact as they were because they are not operators of the state oscillator operators are these we are just position space dependent quantity and extra degrees of freedom dependent quantities okay so all only the operators which get squeezed between the vacuum state is a and a $b^\dagger$ here for example and we know this operator annihilates the vacuum so I am going to get a zero from this and b annihilates the vacuum as well that means $b^\dagger$ $b^\dagger$ with a bra zero is also zero this is just a hermitian conjugate of that so therefore this portion is zero as well so both the portions this portion and the left portion of that are zero individually so I get a zero expectation value of the field operator in their vacuum is zero there is nothing to be expected in the vacuum this is a standard that's why the name vacuum is there is nothing in the field however this is not as same as saying the field is empty despite having no expectation there still can be things expectation is just like a mean value now we know of various distribution which have a mean value zero for example a gaussian centered about zero will have its mean value zero but its other properties are non-zero if something's mean is zero it is not true that it is completely zero or everywhere else so empty field would be completely zero everywhere else non-zero non-empty uh non-empty field with zero expectation could be like a gaussian therefore one should need to look at whether it is only the expectation zero or its other expectation values are also zero for example variance whether it is a non-zero variance. This distribution which is completely zero will have zero variance as well. However, this

kind of distribution will have a zero mean but non-zero variance. So, let us try to compute things. So, variance of any operator we know in quantum mechanics, variance of any operator would be variance square let us say or variance is defined as a root mean square itself. So, that would be operator square's expectation – expectation square. So that means a double operator structure is required. First here I have computed the single operator whose square is supposed to come here and then two operators O and O has to be computed in the same state.Now since we are talking about field operators, these are position dependent quantities. So better quantity would be, more richer quantity would be two different location operators φ x and φ y. The limit x tending to y will generate variances like this, but it has a more richer structure that I can put two operators side by side in terms of two different position values, position and space, time value. So therefore, I am going to obtain this quantity which is a more general thing than the variance. It contains all sorts of fluctuations, not only same point fluctuations, but correlation point, meaning from this point to that point, x and y being different, how are they correlated kind of thing. So let us compute this object and see whether it is 0 as well. If it is 0, that means the field has no correlation, no variance as well and it is flat kind of thing.. So, let us try to do that. So, again I will write down the two φ's as two integrals of oscillator breakup one, oscillator breakup two. p comes with the first oscillator, E′, label p' comes with the second oscillator. Again, if you club together all the terms, you will have four kind of terms, this going with the two terms of here, this second of the file and this also undergoing with the two terms of the second file. So, ultimately I will have a four kind of terms and therefore four kinds of operators which will be coming about. First this a operator will combine with this a operator, then this a operator will combine with a† operator here. Then there is this a† operator which will combine with this a operator here and lastly this a† operator will combine with this a† operator over here so I will have operator a a operator $aa^{\dagger}$operator $a^{\dagger}a$ and operator $a^{\dagger}a$† all of these quantities are getting skewed between batches so that means all these operators over here will be done expectation with respect to the batch. Now I know that this a acting on 0 will kill it. a† is not clear what will happen to them. So let us leave them around.

$$\langle 0| \hat{\varphi}(\vec{x})|0\rangle = \sum_q \int \frac{d^3\vec{p}}{\sqrt{2\omega_{\vec{p}}}} \langle 0| \hat{a}_p^q e^{ip\cdot x} \chi_q + \hat{b}_p^{q\,\dagger} e^{-ip\cdot x} \tilde{\chi}_q |0\rangle$$

$$= \sum_q \int \frac{d^3 p}{\sqrt{2\omega_p}} \left[\langle 0| \hat{a}_p^q |0\rangle e^{ip\cdot x} \chi_q + \langle 0| \hat{b}_p^{q\,\dagger} |0\rangle e^{-ip\cdot x} \tilde{\chi}_q |0\rangle \right]$$

✓ $= 0$

✦   But that is not empty field

$$\langle 0| \hat{\phi}(x)\hat{\phi}(y) |0\rangle$$

$$\langle 0| \begin{matrix} a \fbox{a} & a^\dagger \\ a & a^\dagger \\ a^\dagger & a \\ a^\dagger & a^\dagger \end{matrix} 0 \rangle$$

$$= \int \frac{d^3\vec{p}}{\sqrt{2\omega_{\vec{p}}}} \int \frac{d^3\vec{p}'}{\sqrt{2\omega_{p'}}} \langle 0| \left[(\hat{a}_p e^{ip\cdot x} + \hat{a}_p^\dagger e^{-ip\cdot x})(\hat{a}_{p'} e^{ip\cdot y} + \hat{a}_{r'}^\dagger e^{-ip'\cdot y}) \right] |0\rangle$$

$$= \int \frac{d^3\vec{p}}{\sqrt{2\omega_p}} \int \frac{d^3\vec{p}'}{\sqrt{2\omega_{p'}}} \langle 0| \hat{a}_{\vec{p}}\, \hat{a}_{r'}^\dagger |0\rangle e^{ip\cdot x - ip'\cdot y}$$

$$\delta(\vec{p} - \vec{p}') \implies \qquad \omega_{\vec{p}}' = \omega_{\vec{p}}$$

$$= \int \frac{d^3\vec{p}}{2\omega_{\vec{p}}} e^{ip\cdot(x-y)} \neq 0$$

There are quantum correlations

$$\langle 0| \phi(\vec{x})\phi(\vec{x}')|0\rangle \neq \langle 0| \phi(x)|0\rangle \langle 0| \phi(x')|0\rangle$$

$$\langle 0|\hat{\varphi}(\vec{x})|0\rangle=\sum_q \int \frac{d^3\vec{p}}{\sqrt{2\omega_{\vec{p}}}}\left[\langle 0|\hat{a}_p^\dagger|0\rangle e^{ipx} e^{ip\cdot x}\chi_q+\langle 0|\hat{b}_p^\dagger|0\rangle e^{-ipx} e^{ip\cdot x}\tilde{\chi}_q\right]$$

$= 0$

⋆ But that is not empty field.

$$\langle 0|\hat{\phi}(x)\hat{\phi}(y)|0\rangle$$

$$= \int \frac{d^3\vec{p}}{\sqrt{2\omega_{\vec{p}}}} \int \frac{d^3\vec{p}\,'}{\sqrt{2\omega_{\vec{p}}\,'}} \langle 0|(\hat{a}_p e^{ipx}+\hat{a}_p^\dagger e^{-ipx})(\hat{a}_p e^{ipy}+\hat{a}_p^\dagger e^{-ipy})|0\rangle$$

$$\int \frac{d^3\vec{p}}{\sqrt{2\omega_{\vec{p}}}} \int \frac{d^3\vec{p}\,'}{\sqrt{2\omega_{\vec{p}}\,'}} \langle 0|\hat{a}_p \hat{a}_p^\dagger|0\rangle e^{ip\cdot x - ip'\cdot y}$$

$$\delta(\vec{p}-\vec{p}\,') \quad \Rightarrow \quad \omega_p = \omega_p{}'$$

$$\int \frac{d^3\vec{p}}{\sqrt{2\omega_{\vec{p}}}} e^{ip(x-y)} \neq 0$$

$$\langle 0|\hat{\phi}(\vec{x})\phi(\vec{x}\,')|0\rangle \neq \langle 0|\hat{\phi}(\vec{x})|0\rangle\langle 0|\hat{\phi}(\vec{x}\,')\phi(\vec{x}\,')|0\rangle$$

This $a^\dagger a$ within vacuum will also be 0 because a will kill this state. And $a^\dagger a\dagger$ in the vacuum again will be killed off because $a^\dagger$ 0 from the left is 0. So only surviving term will be $aa^\dagger$. Okay and now I have this kind of structure what do we do whatever we we had done previously I will convert this $aa^\dagger$ into $a^\dagger a$ and a delta function delta function will come from the commutator so I will flip this with the help of a commutator and I will get a $\delta(p - p')$. So now just for illustration, I am working with a scalar field. q is put to 0. There is no Q. But if you are thinking about a spinal field or a vector field, be mindful of there is a q also coming about over here. And there is summation over q as well. The two qs, q and q's. So be careful of that fact. Otherwise, all these things will go along together. This is just oscillator story, which oscillator with quantum number, which we have to sum over collectively. So what I have done, I have taken this pair over here, flipped it and a delta function comes about and then squeeze the whole quantity in the vacuum. When I squeeze the whole quantity in the vacuum, again this $a^\dagger a$ inside vacuum which we have known already is 0. I will be left only with a $\delta(p - p')$ because the vacuum state is normalized. So this expectation of $aa^\dagger$ structure which was non-zero is non-zero with a delta function. It will give me $a\delta(p-p')$. So one of the p' integrations can be computed using this delta function. What will it do? It will replace every p' to p, $\vec{p}\,' \to \vec{p}$. If that happens, ω_p which is just a magnitude of $\vec{p}$ and ω_p' which is the magnitude of $\vec{p}\,'$ will also become the same. The three vector are supposed to get same because of the delta function. The fourth component of the vector will become equal to each other because the zeroth component entirely depends on the three component, the remaining three component, the spatial part. The delta function's integration can be used. to get rid of one integration and converting this p to p' and ω_p to omega'. So, ultimately I will be left with this and this quantity will become the same, this and this quantity will become the same and ultimately I will be left with $d^3\vec{p}$ to $\omega_p e^{ip(\vec{x} - \vec{y})}$, which is not 0, you can prove that this is not a 0, this survives. for any x − y this is not a zero quantity we will compute it explicitly what it is later on but you can convince yourself that this is a non-zero number so it is not a case of an empty field because empty field would have a zero variance as well if I take x going to y that is the same point correlation this will become d^3t/ω_p some integration to compute which will show you some number which is divergent but that is another property of quantum field same point quadratic operators or higher operators are divergent and there is a regularization schemes people have to do on the same point correlators those stories are separate that is the story of the fields and how the ue story of the field unfolds we are not concerned over here because most of the time in this course we will not be concerned with the same point variance but the correlator structure which will emerge handy in our computations later on. So, you see for non-zero x − y, I will have a non-zero number.Even for same number x is equal to y, this is a non-zero number, but a

very huge number. Okay. So, therefore, I have a non-zero correlation in the field, non-zero variance in the field. That means it is not an empty field. It contains something despite being called vacuum because expectation wise individually, it has nothing. If I compute the field operator, I do not get anything. So bottom line is the two point correlator is not just the product of the single time operators expectation this happens for real numbers or classical quantities right if I have of some quantity which takes value q_t at time t classically and q some other value qt' then I know the product of them $q_t q_{t'}$ if this is 0 and this is also 0 one of them is 0 let us say then the product is also 0 classical operators classical quantities satisfy like this if either of them is 0 their product is 0. However, the product operators expectation is not 0 when either of the operators expectation is individually 0. So, this correlator does not factorize like this. That is the hallmark of a quantum theory. They do not get factorized like classical variables. So, the vacuum state is not a classical state in that sense. it has a quantum structure despite being empty it does not follow the line of a classical thing it has its own hidden quantum fluctuations inside it and they might be important for certain processes we will see more many of the examples later on but keep in mind despite being named vacuum it is not completely free of anything it is not an empty state it has correlations but expectations are no are zero all right So, now let us explicitly compute the quantity which we were claiming it to be non-zero and see for ourselves that depending upon x – v is separation. This x – v by the way is space time separation, it is not the spatial separation. Remember this dot product was between four vectors. So, space time separation, how does the two point correlator become zero or non-zero? We have our reasons to believe that it is a non-zero quantity.
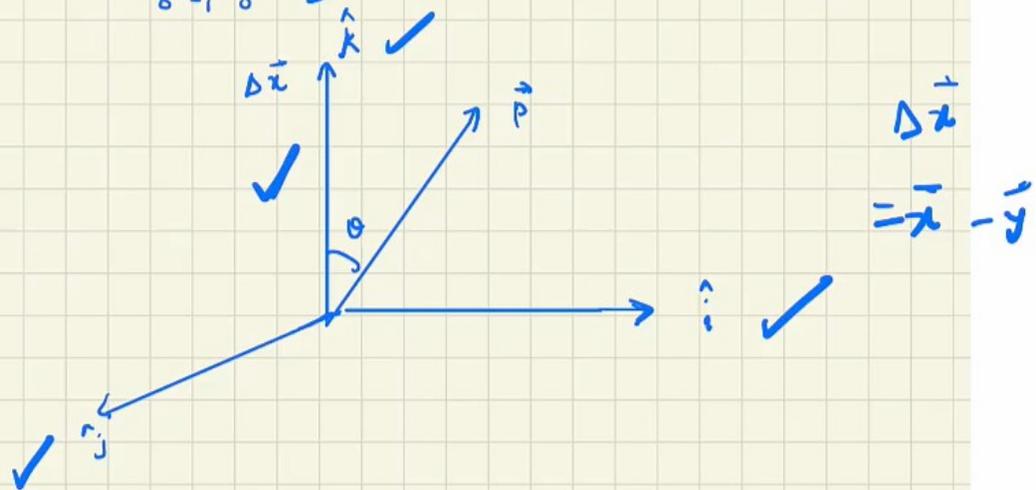
Computation of quantum correlation

$$\langle 0|\, \phi(x)\, \phi(y)\, |0\rangle = \frac{1}{(2\pi)^3} \int \frac{d^3 p}{2\omega_p}\, e^{-i\omega_p(t-t') + i\vec{p}\cdot(\vec{z}-\vec{y}')} \equiv G(x,y)$$

For massless field

$$\omega_p = \left(\sqrt{P_x^2 + P_y^2 + P_z^2}\right) c = p\, c$$

$$G(x,y) = \frac{1}{(2\pi)^3} \int_0^{2\pi} \int_{-1}^{\infty} \int_0^{\infty} \frac{P^2\, dP\, d(\cos\theta)\, d\phi}{2P} \left[e^{-i P \Delta t + i\vec{P}\cdot\Delta\vec{z}} \right]$$

$$\Delta\vec{z} = \vec{x} - \vec{y}$$

★ Given $\vec{x}$ and $\vec{y}$; we can orient $\Delta\vec{z}$ along $\hat{k}$ axis.

$$\vec{P}\cdot\Delta\vec{z} = P \Delta x \cos\theta$$

$$G(x,y) = \frac{2\pi}{(2\pi)^3} \int_{-1}^{1} d\cos\theta \int_0^{\infty} \frac{P}{2} dp \left[e^{-i P \Delta t}\, e^{i P \Delta x \cos\theta} \right]$$

Computation of qunatum correlation

$$\langle 0|\hat{\phi}(x)\phi(x')|0\rangle = \frac{1}{(2\pi)^3} \int \frac{d^3 p}{2\omega_p}\, e^{-I\omega_p(t-t') + i\vec{p}\cdot(\vec{x}-\vec{y})}$$

$$\equiv \quad G(x, y)$$

For massless fields $\quad \omega_p = (\sqrt{P_x^2 + P_y^2 + P_z^2}) = pc$

$$G(x, y) = \frac{1}{(2\pi)^3} = \frac{\int\limits_0^{2\pi} \int\limits_{-1}^1 \int\limits_{-\infty}^\infty p^2 \, dp}{2p} \, d(\cos\theta) \, d\phi \, [e^{-ip\Delta t + i\vec{p}\cdot\vec{\Delta x}\cos\theta}]$$

$$G(x, y) = \dot{\iota}$$
$$\frac{2\pi}{(2\pi)^3} \int\limits_{-1}^1 d\cos\theta \int\limits_{-\infty}^\infty \frac{p \, dp}{2} [e^{-ip\Delta t} e^{-ip\Delta x \cos\theta}]$$

So, let us go ahead and try to do this computation, try to do this computation explicitly. So, remember when we tried obtaining the two-point correlator, we landed up with this integral which we have to perform. So, let us go ahead and perform this integral. We have a $d^3\vec{p}/\sqrt{2\omega_p}$ in the division, $(1/2\pi)^3$ should be there. Actually, I have missed out writing $d^3\vec{p}/\sqrt{2\omega_p}$ everywhere. So, each field comes with a $(1/2\pi)^3$. that we had seen long back as well. So, I am just maintaining it. So, everywhere I have a single field will come with $(1/2\pi)^{3/2}$. a double field will come with $(1/2\pi)^3$. Because two fields will have the square root quantities canceling each other and $(1/2\pi)^3$ will be covering. So, I can put it over here. This is overall a constant. Do not worry. I am just trying to keep it consistent with the notations of many textbooks. So, if I write it like that $d^3\vec{p}/\sqrt{2\omega_p}$ would be there and p dot x which was appearing over here. Remember in our notation temporal part gets a negative sign. So, I will have a $p^0\delta(x-y)$,it was. So, I am going to write this as t – t' and then the spatial part comes with a usual dot product in the spatial vectors. So, p dot x – y it was. So that is how the p dot x would be written. And symbolically it is given a structure g . This is called the correlation function or sometimes the Wittmann function. So Wittmann function. Now you can see from the face of it everywhere p is undergoing integral, but p is depending on t – t' and x – y. It is not depending t and t' individually in a arbitrary fashion, it only depends on t – t'. And in spatial part it depends on $x - y$ which was the hallmark initially which we have written that it depends on four vector $x - y$. It is not a function of x + y, it is not a function of x^2+y^2 or any other function. It is a function of x and y in a particular way which is $x - y$. So, we could have written g For vacuum state, it is just gx – y. For other states, it might not so happen because that depends on what integral I will end up with. This integral is true only for the vacuum state. This integral is true only for the vacuum state. For other states, we have to see out of these four operators which we had seen over here, only $a^\dagger$ was surviving. That gave me this. It might so happen that for other operators, other states many of these operators are non-zero and ultimately I do not know whether I have a structure of as clean as this. This is 2 only for vacuum. Only for vacuum the Wittmann function or two-point function despite coming individually with each field at the end of the day becomes a function of $x - y$. Keep that in mind and we will see an example later on that in the next class that this idea that g is a function of x – y is only true for vacuum state. So, I am not writing it as x – y, I am just maintaining it x comma y. Now, for massless field we know ω_p is related to the magnitude of the spatial vector times c that we have seen before as well. I am not writing + – because + – part was already taken care when we were writing these quantities here and here. These included the + – part. So whatever is appearing here has a definite signature, positive. So that is why thI am using c is equal to 1 unit as before . e^{p_0} which is ω_p has a positive signature over here. So let us write it down. So what what will happen $d^3\vec{p}$ downstairs I should have a $2pc$ but I am forgetting about c c is equal to c is being put to one and this $d^3\vec{p}$ integral can be written as in spherical polar coordinates as the magnitude of the $\vec{p}$, $p^2 \, dp$, d $\cos\theta$ and d φ, where $\cos\theta$ d φ, θ and φ are angles in x, y, z axis or i, j, k axis. So, I can think of a arbitrary coordinate system,

some direction is I direction, some direction is j direction, some direction is k direction. And in that case, the $d^3\vec{p}$ can be written like this. In this coordinate systems, angular spherical polar coordinates. Now I can, what I can do, what appears in this integral over here is this $\delta(x)$ vector. $\delta(\vec{x} - \vec{y})$ So, I have decided to call t – t′ to $\delta_t(\vec{x} - \vec{y})$ is $\delta(\vec{x})$. Now given you can think of any point in space x, any other point in space which is y. So, this will be x, this will be y. in your initial coordinate system. So, $x - y$ will be some other vector. Now, what you can think of this new direction $x - y$ as your new z-axis or new k-axis. So, that is what we have done here. We have aligned our k-axis with $\delta(x)$ direction. Once point x and y are given to you this is four dimensional point x and four dimensional point y out of which I will plug out pick out what is the position space x and position space y then I find out where is the x – y direction x – y vector will be some direction I start calling this as my new z direction so that is what we have done looking at $\delta(x)$ vector we have computed the direction of the z axis in the new basis. And therefore, I am going to write down the angular integrals and other things assuming that $\delta(x)$ is giving me the direction of z. So, if that happens then the $\vec{p} \cdot \vec{x}$ appearing over here is actually the p is dot product with the z direction. So, it will become $p \cdot p\delta(x)$ times $cos\theta$. Remember this is pointing in the z direction its magnitude is $\delta(x)$ So, magnitude I am writing this much. So, p dot x will be p x $cos\theta$, $p\ \delta(x)\ cos\theta$. So, so far so good. I will just plug this quantity here as $p\ \delta(x)\ cos\theta$. Now, I have a d φ integral to compute.   Nothing in this integral depends on φ. It depends on $cos\theta$. It depends on $\delta(t)$. Nowhere φ has appeared in this coordinate system. So, d φ integral in this coordinate setup will just go along without unperturbed and 2π will come out. I will be left with only d $cos\theta$ integral and dp integral. p is appearing here, $cos\theta$ is appearing here. So, that we have to do slightly more carefully. So, first let us do the d $cos\theta$ integral. You can call this $cos\theta$ as y, then effectively I have integral of dy and $e^{ip(\hat{x} - \hat{y})}$. So, if you do that you will obtain this integral which is required as $e^{ip\ cos\theta\ \delta(x)}/ip\ \delta(x)$ between – 1 to 1 and you can verify for yourself you will get this structure $e^{ip\ \delta(x)} - e^{-ip\ \delta(x)}/ip\ \delta(x)$. So next we can plug this quantity of integral into the expression which I had. Now only this integral is left, this function is left and the integration of this quantity and that quantity has been performed. So outside I will get 1 over 4 π^2, dp integral is supposed to be surviving but see there is a p coming from here which will cancel the, which will cancel the p. which was surviving over here. This p will be cancelled by the integration will give me this result, this p in the denominator cancels the p upstairs. What will extra it will bring? i times $\delta(x)$ will be extra. So, twice $i\ \delta(x)$ will be in the denominator. So, the left over integral is dp from 0 to infinity $e^{-ip\ \delta(t)}$ which was the other function which we have not touched and the $cos\theta$ integral has given me this piece.

$$\int_{-1}^{1} d\cos\theta \; e^{ip\Delta x \cos\theta} = \left. \frac{e^{ip\cos\theta \,\Delta x}}{ip\,\Delta x} \right|_{-1}^{1}$$

$$= \frac{e^{ip\Delta x} - e^{-ip\Delta x}}{(ip)\Delta x} \quad \checkmark$$

$$\therefore \; G(x,y) = \frac{1}{4\pi^2} \int_{0}^{\infty} \frac{dp}{2ip\,\Delta x} \; e^{-ip\Delta t} \left(e^{ip\Delta x} - e^{-ip\Delta x} \right)$$

$$= \frac{1}{8\pi^2 i \,\Delta x} \left[\int_{0}^{\infty} dp \left(e^{-ip[\Delta t - \Delta x]} - e^{-ip[\Delta t + \Delta x]} \right) \right]$$
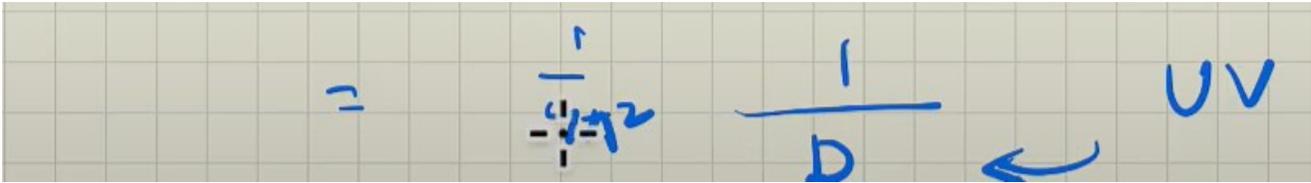
In order to compute its 'convergent' limit

$$\Delta t \;\longrightarrow\; \lim_{\varepsilon \to 0} \left(\Delta t - i\varepsilon \right)$$

$$(x,y) = \lim_{\varepsilon \to 0} \frac{1}{8\pi^2 i \,\Delta x} \left[\frac{e^{-ip(\Delta t - \Delta x)} \, e^{-p\varepsilon}}{-i[(\Delta t - i\varepsilon) - \Delta x]} + \frac{e^{-ip(\Delta t + \Delta x)} \, e^{-p\varepsilon}}{i[(\Delta t - i\varepsilon) + \Delta x]} \right]_{0}^{\infty}$$

$$= \lim_{\varepsilon \to 0} \frac{1}{8\pi^2 \,\Delta x} \left[\frac{e^{-ip(\Delta t - \Delta x) - p\varepsilon}}{[(\Delta t - i\varepsilon) - \Delta x]} - \frac{e^{-ip(\Delta t + \Delta x) - p\varepsilon}}{[(\Delta t - i\varepsilon) + \Delta x]} \right]_{0}^{\infty}$$

$$= \lim_{\varepsilon \to 0} \frac{1}{8\pi^2 \,\Delta x} \left[\frac{1}{(\Delta t - i\varepsilon) + \Delta x} - \frac{1}{(\Delta t - i\varepsilon) - \Delta x} \right]$$

$$= \lim_{\varepsilon \to 0} \frac{1}{8\pi^2 \,\Delta x} \left[\frac{-2\Delta x}{(\Delta t - i\varepsilon)^2 - \Delta x^2} \right] = \frac{1}{4\pi^2} \frac{1}{(-\Delta t^2 + \Delta x^2)}$$

So, let us combine them both. So, if we combine these quantities, I get $e^{-ip\,\delta(t) - \delta(x)}$ and $e^{-ip\,\delta(t)} + \delta(x)$. Notice one thing that $\delta(t)$ appears with negative sign in both the terms. In this term, it is coming with overall positive term signature and in this term, it is coming with overall negative sign. This is like e^{ikx} integral, right? Integration over dk from 0 to infinity. x happens to be this function here and this function in the second term. But the structure is very similar, e^{-ikx} kind of thing. And you know e^{-ikx} gives back e^{ikx} with $aix - ix$ in the denominator where k runs from 0 to infinity. So exponential gives you back the exponential integral and k quantity runs from 0 to infinity. So, infinity in the exponential will give you $e^{-i\infty x}$ or in our case $\delta(t) - \delta(x)$ for the first integral and $\delta(t)$ + delta I for the second integral.And this will be highly fluctuating function. This is the frequency of infinity. This will be rapidly fluctuating functions. And it is difficult to estimate its value because it is highly oscillatory function. In any finite time duration, it will do so many oscillations that virtually everything will be cancelled out and you will be left with some finite mean value it would have. so it is not a convergent kind of thing this is not a convergent result it is oscillating about some point so what we can do in order to make it convergent we can take its limiting convergent behavior that is to say whatever is the common signature term which is appearing in both of them I will go one step back and call that $\delta(t)$ I am going to replace with $\delta(t) - i\varepsilon$ okay if that happens Both the terms get this $- i\varepsilon$ extra from the $\delta(t)$ part and both the terms obtain $e^{-p\delta}\,e^{-p\varepsilon}$ at the end of the day I am going to put ε tending to 0. Because I just want to get the result for only $\delta(t)$. But as of now, I will just insert an extra $- i\varepsilon$ to make the integrals convergent. Now, I have no problem because both the integrals e^{-ikx} and $e^{-ikx'}$, where this is x' and this is x, both have this extra $e^{-p\varepsilon}$. I have assumed ε is greater than 0 let us say. In that case, when I take the upper limit infinity, this exponentials will kill the upper limit. $e^{-\infty}$ is 0. So therefore, at the upper limit, the highly oscillatory kind of thing averages out to 0. There is nothing left from the upper thing. While for the lower limit, the things will survive. The lower limit, the numerators will become 1 for both of them and remember there is a sign difference. $- i$ appears for the first term and $+ i$ appears for the second term.And there is a $+ \delta(x)$ here and a $- \delta(x)$ here. So only lower limit of this thing will matter and that is what we compute. we will get this limit delta e tending to 0, because limit ε tending to 0, because that is what we want at the end of the day. Outside this 4 π^2 times twice I $\delta(x)$, they are giving me $8\pi^2 i\delta(x)$ which I am going to maintain. And the integrals are going to throw up these quantities, $i - i$ kx and so on. So, towards the lower limit 0, I will be left with this quantity which is surviving 1 over $\delta(t) - i\varepsilon + \delta(x)$ in the lower limit this will become positive sign this will come first and the other thing will come second this will come here at the lower limit 1 over $\delta(t) - i\varepsilon - \delta(x)$. You can combine these things smoothly and you will get the structure that is $-$ twice of $\delta(x)$ and denominator $- \delta(t) - i\varepsilon$ whole square $- \delta(x)^2$ this is spain and simple algebra and then you take ε tending to zero now you see a finite piece survives and that is non-zero this happens to be 1 over 4 π^2 times the invariant space time distance d invariant space time distance if you remember our discussion ds² was something like this $- dt^2 + dx^2$ So, this is finite ds and dx were infinitesimal and this is $\delta(t)$ and $\delta(x)$, this can be two for two separated physically large separated points as well, but still it is Lorentz invariant quantity. So, this is the structure of the two-point function or the correlator structure is 1 over d and what is the space-time distance. If you try to bring the two point to the same, for example, getting the variance.And therefore, this will blow up that is the divergent part we were talking about initially that any two point

or quadratic or higher order operators in quantum field theory has a divergent pole always that is called the UV pole. UV meaning short distance pole, ultraviolet pole, the small wavelength limit or very high frequency limit. So, this is a divergent structure in the two points being the same, φ^2 kind of thing. So, φ^2 x can be viewed as φ x, φ y and limit x tending to y and that is where the d will shrink to 0.

Otherwise for finite x and y separation, this is my non-zero number, the physical d, physical point separation between two points, physical distance separation and therefore, the correlator correlation is non-zero over there. Even for same point correlation is non-zero, it is divergent. But all other places as well even at very very far away distance, the correlation is non-zero. So, this again leads to another kind of EPR kind of puzzle, how correlations are surviving at very very large distance. $c\delta(t)$. That means there are no physical signaling between these points. Points are so separated that their distance in $\delta(t)$ amount cannot be covered by speed of light as well. There is no causal communication. Still correlation survives. Correlation of quantum fields is not causally, not a causal effect. They are not generated from one point and sent to another point because nothing can travel faster than speed of light and this survives even after the domain of the speed of light. So, this is just a genuine quantum of structure, this is not classical thing, nothing is classically travelling from one point to another, this is just inbuilt correlators in them.

○ For electromagnetic field

$$\langle 0| A^i(x)\, A^j(y)\, |0\rangle$$

$$= \frac{1}{(2\pi)^3}\int \frac{d^3\vec{p}}{\sqrt{2\omega_p}}\int \frac{d^3\vec{p}'}{\sqrt{2\omega_{p'}}}\, \langle 0|\left(\hat{a}^i_{\vec{p}}\, e^{ip\cdot x} + \hat{a}^{i\dagger}_{\vec{p}}\, e^{-ip\cdot x}\right)\left(\hat{a}^j_{\vec{p}'}\, e^{ip'\cdot y} + \hat{a}^{j\dagger}_{\vec{p}'}\, e^{-ip'\cdot y}\right)|0\rangle$$

$$= \frac{1}{(2\pi)^3}\int \frac{d^3\vec{p}}{\sqrt{2\omega_p}}\int \frac{d^3\vec{p}'}{\sqrt{2\omega_{p'}}}\, \langle 0|\,\hat{a}^i_{\vec{p}}\, \hat{a}^{j\dagger}_{\vec{p}'}\, |0\rangle\, e^{ip\cdot x - ip'\cdot y}$$

Using $\quad [\hat{a}^i_p, \hat{a}^{j\dagger}_p] = \left(\delta^{ij} - \frac{p^i p^j}{|P|^2}\right)\delta^3(\vec{p}-\vec{p}')$

$$G^{ij}(x,y) = \frac{1}{(2\pi)^3}\int \frac{d^3\vec{p}}{2\omega_{\vec{p}}}\left(\delta^{ij} - \frac{p^i p^j}{|P|^2}\right) e^{iP\cdot(x-y)}$$

☆ $\langle 0| E^i(x)\, E^j(y)\, |0\rangle$

$$= \frac{1}{(2\pi)^3}\int \frac{d^3\vec{p}}{\sqrt{2\omega_{\vec{p}}}}\int \frac{d^3\vec{p}'}{\sqrt{2\omega_{\vec{p}'}}}\,(-i\omega_p)(-i\omega_{p'})\, \langle 0|\left(\hat{a}^i_{\vec{p}}\, e^{ip\cdot x} - \hat{a}^{i\dagger}_{\vec{p}}\, e^{-ip\cdot x}\right)\left(\hat{a}^j_{\vec{p}'}\, e^{ip'\cdot y} - \hat{a}^{j\dagger}_{\vec{p}'}\, e^{-ip'\cdot y}\right)|0\rangle$$

$$= \frac{1}{(2\pi)^3}\int \frac{d^3\vec{p}}{\sqrt{2\omega_{\vec{p}}}}\int \frac{d^3\vec{p}'}{\sqrt{2\omega_{\vec{p}'}}}\,\omega_p\,\omega_{p'}\left(\delta^{ij} - \frac{p^i p^j}{|P|^2}\right)\delta^3(\vec{p}-\vec{p}')\, e^{ip\cdot x - ip'\cdot y}$$

$$= \frac{1}{(2\pi)^3}\int d^3\vec{p}\,\frac{\omega_p}{2}\left(\delta^{ij} - \frac{p^i p^j}{|P|^2}\right) e^{iP\cdot(x-y)}$$

○ For electromagnetic field

$\langle 0| A^i(x)\, A^i(y)|0\rangle$

$$\frac{1}{(2\pi)^3}\int \frac{d^3\vec{p}}{\sqrt{2\omega_{\vec{p}}}}\int \frac{d^3\vec{p}'}{\sqrt{2\omega_{\vec{p}'}}}\langle 0|(\hat{a}^i_p e^{ipx} + \hat{a}^{i\dagger}_p e^{-ipx})(\hat{a}^i_p{}' e^{ip'y} + \hat{a}^{i\dagger}_p{}' e^{-ip'y})|0\rangle$$

$$\frac{1}{(2\pi)^3}\int \frac{d^3\vec{p}}{\sqrt{2\omega_{\vec{p}}}}\int \frac{d^3\vec{p}'}{\sqrt{2\omega_{\vec{p}'}}}\langle 0|\hat{a}_{\vec{p}}\,\hat{a}^{j\dagger}_{\vec{p}}{}'|0\rangle e^{i\,px - ip'y}$$

Using $\quad [\hat{a}_p, \hat{a^{j\dagger}}] \;=\; (\delta^{ij} - \dfrac{p^i p^j}{|p|^2})\delta(\vec{p} - \vec{p}\,')$

Using $\quad G^{ij}(x,y) = \dfrac{1}{(2\pi)^3}\displaystyle\int \dfrac{d^3\vec{p}}{\sqrt{2\omega_{\vec{p}}}}\int \dfrac{d^3\vec{p}}{\sqrt{2\omega_{\vec{p}}}}(\delta^{ij} - \dfrac{p^i p^j}{|p|^2})\delta(\vec{p} - \vec{p}\,')e^{ip\cdot(x-y)}$

$\star \quad \langle 0|\hat{E}^j(x)E^j(y)|0\rangle$

$=$

$\dfrac{1}{(2\pi)^3}\displaystyle\int \dfrac{d^3\vec{p}}{\sqrt{2\omega_{\vec{p}}}}\int \dfrac{d^3\vec{p}\,'}{\sqrt{2\omega_{\vec{p}}\,'}}(-i\omega_p)(-i\omega_p')\langle 0|(\hat{a}_{\vec{p}}^i e^{ipx} - \hat{a}_{\vec{p}}^{i\dagger}e^{ipx})(\hat{a}_{\vec{p}}^j{}' e^{ip'y} - \hat{a}_{\vec{p}}^{j\dagger}{}' e^{ip'y})|0\rangle$

$=-$

$\dfrac{1}{(2\pi)^3}\displaystyle\int \dfrac{d^3\vec{p}}{\sqrt{2\omega_{\vec{p}}}}\int \dfrac{d^3\vec{p}\,'}{\sqrt{2\omega_{\vec{p}}\,'}}\omega_p\omega_p'(\delta^{ij} - \dfrac{p^i p^j}{|p|^2})\delta^3(\vec{p} - \vec{p}\,')e^{ip\cdot(x-y)}$

$=\quad \dfrac{1}{(2\pi)^3}\displaystyle\int \dfrac{\omega_p}{2}(\delta^{ij} - \dfrac{p^i p^j}{|p|^2})e^{ip\cdot(x-y)}$

And this property which is very unclassical and very significant will become very important when we start discussing atoms talking to various kinds of field. Now just to summarize things and wind up for other kinds of field you can do the same computation which we have done for scalar field you can straight away do for vector fields as well you just have to write a I and a j with their appropriate indices I is here and I is here so I am writing a I and a j not vector a and vector v that also you can do if I write vector a at location x and vector a at location y that also become can be computed by writing this object as aI ei, eI is the basis vector this operator and the similar thing for other one aj ej summation over j is implied. So, I am first writing just aI aj kind of thing you can multiply eI ej unit vectors to make the full vector correlator out of it. So, the component of full vector correlators will be a I a j kind of thing. So, just let me write it down once more. If I compute the full a x a y correlator, I will get a structure like summation over I summation over j, then this a I a j correlator which we are trying to compute and then $\hat{e}_i \quad \hat{e}_j$ unit vectors. So, this piece needs to be calculated and that is what we are doing here. I will write a i's like this, a j's like this. The same story will go through. Only one set of operator will be surviving across vacuum, a and $a^\dagger$. All other operators will be squeezed to 0 between vacuum. And this time you have to be just careful to use the commutation relation appropriately. Previously we had flipped this with a use of a $\delta(p-p')$. Now you will have indices.Those indices for example for vector cases we have realized that if they are the two-dimensional plane indices then this is delta rs.   But if it is three-dimensional I and j they should satisfy the restricted completeness relation. But if it is three-dimensional I and j they should satisfy the restricted completeness relation. So, this is the commutator which we had seen previously. We had previously written a lambda p, a $p'^\dagger$ lambda$'$ was delta lambda lambda$'$ and $\delta(p) - p'$. if you compare our nodes from the electrodynamics quantization. Here, lambdas were supposed to be polarization vectors along the plane. But if you insist that it should be on the three dimensions full A, then this $\delta_{\lambda\lambda'}$ will be related to the three-dimensional completeness relation like this. This is what we had seen previously as well. So, you do the same thing, you will get the correlator here ultimately like this. So, gij which is being computed is becoming $\delta_{\lambda\lambda'}$. So, this is the extra piece you have to be careful under integration, otherwise this integration we have just computed. So, you have to redo this computation with an extra $\delta_{\lambda\lambda'}$ thrown up, first term will become exactly the term which we have computed, second term you have to be slightly more careful, extra π pj and p^2 pr has arrived. So, do some business slightly carefully to obtain that. Similarly, for electric field we can do the same thing.

Electric field remember was obtainable from the derivative of this with a negative sign. Remember this is in the so called Coulomb gauge and in this Coulomb gauge electric field has to be obtained from the temporal derivative and temporal derivative will throw up $-i\,\omega_p$ kind of terms as a common factor with a negative sign inserted in between. So that is what the decomposition we had written previously as well. Again do the same algebra, everything else will go through like whatever we had done for AIA. Then ultimately electric field correlators can also be written like this over here. Here ω_p has appeared upstairs, previously it was appearing downstairs, this has happened double ω_p coming from derivative of each of the As. This E and this E are made from time derivative of this and time derivative of that, that throws off extra $\omega_p$airs and ultimately this brings ω_p from downstairs to upstairs. So, therefore this is somewhat of a new computation which is, but again you can do the same analysis. The d^3integral can be broken into $2\pi d cos\theta$ and p^2dp kind of integral and one has to compute that. And just writing it just for formal sake of structure that we know how to compute the vacuum correlation for electromagnetic field both for vector potential as well as for the electric field. And similarly one can write down for magnetic field as well. So, we know none of the fields vacuum are really empty, they are made up of the correlations, they are buzzing with correlations and later on we will see that even for the empty spaces state as empty as vacuum, it is not completely empty, we would have artifacts of these correlations which will be visible in our physical considerations. So, for vacuum I stop here, in the next class we will discuss about non-vacuum state .