Foundation of Quantum Theory: Relativistic Approach

Electromagnetic Field Quantization 1.3

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Quantum Electrodynamics

Lecture- 28

So we resume today the discussion of the quantization of electromagnetic field and in today's class we will try to see achieve the quantization procedure completely So, recall in the last discussion session, we had the handle on our Lagrangian or the action of the theory, which used to give me the correct equations of motion, the $\Box A_\mu$ is equal to 0, which constitutes the two dynamical sets of Maxwell's equations.

$$S = -\frac{1}{4} \int d^4x \, (F_{\mu\nu} F^{\mu\nu})$$

$$= -\frac{1}{4} \int d^4x \, (\partial_\mu A_\nu - \partial_\nu A_\mu) \eta^{\mu\alpha}\eta^{\nu\beta}(\partial_\alpha A_\beta - \partial_\beta A_\alpha)$$

$$= -\frac{1}{4} \int d^4x \Big[[\partial_\mu (A_\nu F^{\mu\nu}) - \partial_\nu (A_\mu F^{\mu\nu})] - A_\nu \partial_\mu F^{\mu\nu} + A_\mu \partial_\nu F^{\mu\nu} \Big]$$

$$\boxed{S = \frac{1}{2} \int d^4x \, [\vec{E}^2 - \vec{B}^2]}$$

$$\frac{\delta S}{\delta A^\alpha} = \partial_\mu F^{\mu\nu} \delta_{\nu\alpha} - \partial_\nu F^{\mu\nu} \delta_{\mu\alpha}$$

$$\Rightarrow \partial_\mu F^{\mu\alpha} = 0$$

$\Rightarrow$ $\partial_0 A_0$ never appears in the Lagrangian

$$\Pi^i = \frac{\partial \mathcal{L}}{\delta \dot{A}_i} = -F^{0i} = E^i = -E_i$$
$$= -(\partial_0 A_i - \partial_i A_0)$$

Electric field is not a four vector

$$\{A_i(\vec{x}), E^j(\vec{x}')\} = \delta(\vec{x}-\vec{x}')\,\delta_i^j$$

$$\mathcal{H} = \Pi^i \dot{A}_i - \mathcal{L}$$

$$= \Pi^i (\pi^i + \partial_i A_0) - \mathcal{L}$$

$$= \frac{1}{2}(\vec{E}^2 + \vec{B}^2) + (\partial_i A_0) E^i$$

$$\boxed{H = \int d^4x \left[\frac{1}{2}(\vec{E}^2 + \vec{B}^2) - A_0 \, \partial_i E^i\right]}$$

$$S = \frac{-1}{4} \int d^4 x (F_{\mu\nu} F^{\mu\nu})$$

$$= \frac{-1}{4} \int d^4 x (\partial_\mu A_\nu - \partial_\nu A_\mu) \eta^{\mu\alpha} \eta^{\nu\beta} (\partial_\alpha A_\beta - \partial_\beta A_\alpha)$$

$$= -[\frac{1}{4} \int d^4 x [(\partial_\mu (A_\nu F^{\mu\nu}) - \partial_\nu (A_\mu \partial_\nu F^{\mu\nu}))] - A_\nu \partial_\nu F^{\mu\nu} + A_\mu \partial_\nu F^{\mu\nu}]$$

$$S = \frac{1}{2} \int d^4 x [\bar{E}^2 - \bar{B}^2]$$

$$\frac{\delta S}{\delta A^\alpha} = \partial_\mu F^{\mu\nu} \delta_{\nu\alpha} - \partial_\nu F^{\mu\nu} \delta_{\mu\alpha}$$

$$\Rightarrow \quad \partial_\mu F^{\mu\nu} = 0$$

⇒ ∂_0 A_0 never appears in the Langrangian

$$\Pi^i = \frac{\delta L}{\delta \dot{A}_i} = -F^{0i} = E^i = -E^i = -(\partial_0 A_i - \partial_i A_0)$$

$$\{ \quad A_i(\vec{x}), E^j(\vec{x}\,') \quad \} = \quad \delta(x - x') \delta_i^j$$

$$H = \Pi^i \dot{A}_i - L$$

$$= \Pi^i (\pi^i + \partial_i A_0) - L$$

$$= \frac{1}{2} (\vec{E}^2 + \vec{B}^2) + (\partial_i A_0) E^i$$

$$H = \int d^4 x [\frac{1}{2} (\vec{E}^2 + \vec{B}^2) + A_0 \partial_i E^i]$$

So, recall in the last discussion session, we had the handle on our Lagrangian or the action of the theory, which used to give me the correct equations of motion, the $\Box A_\mu$ is equal to 0, which constitutes the two dynamical sets of Maxwell's equations. And we can see that the action of interest was this object, where , were contracted amongst each other with – 1 by 4 factor and with some massaging of the equations which we have seen previously due to their anti-symmetric nature and the contraction with respect to the $\eta_{\mu\nu}$'s which come handy in writing this one kind of things one can rewrite the whole action in terms of boundary terms as well as terms dependent on the a's directly previously when I am writing all the terms are appearing contain the derivative of amuse there is no free term which has a derivative which which is free of the derivative of amuse so since in the variational approach I want to get an action which has a free amu I can convert it into a boundary terms – something which contains a free $_\mu$ and with respect to the variations of A_μ we obtain the equations of motion which is the $\Box A_\mu$ is equal to zero which is equivalently ∂_μ is equal to 0 or $\partial_\mu F_{\mu\alpha}$ is equal to 0, where α is a free index. So that we have seen in the previous class. An exercise, a set of side exercise for you was to clearly verify that the action can also be written in terms of dynamical degrees of freedom which are physically observable, electric field and magnetic field, which is nothing but $E^2 - B^2$. So remember, we are writing these things in the units of ε_0 is equal *to* $4\Pi F_0$ is equal to 1 and c is equal to 1 as well. So otherwise there will be some dimension full constants in this section. So in the previous class itself we realized that $\partial_0 A_0$. never appears in the Lagrangian. So, its conjugate momenta corresponding to A_0. is 0. But for remaining three A s, their time derivative appears in the Lagrangian. So, we can define the momenta corresponding to them. And we obtain that the electric field is the conjugate momenta corresponding to AI. So, AI electric field pair has a Poisson bracket structure. Okay. Using that, I can write down in the phase space what would be the Hamiltonian which governs the theory. And we obtained a piece of a hamiltonian which was the field strength the energy density of electric field energy density of magnetic field up to a piece which we were worrying about what does it mean because presence of that term with a – sign was not very convincing for us because this runs a risk of converting the total Hamiltonian negative and maybe then we would have to demand new structures in the phase space like what we had done for the Dirac fields. So in this class let us try to see how to handle that and ultimately it turns out that it is not very difficult term to handle at least on the equations of motions of this theory. Because if I take, let us say the Lagrangian or even the Hamiltonian, if you want to use Hamilton JacobI equations, both of them are equivalent. So, take any of them. Take either the action in terms of Lagrangian or you take the Hamiltonian and obtain the Hamilton JacobI equation. That means the equation of motion with respect to A_0. You do $\partial_s/\partial A_0$. or you put $\partial H/\partial A_0$.. Both of these things should give us some equations of motion which would uh effectively tell you about what how the different terms are evolving so in this thing they should tell me the rate of change of the conjugate momenta associated to A_0. $\dot{P}_0$ and similarly here $\dot{P}_0$ if you remember in the phase space or call it Π_0 in the phase space should be written like this okay with a – sign. So, if I want P_0 was supposed to be 0 on the Lagrangian or even the phase space, there was no P_0 which did not arrive, Lagrangian did not depend on $\dot{A}_0$ and that is the characteristic of the physical theory that P_0 is 0. And if I want my P_0 to remain 0, that means I am on the physical surface all the time, my dynamics does not take me away from the electrodynamical description because electrodynamical description is always Π_0 should be 0. If that is the demand we enforce, that means Hamiltonian's derivative with respect to A_0. should also be 0. This would be the requirement of keeping this equation intact under time evolution. If that is the demand we enforce, then we get from the variation of the Hamiltonian or equivalently the variation of the action with respect to A_0. will yield me an equation like this $\partial_i F_{0I}$ is equal to 0 that is equivalent to $\partial_i E_I$ is equal to 0 which can you straight forwardly see from this Hamiltonian approach. So, this is a demand which goes by the names of Gauss's law in vacuum. And this is a enforcing demand of a constraint. The constraint was this. Π_0 was not to be appeared in the physical description of the system. And the demand is that under time dynamics, Π_0 should not turn on. Because that is the description of electrodynamic field. This is the description of the action of the electrodynamic field.

$$S = -\frac{1}{4}\int d^4x \left[A_\nu \partial_\mu F^{\mu\nu} - A_\mu \partial_\nu F^{\mu\nu} \right]$$

$$= \frac{1}{2}\int d^4x \left[A_\mu \partial_\nu F^{\mu\nu} \right]$$

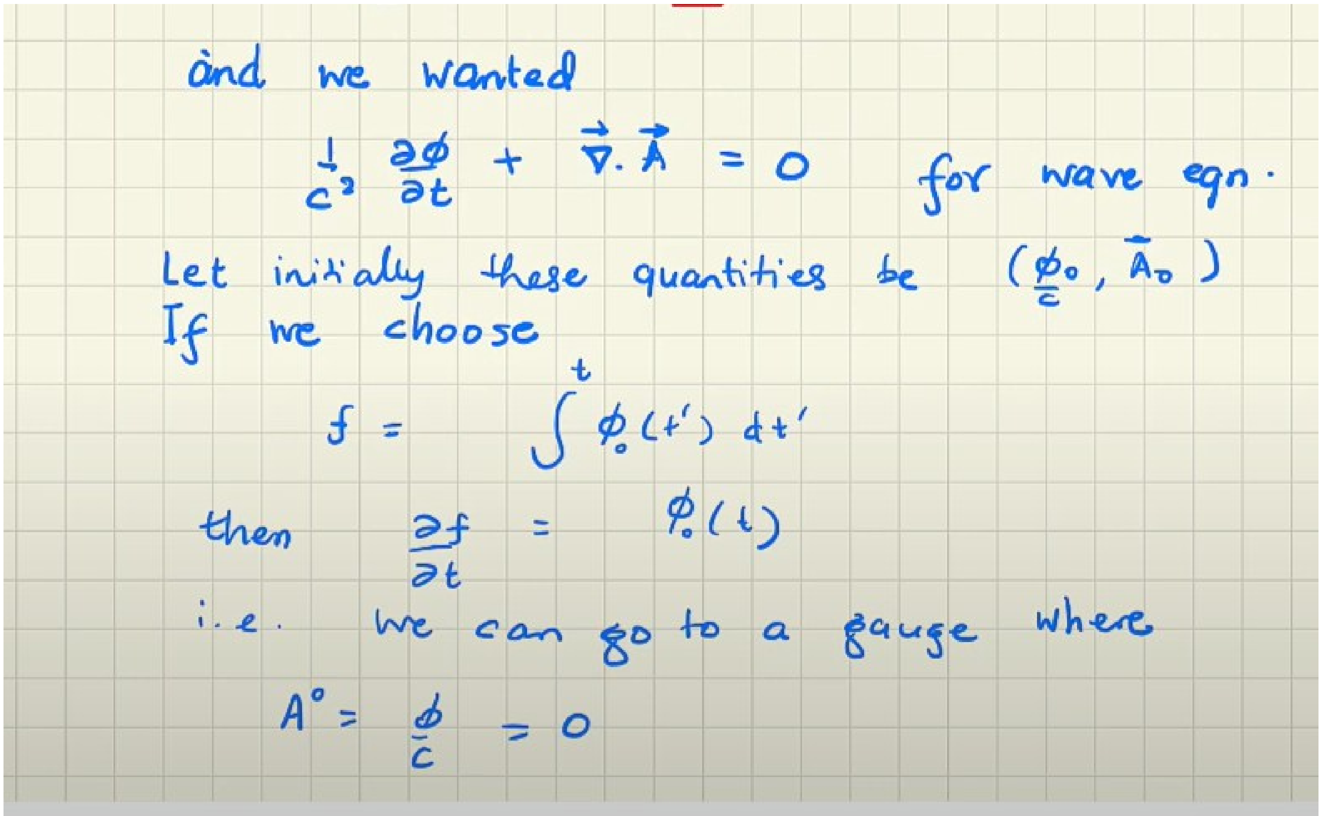
$$\Pi_0^0 = \frac{\delta S}{\delta A^0} = \partial_i F^{0i} = \boxed{\partial_i E^i = 0}$$ } Gauss's law in vacuum

This automatically fixes the Hamiltonian

$$H = \int d^4x \left[\frac{1}{2}(\vec{E}^2 + \vec{B}^2) \right] \geq 0$$

Recall we had freedom of

$$\phi \to \phi - \frac{\partial f}{\partial t}$$



Under time evolution, you do not want to destroy that.So you want Π_0 remains zero. And that is equivalent to this new constraint, which one should enforce that. $\partial_i E_1$ is equal to zero. So both action approach wise, Π_0 is this. Or the Hamiltonian approach was also Π_0 would be – of this. Both the things are equivalent. Putting them to zero will give you this Gauss's law. So Gauss's law is really not a dynamical law, but just telling you the constraint of remaining on the physical surface of the Hamiltonian or on the phase space describing the electromagnetic theory correctly. So, if that happens that means this term is not a worry anymore on those surface where physical dynamics is happening That is you are on the real physical surface in the phase space. So remember let us draw phase space. So there will be some surface initially which was $_0$ is equal to zero. Phase space is made up of $A_1\Pi_1$ pair Π_1A_1 Π_2A_2 Π_3A_3 and $A_0\Pi_0$. All $A_1\Pi_1$.$A_2\Pi_2$ A_3 Π_3 you are free to explore under time evolution they are not constrained to be anything but Π_0 one of them has to be constrained that this is the dynamics this surface Π_0 is equal to zero is a surface which are hyper surface in higher dimension where the electrodynamical theory lives okay so you do not want to leave that surface that is this constraint And therefore, one can drop the last term in the Hamiltonian, which tells you if it is non-zero, that means you have moved away from the physical description of electrodynamics. So, I do not want that. Under time dynamics, I want to remain there only. So, therefore, I can constrain myself to remain on that physical surface that is equivalent to Caussar's law. And under that physical surface, the Hamiltonian becomes just $E^2 + B^2$ with a half. And under that physical surface, the Hamiltonian becomes just $E^2 + B^2$ with a half. And now that is a relief for us because this is a positive semi-definite Hamiltonian. This does not become negative ever. So at least on the physical surface, I am guaranteed to get a meaningful Hamiltonian. And therefore, I should not be worrying about the negativity of Hamiltonian, which bugged me in the Dirac fields case or Spinor field case.

Okay. So from now on, we will work with this Hamiltonian rather than the previous terms, ensuring that on physical surface, on physical surface, we are doing the business remember we are not using equation of motion since we are going to do quantum dynamics of this we should not be using classical equation of motion so you might wonder that why to use classical equation of motion for doing quantum mechanics but remember we are enforcing the what you would have called equations of motion on A_0. this is the equations of motion for A_0. and A_0. remember was not a dynamical variable of our theory.

So, therefore, the A_0. was not a dynamical quantity in our theory. This was just a free fiducial object which has appeared in the Lagrangian. It does not have a real-time dynamic. It is something called gauge degree of freedom. So therefore, I am not fixing the dynamical variables A_1, A_2, A_3 or E_1, E_2, E_3, B_1, B_2, B_3 . Π_0, I should not evolve in time. So with this understanding, we will move forward and try to see how can we explore the remaining degrees of freedom judiciously. So that we get to the quantum portion of the little time. Remember, in doing the business so far, A_0 has been declared to be unphysical, meaning not really physical variable. It is just something which we can exploit, just like its conjugate momenta we have exploited right now. So real degrees of freedom which are going to be witnessable are made up from A_1, A_2, A_3 only. So already now we have a three degrees of freedom describing electrodynamics later on we will see one of them also goes away ultimately electrodynamics is going to be described only with two degrees of freedom okay now further we recall that while we were discussing electrodynamics structure we had a another freedom gauge freedom in redefining my ϕ and a up to some scalar function f. So, ϕ could have been rescaled by taking $\phi - \partial f/\partial t$ for any arbitrary function f. f was not concerned to be any particular function. And also under that, we had realized that a can be shifted to a + the gradient of f. So, under this gradient of f and temporal derivative of f, the A_0 component and A_1, A_2, A_3 components can be shifted by choosing any arbitrary function f. The electrodynamic structure does not change under this selection of A. Also, we wanted that there should be a gauge condition where I kill off the terms which are appearing extra in the dynamics of $\vec{A}$. So, remember $\vec{A}$, the $\Box A$ + some extra quantity were appearing if I just do the computation of curl of curl of A. Those extra terms can be switched off by demanding this. Therefore, we would have thought that we could, we thought that time that we could take the freedom at our hand choosing f in such a way that this whole term can be killed off. And ultimately, I get the wave equation for all the components A_0, A_1, A_2, A_3.

Okay. And this we know can always be done. However, in following discussion, we will do it slightly more carefully in order to make contact with the physical more obviously. Meaning there are various ways of selecting this condition. This goes by the name of a Lorentz gauge. But now we are going to work in something called the Coulomb gauge, which is an equivalent description of this while in two steps. So let us try to see what do I mean. So first, suppose initially I am starting with a pair, pair meaning the scalar and the vector potentials. Vector potential is A_0 and the scalar potential is ϕ0. And I want to choose some function f such that I achieve this goal, let us say. So I would therefore take a function f, which is a function of p and x both. So at any time t, I'm going to use the function f as such. This is my choice. I will take the ϕ_0 which is provided to me. I will integrate it over some time up to time where I want to do my computations and get a function of t and x. This is a function of t and x. This is also a function of t and x. So what I'm doing, I'm just doing the temporal integration up to a time t where I want to know my function and I want to achieve this at different different · different different functions will do that that means different different values of this function will do that that is fine so I will get a functional form in time t so let us do that so do this integration up to the symbol t such that when I do $\partial f/\partial t$ when I take the partial derivative I will see that the partial derivative time is only appearing in the upper limit over here and everything else is a dummy variable it is getting integrated over. So, $\partial f/\partial t$ is just ϕ_0.

What I have done, I have taken the integral of ϕ_0 as my function such that $\partial f/\partial t$ is just ϕ_0, which is fine. Therefore, if I do that, I know after doing the gauge transformation with that f, I will land up on the case where ϕ has vanished. In the new gauge, the ϕ has vanished. Initially, it was ϕ_0. I did this gauge transformation. I ended up Π being zero that is a zero has been achieved to be zero in this new gauge so at least this part of this computation I can go to a frame where Π has vanished completely so then I will be worrying about only that part now separately I could have done the other way as well I could have chosen another gauge. Where I killed off the second term.

Similarly we can choose a gauge

s. t. $\vec{\nabla}\cdot\vec{A} = \vec{\nabla}\cdot\vec{A}_0 + \nabla^2 f = 0$

i.e. we can choose a function f

s. t. $\nabla^2 f = -\vec{\nabla}\cdot\vec{A}_0$

$$f = \frac{1}{4\pi}\int d^3x' \frac{\vec{\nabla}_{x'}\cdot\vec{A}_0(x',t)}{|\vec{x}-\vec{x}'|}$$

Since $\nabla^2_{x'}\frac{1}{|\vec{x}-\vec{x}'|} = -4\pi\,\delta(\vec{x}-\vec{x}')$ if $x, y, z \in \mathbb{R}$

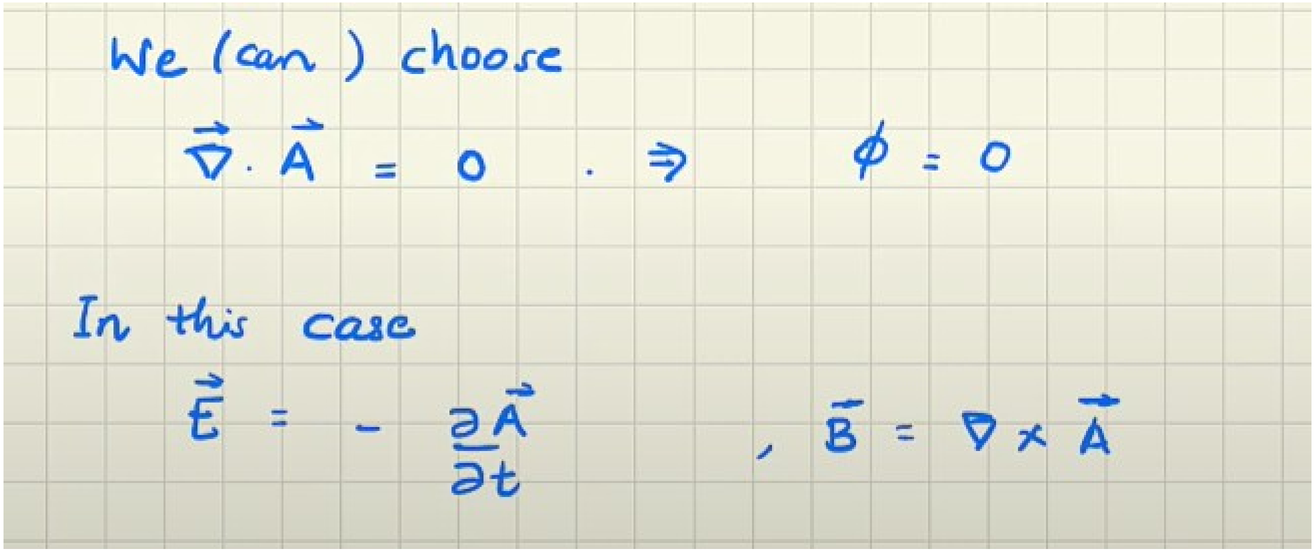
Incidentally these two can be achieved in one gauge condition.

$$\vec{\nabla}\cdot\vec{E} = -\left(\nabla^2\phi_0 + \frac{\partial}{\partial t}\vec{\nabla}\cdot\vec{A}_0\right) = 0$$

$$\phi_0 = -\frac{1}{4\pi}\int d^3x' \frac{\frac{\partial}{\partial t}\left(\vec{\nabla}_{x'}\cdot\vec{A}_0(\vec{x}',t)\right)}{|\vec{x}-\vec{x}'|}$$

If $\vec{A}_0(\vec{x}',t)$ is provided, ϕ_0 will be known.

We (can) choose



That means divergence of A initially would have been something. Let us say when I chose. My initial starting split up was ϕ_0 and A_0 . There was no guarantee that divergence of A_0 is 0. It could be anything. So non-zero number. So if that is the case. Under gauge transformation divergence of A is supposed to change by the Laplacian of F. Remember. The A was itself supposed to change through a gradient term of f. So therefore the divergence of A will change with Laplacian of f. So that is what it would happen and I want it to be zero. I want it to become zero in the new gauge. This is my demand let us say. That means f has to satisfy an equation which is Laplacian of f is – of the divergence of the old a the previous a_0 before the gauge transformation. So this can be taken as a source function for function f so called it some source function J and remember given this structure Laplacian of some function is equal to some source function ends up giving you a solution to the through the green function approach that the function itself will be one upon four $\Pi\cdot$ the source function integrated with the green function of the free space and the green function of free space is one over $x - x'$ so what i'm doing i'm taking the source function J_x make its variable x$'$ dummy divide it by $x - x'$ mod and integrate over the dummy variable x point this I claim this. f which I have written as an integral form I claim is a solution of the equation which I am demanding this equation has this solution that you can verify quickly such that if I take the laplacian take the laplacian with respect to x so remember this laplacian is with respect to x so the laplacian will go and search for x dependency this is x$'$ this is x$'$ this is x$'$ only x dependence dependency is coming over here. And you can, you know from your electrodynamics course, the green function approach, that the Laplacian of 1 over $x - x'$ is just the δ function with $a - 4\pi$. So, that Laplacian acting on this $x - x'$ will give you a δ function in $x - x'$. And that integration will exactly convert it into the equation which we are demanding. Therefore, this is a good solution in free space. The green function in free space is 1 over $x - x'$. Okay, so therefore, I can choose my function this. If you give me initial A_0, I will compute its divergence, put it here, do this integration and whatever comes out, I will declare it to be my F and I will do the gauge transformation by that. And you know, once I have done the gauge transformation, this Laplacian is going to be – of divergence of A_0. Therefore, in the new gauge through this F, the divergence of new A will be 0. So, I could have killed off the divergence term as well. In the previous example, we saw if I chose the integral version of the ϕ_0, I could have killed off the ϕ, which is appearing over here. Or the second approach is that if I knew what is the divergence of A_0, I can kill off that term as well and can go to new gauge where this term is 0. And it so happens, these two conditions are not different. They are one and the same condition under certain context. And that is what one should be able to realize just by staring at the Gauss's law. The divergence of E should be 0. If divergence of E is supposed to be 0, let us say, that should be true in all the frame. Electric field does not change under gauge transformation. So E does not change from E0 to E. E remains the same under both the gauges. So in the previous gauge where I had not selected any F, then the electric field would have been defined in the pair of ϕ_0 and A_0. Okay, and the divergence of electric field would have been obtained from the Laplacian of ϕ_0 and the divergence of A_0. And that should have been zero from Gauss's law. Okay, so that means again I can write down this as a equation of Laplacian of ϕ_0 is equal to – of $\frac{\partial(\nabla . A_0)}{\partial t}$ of divergence of A_0. This equation should have been true because of Gauss's law. Again, use this as a source function, divide it by $x - x'$, integrate it over d^3x', just like we did for this thing. We see that in any frame, any gauge, ϕ_0 can be obtainable from the divergence of A, because divergence of A goes here, sits in the derivative place, its integration is going to fix what is ϕ_0. This is always true in any gauge this will be true.

So therefore if I choose a gauge where divergence of A vanishes then this left hand side will also vanish. That means choosing divergence of A is equal to zero over here go to this gauge will ensure that divergence of A is zero and Gauss's law will ensure that as a result ϕ will also be zero. So therefore I choose this gauge where divergence of A is zero. Π will automatically be 0 and this condition will also be satisfied automatically. So that is the Coulomb gauge. In the Coulomb gauge, we can write down the wave equation just what we wanted. Π becomes 0 and divergence of A becomes 0. If ϕ becomes 0 in the new gauge, this term in the new gauge will not be present. So electric field will be totally obtainable from the temporal derivative of the remaining *A's*. A_0 or ϕ has been put to 0. So therefore only vector quantities are going to determine your physics in this gauge. So electric field will be – of $\partial A/\partial T$ and magnetic field like before will be curled off this way.

In this gauge

$$\frac{1}{c^2}\frac{\partial \phi}{\partial t} + \vec{\nabla}\cdot\vec{A} = 0$$ is automatically true

Thus in this gauge wave equation will be true.

$$\Box A^{\mu} = 0$$

$$A^0 = 0 \quad \Rightarrow \quad \Box A^i = 0$$

$$\Rightarrow \quad A^i = \frac{a_p^i}{\sqrt{2\omega_p}} e^{ip\cdot x}$$ is solution of the wave eqn.

i is now (spatial) vector index.

$$p^0 = \pm\sqrt{(p_x^2 + p_y^2 + p_z^2)}\, c = |\vec{p}|\, c$$

$$\vec{\nabla}\cdot\vec{A} = 0 \quad \Rightarrow \quad \partial_i A^i = 0$$

$$\Rightarrow \quad (a_p^i \, p_i) = 0$$

Again a^1_p, a^2_p ; a^3_p are not all independent

Only two of them are really independent.

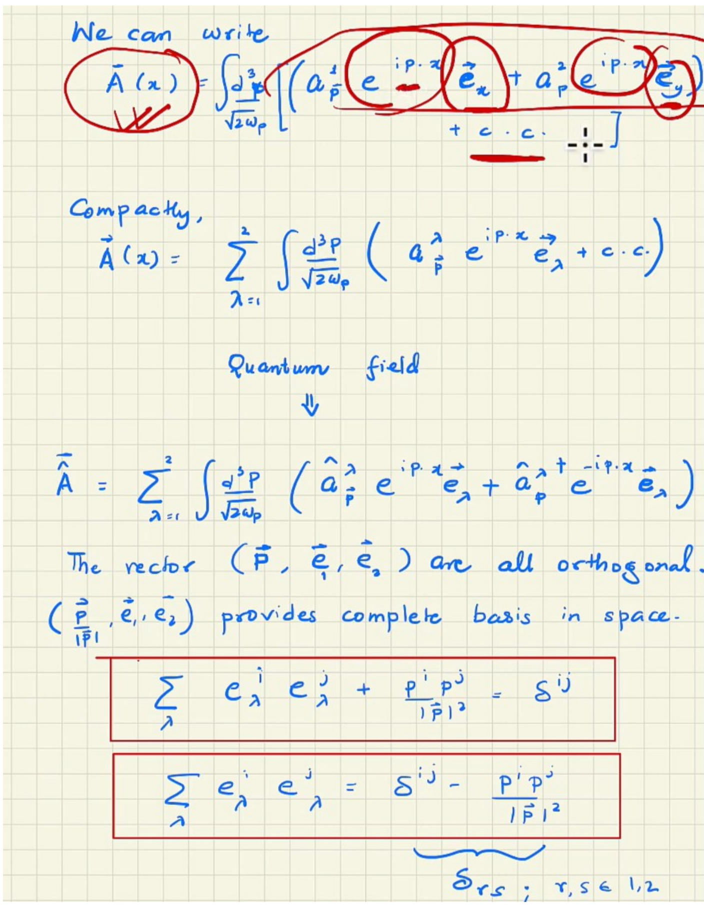
In Fourier space

$$\vec{A}_{\vec{p}} \cdot \vec{p} = 0$$

Thus, if $\vec{p} = \hat{k}_z$, $\vec{A}_p = (A_{px}, A_{py}, 0)$

So far so good. So let us go ahead and try to see that once we have set up the wave equation because in this case the condition of getting wave equation is automatically satisfied so we can indeed write down $\Box A_\mu$ is equal to zero it so happens that a zero is zero so therefore that will not be a physical quantity physical quantity will be the remaining non-zero aI so what we really are after is the solution for the $\Box A_i$ is equal to zero now we have a three component object AI whose wave equation we are trying to solve so previously we had solved for scalar field single component object and the four component object for tirac field now we have at hand in this gauge three component object A_i whose wave equation is the story of electrodynamics okay this is fine and let us go ahead we already know I will not waste time in again obtaining the solution of the box equation because we have solved it too many · so I know ultimately it so happens that on the exponential, the plane wave structure comes about. – Et + vector p $\dot{x}$ vector x. So I am collectively writing as a four dimensional dot product $p\dot{x}$. With + $p\dot{x}$, I mean the energy comes with a negative sign. With – $p\dot{x}$, I will say the energy comes with a positive sign. So therefore, I know it has two solutions, either with positive frequency or energy, you call the energy in terms of P mus, P0, not the Hamiltonian energy of this field. So this plane wave energy, not the field's energy. So this P can have two solutions. This is massless Klein-Gordon equation. There is no mass term. So P_0 is related to the remaining P1, P2, P3 with this relation. magnitude of P · c is P0. And it will come with a sign, + – sign actually, + – sign of both. So it can have a positive energy solution, positive frequency solution, or negative frequency solution as before. That is fine.

Now we have as well seen that divergence of A is supposed to 0, which can be written as $\partial_i A_i$is equal to 0. If I do that and take one plane wave from here with a definite moment of P, doing this derivative will show me that A I the the quantity which is appearing over here is getting dotted with p so this is the component first component is obtained by first component of this second component is obtained by second component of this and third component of this is obtained by third component of this · the exponential function but vector property of a is captured by vector properties of small a small a as the vector quantity it has component and multiplied with this plane wave gives you a $\vec{A}$i. Once I take the divergence of this a, you will see that the p will get dotted, p coming from the special derivative of this exponential function will bring down a p and dotted with this $\vec{A}$, $a{\cdot}p$ will appear, should be zero, which in component form can be written as a of I index gets contracted with ap_i index, that should be zero. So you see that the $\vec{A}$, which is appearing over here and giving a vector character to the capital A, the gauge field, is orthogonal to the direction of a propagation of your wave. So therefore, we initially wanted to start the discussion with three component object Ai. It so happens that all three components are not independent because $\vec{A}$ cannot be arbitrary. $\vec{A}$ has to maintain a relation that it is orthogonal to vector P. That means $\dot{A}$ this is equal to 0. This suggests that if you know the propagation direction, then I have a relation between A_1, A_2, A_3. If you know what is P and you know 2 of A_1, A_2, then A_3 will be automatically determined. So all three quantities are not independent quantities. Only two of them are really independent. So if I choose my axis such that my direction of propagation is Z direction, then I know that the $\vec{A}$ can lie only in XY plane because it has to be perpendicular to the P direction. That means it will have two components only. And therefore the $\vec{A}$ also for a given plane wave can only lie along xy direction so therefore if p is chosen along z direction a of p will only along b x or y so it will reduce the problem to ultimately two degrees of freedom so therefore the story of electrodynamics which we thought is a three degree of freedom object looking at the structure. The divergence of A is equal to zero condition is actually suggesting me that in this gauge only two degrees of freedom remain physical. Therefore, we discuss a theory. Electrodynamics theory is a theory of A_i where only I can take two values or call it A_R A_R. lies in the plane. So this is a plane and this is a propagation direction P. So your A can take any direction in this plane. So this is the condition r, r can take value one two in the plane so ap can be written with the ap can be written as ar er where er is a unit vector in the plane okay so let us write it down as a full theory as we know only it could have two components along x or y direction the $\vec{A}$ the full $\vec{A}$ can only project along x direction or y direction so therefore the full a which we are looking for the capital a the gauge field is some vector · plane wave multiplied in xy plane and since we want the gauge field to be the vector field vector potential to be a real quantity because magnetic field and other things are going to be obtainable from this temporal derivative of this will give me electric field spatial curl will give me the magnetic field so therefore better this not be a complex quantity So therefore, I will add a complex conjugate to this.



We can write

$$\vec{A}(x) = \int \frac{d^3p}{\sqrt{2\omega_p}}\left[\left(a_p^1 e^{ip\dot{x}}\vec{e}_x + a_p^2 e^{ip\dot{x}}\vec{e}_y\right) + c.c\right.$$

Compactly,

$$\vec{A}(x) = \sum_{\lambda=1}^{2}\int \frac{d^3p}{\sqrt{2\omega_p}}\left(a_{\vec{p}}^{\lambda} e^{ipx}\vec{e}_\lambda + c.c.\right)$$

$$\hat{\vec{A}} = \vec{A}(x) = \sum_{\lambda=1}^{2}\int \frac{d^3p}{\sqrt{2\omega_p}}\left(\hat{a}_{\vec{p}}^{\lambda} e^{ipx}\vec{e}_\lambda + a_{\vec{p}}^{\lambda\dagger} e^{ipx}\right.$$

The vector $(\vec{P}, \vec{e}_1, \vec{e}_2)$ are all orthogonal.

$(\frac{\vec{P}}{|P|}, \vec{e}_1, \vec{e}_2)$ Provides complete basis in space.

$$\sum_\lambda e_\lambda^i e_\lambda^j + \frac{P_i P_j}{|P|^2} = \delta^{ij}$$

$$\sum_\lambda e_\lambda^i e_\lambda^j = \delta^{ij} - \frac{P_i P_j}{|P|^2}$$

So I will obtain this. Plane wave as you know is not of real quantity. Its complex conjugation has to be added. So I take this solution, the first line of this object. I take this solution. This is a valid solution of wave equation which points in the xy plane. Take its complex conjugate. I get the full real vector potential A and I want to do business with that. Compatibly, the whole thing can be written. Since this has two objects appearing over here, I can collectively call it e_λ , where λ takes two values, 1 and 2, 1 for x, 2 for y. So I can write down the whole lengthy kind of expression as this as simple as AP , where λ will again tell me what coefficient I'm coming with the direction e_x, and 2 will tell me what is the projection along direction e_y . So therefore, compactly the vector potential is written like two component vector and its complex conjugation like this. So this is my dynamical description of electromagnetic theory. And now we know once we have solutions of equations of motion and everything, and we know it's a oscillator story in momentum space, everything of the scalar field or the Dirac field which we have learnt is applicable to these two degrees of freedom as well counting. So therefore, I can quantize the systems as a harmonic oscillator system in momentum space again and in that process these things will become operators. So I cut short the story That is the same story which we have learned far too many · by now. And we know now that the vector potential can be written as operator this here. And it's Hermitian conjugate because it has become operator. So this will become

Hermitian conjugate. So a daggers will appear. Again, take value 1 and 2 for two polarizations in the plane. P is the momentum corresponding to the oscillator which we are talking about. And this is the mode function which is satisfying the oscillator structure. In addition, there will be a polarization vector which is coming about which tells you in which direction is indeed pointing depending upon how many excitations along the x direction how many excitations along the y direction collectively I will get to know how many what is the overall polarization of this quantum field if I put it under some state if take its expectation under some state that means that will give me expectation of this and expectation of this operator that will decide my polarization and as you can see this is these are operators, the expectation will have some variances as well fluctuations as well so therefore whenever I say expectation of the vector potential is this much I am only talking about polarization has an expectation expectation with behavior in that direction there will be fluctuations associated with it due to quantum character of a and a dagger.

Okay so now I have become this has become a vector potential but quantum mechanical in its nature so I do not have a definite direction of polarization I have only expectation and variances however the story which we have learned that propagation direction P and the plane of the polarizations are all orthogonal to each other. So therefore, in three-dimensional space, the propagation directions unit vector, which is $\frac{\vec{P}}{|P|}$, e_1 and e_2, provide you complete basis in the three-dimensional space. That means if I take the outer product of the three unit directions, that should sum up to identity. So that is the mathematical statement. The basis vectors in the plane and the basis vector along the propagation direction some or outer product of that is equivalent to δ_{ij} this is the that statement the total e in the three dimensional space summed over I I I running from one to three should be identity which is this box and from here, you can see that in plane, if I just focus on what is the relation of the plane basis vector, in the plane, the two basis vectors sum up to $\delta_{ij}-$ the third direction. So, this is the projection operator into the plane. That is the plane's orthogonality relation. Any vector in the plane can completely be described by these two vectors only. Completeness relation in the plane completeness relation in the plane can be written as completeness relation of the full three space – the unit direction of the third thing so that means $e_i\, e_r\, e_s$ outer product so r coming from one to two is equal to identity – the e3 outer product okay so that is this equation where rs you can write down this thing as a δ rs where you have to make sure convince yourself that r and s which are appearing over here are only indices pointing along the component of vectors in the plane of the polarization in the plane of the polarization okay so now once that structure is set up the remaining job is very straightforward. Now we have a description of A which has pointing in the plane of polarization which is orthogonal to the plane of the projection, plane of the propagation of the wave. The $\vec{A}$ field is completely known and from this A field we know how to compute the magnetic field and electric field in this gauge because in this Coulomb gauge A_0 is zero. The electric field is entirely the temporal derivative of this. You can take the A which is given over here. Do the temporal derivative temporal derivative will hit only this part because everything else is time independent and its hermitian conjugate over here as well actually this and this the temporal derivative will be hitting and hitting this temporal derivative will give you $-i\omega p$ from this derivative remember this involved $-i\omega p$ or E if you want to call it $+ ip\dot{x}$. And while it's Hermitian conjugate, we have the signs flipped. So when I do the derivative with respect to this, a – *iota* will come out. And there should be a – overall. So I should put $-A$ here overall.

As we know in this gauge

$$\vec{E} = -\frac{\partial \vec{A}}{\partial t} = \sum_{\lambda=1}^{2} \int \frac{d^3\vec{p}}{\sqrt{2\omega_p}} \left[(-i\omega_p)\, \hat{a}_{\vec{p}}^{\lambda}\, e^{ip\cdot x}\, \vec{e}_\lambda + (i\omega_p)\, \hat{a}_{p}^{\lambda\dagger}\, e^{-ip\cdot x}\, \vec{e}_\lambda \right]$$

$$\vec{B} = \vec{\nabla} \times \vec{A} = \sum_{\lambda=1}^{2} \int \frac{d^3\vec{p}}{\sqrt{2\omega_p}} \left[(i\vec{p} \times \vec{e}_\lambda)\, \hat{a}_{\vec{p}}^{\lambda}\, e^{ip\cdot x}\, \vec{e}_\lambda - (i\vec{p} \times \vec{e}_\lambda)\, \hat{a}_{\vec{p}}^{\lambda\dagger}\, e^{-ip\cdot x}\, \vec{e}_\lambda \right]$$

We would expect

$$[A_i(\vec{x}), E^j(\vec{y})] = i\,\delta(\vec{x}-\vec{y})\,\delta_i^j$$
$$= i \int d^3\vec{p}\; e^{i\vec{p}\cdot(\vec{x}-\vec{y})}\,\delta_j^i$$

But A_i or E^j are constrained to not take values in $\vec{p}$ direction, i.e. they are made up from two dimensional plane waves

$$\vec{\nabla}\cdot\vec{E} = 0 \;,\; \vec{\nabla}\cdot\vec{A} = 0$$

$$\therefore \; [A_r(\vec{x}), E^s(\vec{y})] = i\int d^3\vec{p}\; e^{i\vec{p}\cdot(x-\vec{y})}\,\delta_r^s$$

$$[A_i(\vec{x}), E^j(\vec{y})] = i\int d^3\vec{p}\left(\delta_j^i - \frac{p^i p_j}{|p|^2}\right) e^{i\vec{p}\cdot(\vec{x}-\vec{y})}$$

As we know in this guage

$$\vec{E}-\frac{\partial\vec{A}}{\partial t}=\sum_{\lambda=1}^{2}\int\frac{d^3p}{\sqrt{2\omega_p}}\left[(-i\omega_p)a_{\vec{p}}^{\lambda}e^{ipx}\vec{e}_\lambda+(i\omega_p)a_{\vec{p}}^{\lambda\dagger}e^{-ipx}\vec{e}_\lambda\right]$$

$$\vec{B}=\sum_{\lambda=1}^{2}\int\frac{d^3p}{\sqrt{2\omega_p}}\left[(ip\times e_{\vec{p}}^{\lambda})a_{\vec{p}}^{\lambda}e^{ipx}\vec{e}_\lambda-(ip\times e_{\vec{p}}^{\lambda})a_{\vec{p}}^{\lambda\dagger}e^{-ipx}\vec{e}_\lambda\right]$$

We would expect

$$[A(\vec{x}),E^j(\vec{y})]=i\delta(\vec{x}-\vec{y})\delta_i^j$$

$$=\quad i\int d^3\vec{P}e^{i\vec{P}(\vec{x}-\vec{y})\delta_i^j}$$

But *A, or* E^j are constrained to not take values in $\vec{P}$ direction, ie, they are made up from two dimensional plane waves.

$$\nabla\cdot\vec{E}=0,\nabla\cdot\vec{A}=0,$$

$$[A_\gamma(\vec{x}),iE^\gamma(\vec{y})]=i\int d^3\vec{p}e^{(i\vec{p}(\vec{x}-\vec{y}))}\delta_r^s$$

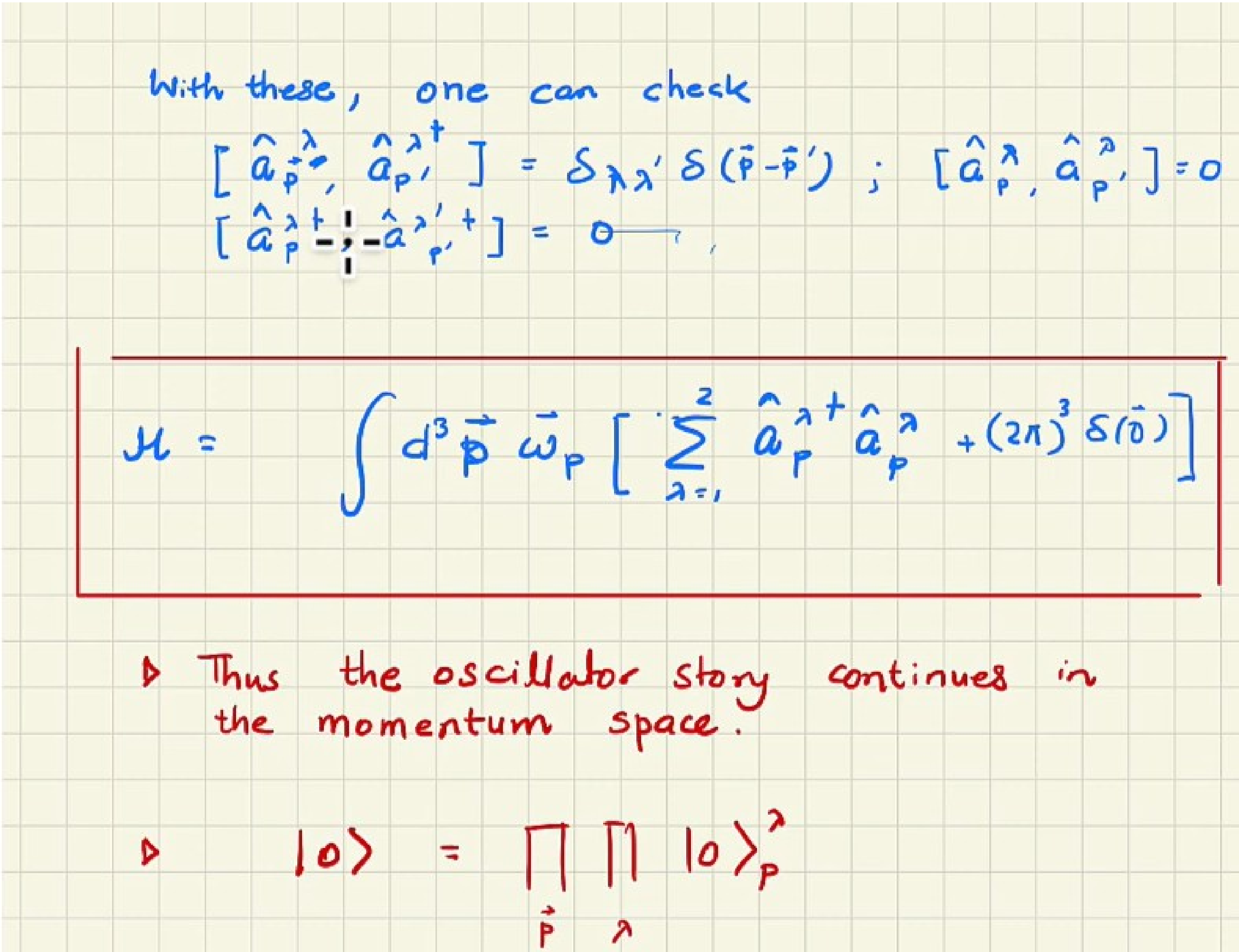
$$[A_\gamma(\vec{x}),iE^\gamma(\vec{y})]=i\int d^3\vec{p}(\delta^{ij}-\frac{P_iP_j}{|P|^2})e^{(i\vec{p}(\vec{x}-\vec{y}))}$$

That will come down and + $i\omega p$ will come from the second terms derivative which had a – ip $\dot{x}$ because this has a + $i\omega p$ remember the structure is slightly reversed when I put a – sign here that means frequency or the energy function of the plane wave is appearing with a positive sign in the exponential while in this case p $\dot{x}$. And using the definition that energy function is coming or the frequency is coming with a – sign so the derivatives will throw up – $i\omega t$ $i\omega p$ or + $i\omega p$ the temporal derivative so this is how you will be able to compute the electric field this has this structure there is a relative sign difference similarly when I have to obtain the magnetic field it is curl of a curl of a will also be obtainable from the curl action, The curl action will just go and hit the special derivative hiding in this exponential here and there. And this will give you ip because remember this has a – $i\omega pt$ + $ip\dot{x}$. So this time when I do curl, the derivative option will bring down ip back and that has to undergo the curl with respect to the vector which is over here.

So I have written it twice. This should not be appearing and this should not be appearing. This curl is here and this curl is here the curl action is going to bring down the p's out and we will be doing the cross product which is the character of which is the character of the curl with the remaining vector which is downstairs e_λ okay so that is how we compute electric field and magnetic field from the vector potential a it's two components. Now we already knew the a_i *s and* e_j *s* are supposed to be conjugate pair that means we should expect that their commutation relation which should be upgraded version of their Poisson bracket should give me this object $\delta(x-y)$ they should be on same location and δ_{ij} which is this thing this is the reminiscent of let us say ϕ_i and ϕ_j if I do at x and y. This is what this should become. My field are now A_i's and conjugate moment are the electric field as we have learned. So, conjugation relation or the commutation relation should be this. That is what our expectation should have been. But one should take it slightly carefully. It is slightly modified relation of commutation that we will see in a minute. This δ function which is appearing over here can be written as a d^3p integral of the plane wave. This is as simple, right?

Now, we know that if this relation is true, I have to further enforce that this A_i is or the component of Ej, electric field are not in the direction of propagation. P should be orthogonal to this or that. That information has not gone in writing this δ function over here. The δ function does not care. It is just saying in all direction plane waves with equal magnitude. That is not true for our case because direction of P should be suppressed. It is plane wave only in the plane orthogonal to the direction of propagation. That means divergence of E and divergence of A should be 0. That has to be enforced. So that means all components I and j, it cannot take all the three components. Whatever is appearing over here should rather be r and s, and that should be *rs* here. So this should be the structure. If I force my vectors in the plane wave, in the plane of the orthogonal to the direction of proposition, then this orthogonality should appear with a δ rs in the momentum part. Remember, each plane wave is not a free plane wave in all directions. This is plane wave only along some plane and orthogonality along that plane should only be appearing. That means as we discussed δ_{rs} which was the orthogonality relation of the plane should be written as $\delta_{ij} - \Pi_{pj}$ divided by p^2. So that is how my commutation relations are modified. If I insist in writing the three component A_i A_j with forgetting that one of them will be unphysical again it can be killed off by going to a particular it can be killed off by demanding the orthogonality relation with the propagation. If I choose my propagation direction to be Z, the third component will be zero. So therefore, the commutation relation should reflect that and that is this equation or if I insist in writing the three-dimensional vector, that means whatever is appearing here should be the inform should be carrying the information that ultimately it is a two-dimensional story. Therefore, this $\delta_{ij}\Pi_{pj} /p^2$ should replace the naive δ_{ej} expectation where there was no constraint. Wrote like this there would be no constraint that p or a f a or p e or p are orthogonal to each other this will be just two quantities who's who are from canonically conjugate to each other apart from that electrodynamic story is not just the story of conjugate pair it is the story of conjugate pair which are not free to take all directions value and they are constrained to lie in the plane. And that is the statement that the weightage of the plane wave in the full three-dimensional space is reduced. It is projected along the plane only. So, this is the projection operator in the plane in some sense. Okay. So, therefore, one has to be careful that our commutation relation is this rather than the couple of lines ago where we write quote like this. This would have been a commutation relation of unconstrained *A and E*. This is a commutation relation of constrained A and E which are supposed to lie on a particular plane only.

Okay. So then we are through. We have moved all the structure at our hand. Now you can verify that if I write A and E and they have to satisfy this commutation relation which is just coming from the character of the electrodynamical field. W e have demanded what we had demanded that we are in a particular gauge where a zero is zero divergence of a is zero and therefore divergence of e is also zero we are also demanding that in this gauge the wave equation part is automatically true now we are saying that we can project our basis in a particular direction that z direction is the momentum, the propagation direction of the wave. Under this choice, only two degrees of freedom are left, which are R and S. And one can write down the commutation relation like that or in terms of, if I insist on writing the three-dimensional $\vec{A}$, then this is the relation. And with that, if I want to satisfy this AEJ commutation relation, which is outcome of our demand of choices of orthonormality, and feed in this form of A and this form of E, this should satisfy the commutation relation at hand. You will be able to verify, use the definitions of A and E, which are given over here. The two d^3p integrals will appear. There will be some A s, E s, which will be appearing in both of them. And E s has a completeness relation like that. Use everything at your hand. This is a 20-minute exercise, but this is straightforward.You will be able to verify that indeed the a and a daggers again satisfy the same commutation relation they were satisfying for a harmonic oscillator all along which we have learned. So a and a dagger which we have written as operators as of now indeed satisfy the same commutation relation that of an operator. So therefore harmonic oscillator structure story is consistent with the story which we have developed. So ap of now we have two types of a's one for is equal to one and another for is equal to two.



With these, one can check

$$[\hat{a}^{\lambda}_{\vec{p}}, \hat{a}^{\lambda\dagger}_{\vec{p}}] = \delta_{\lambda,\lambda'}\delta_{p-p'} \quad ; \quad [\hat{a}^{\lambda}_{\vec{p}}, \hat{a}^{\lambda}_{\vec{p}}{}'] = 0$$

$$[\hat{a}^{\dagger}\vec{p}, \hat{a}^{\lambda\prime\dagger}_{\vec{p}}{}'] = 0$$

$$H = \int d^3\vec{P}\,\vec{\omega}_p \left[\sum_{\lambda=1}^{2} \hat{a}^{\lambda\dagger}_{\vec{p}}, \hat{a}^{\lambda}_{\vec{p}} + (2\pi)^3 \delta(0)\right]$$

► Thus the oscillator story continues in the momentum sapce.

► $|0\rangle = \prod(\vec{p}) \prod(\lambda) |o\rangle^{\lambda}_{P}$

Remember for scalar field I had only one type of ap. For the Dirac field I had ap as a function of s two types of ap this time also I have two · of a p with λ now this is not spin but polarization okay ultimately that can be tied to the spin as well of the photon but photon being a massless spins one particle can be argued to have only two components that is not the discussion topic of this course but one way of looking at that is that way as well. So therefore, like Dirac operators, Dirac fields, this field also comes with two different types of oscillator, polarization 1, polarization 2. And Dirac fields, spin 1, spin 2, meaning spin half, spin – half was there. So analogous to that, but not the same thing. This is a vector field, that was a Spinor field. And all others commutator among themselves, a with a and a dagger with a dagger are 0. So all the thing which we have learned for Dirac thing apart from its basic underlying algebra goes through and you know this time you will ultimately end up getting the harmonic oscillator Hamiltonian again back as well. So h will be this just like we have done the number operator + an extra with a positive sign the divergence term which we have been getting all along as well this time it is again a bosonic will you will get a + over here and ultimately in the normal ordered form if you have to discuss you can discard this term this is all the excitations which we will discuss will be above this minimum value the hamiltonian can take the excitations are supposed to be above this piece which is divergent in nature and the same game like a vacuum of the theory is vacuum of all oscillators round state. The oscillators are of various different momenta and they come with different polarization angle as well polarization component . Previously in spin case it was P and spin product which oscillator with what spin now we are talking about which oscillator with what polarization. So this is the same story after all and therefore the all this techniques what we have learned will be applicable here as well so we have a similar to Dirac field story a two component object but vector component there it was a spinal component and it is coming with a polarization vector the wave is proportional to e with some operator over here previously the Dirac field was proportional to two component object with a complex Spinor coming along. So, E has morphed into the, the Spinor has been morphed into E , operator is still two component operator which is, previously it was spin dependent operator at a particular peak, this time it is polarization dependent operator at a particular peak. So therefore, whatever we have learned for scalar field and Dirac field goes through and we have the same structure and we know how to quantize the electrodynamics field in the same way that we have been quantizing all along. So I stopped the discussion over here. This is given you just a glimpse of how things are quantized to electromagnetic field and that is what we adopt to. And we will move forward from in the next class onwards. Once we have seen how to quantize different fields, we will know about what is the properties of interesting states in that and what is the expectation value of various things which will come handy when we try to couple these fields to the atoms which we had discussed in the first part of the course.