

Foundation of Quantum Theory: Relativistic Approach

Electromagnetic Field Quantization 1.1

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Electromagnetic field

Lecture- 26

So starting this set of discussions today, we will see the structure of electromagnetic field in terms of field theory and then we'll see its quantization in the way we have seen the quantization of a scalar field as well as for a Spinor field. So, in order to go up to the phase-space structure, the Hamiltonian and other things we need to know the Lagrangian and whether before that we would need to write a structure which is Lorentz invariant as well. Whatever we demanded for scalar field and the Spinor field should be true here as well.

Electromagnetic field Quantization

The working formulae governing electrodynamics

$$\vec{\nabla} \cdot \vec{D} = \frac{\rho_{\text{free}}}{\epsilon_0}$$

$$\vec{\nabla} \cdot \vec{B} = 0$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\vec{\nabla} \times \vec{H} = \mu_0 \left(\vec{J} + \epsilon_0 \frac{\partial \vec{D}}{\partial t} \right)$$

Since, $\vec{\nabla} \times \vec{E} \neq 0$ we can not say

$\vec{E} = -\vec{\nabla} \phi$ which was true for electrostatics

$$\Rightarrow \vec{\nabla} \times \vec{E} = -\frac{\partial}{\partial t} (\vec{\nabla} \times \vec{A}) \quad \left\{ \begin{array}{l} \text{because } \vec{\nabla} \cdot \vec{B} \text{ is} \\ \text{still zero in ED} \end{array} \right.$$

$$\Rightarrow \vec{\nabla} \times \left(\vec{E} + \frac{\partial \vec{A}}{\partial t} \right) = 0$$

$$\therefore \vec{E} + \frac{\partial \vec{A}}{\partial t} = -\vec{\nabla} \phi$$

$$\vec{E} = -\nabla \phi - \frac{\partial \vec{A}}{\partial t}$$

In vacuum,

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0}$$

$$\Rightarrow -\nabla^2 \phi - \frac{\partial}{\partial t} (\nabla \cdot \vec{A}) = \frac{\rho}{\epsilon_0}$$

Even in this case \vec{A} has a symmetry that

$$\vec{A} \rightarrow \vec{A} + \vec{\nabla} f(t, \vec{x})$$

In order for \vec{E} to remain invariant, ϕ also has to change in gauge transformation in a way

$$\phi \rightarrow \phi - \frac{\partial f}{\partial t}$$

s.t. $\vec{\nabla} \phi + \frac{\partial \vec{A}}{\partial t}$ remains invariant

Thus, we can choose an $f(t, \vec{x})$ s.t.

$$\vec{\nabla} \cdot \vec{A} + \frac{1}{c^2} \frac{\partial \phi}{\partial t} = 0$$

In order for \vec{E} to remain invariant, ϕ also has to change in gauge transformation in a way

$$\phi \rightarrow \phi - \frac{\partial f}{\partial t}$$

s.t.

$$\nabla\phi + \frac{\partial \vec{A}}{\partial t}$$

remains invariant

Thus, we can choose an $f(t, \vec{x})$ s.t.

$$\nabla \cdot \vec{A} + \frac{1}{c^2} \frac{\partial \phi}{\partial t} = 0$$

$$-\nabla^2 A + \frac{1}{c^2} \frac{\partial^2 \phi}{\partial t^2}$$

In this gauge

$$-\nabla^2 \phi - \frac{\partial}{\partial t} \left(-\frac{1}{c^2} \frac{\partial \phi}{\partial t} \right) = \frac{\rho}{\epsilon_0}$$

$$-\nabla^2 \phi + \frac{1}{c^2} \frac{\partial^2 \phi}{\partial t^2}$$

$$\Rightarrow \nabla^2 \phi - \frac{1}{c^2} \frac{\partial^2 \phi}{\partial t^2} = -\frac{\rho}{\epsilon_0}$$

$$\Rightarrow \square \phi \equiv \eta^{\mu\nu} \partial_\mu \partial_\nu \phi = -\frac{\rho}{\epsilon_0}$$

And the vector potential

$$\square \vec{A} = -\mu_0 \vec{J}$$

Electromagnetic field Quantization

The working formulae generating electrodynamics

$$\nabla \cdot \vec{D} = \frac{\rho}{\epsilon_0}$$

$$\nabla \cdot \vec{B} = 0$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\nabla \times \vec{H} = -\mu_0 \left(\vec{J} + \epsilon_0 \frac{\partial \vec{D}}{\partial t} \right)$$

Since $\nabla \times \vec{E} \neq 0$ we can not say $\vec{E} = -\nabla \phi$ which was true for electrostatics.

$$\Rightarrow \nabla \times \vec{E} = -\frac{\partial \vec{A}}{\partial t} \quad \{\text{because } \nabla \cdot \vec{B} \text{ is still zero in ED}\}$$

$$\Rightarrow \nabla \times \left(\vec{E} + \frac{\partial \vec{A}}{\partial t} \right) = 0$$

$$\therefore \left(\vec{E} + \frac{\partial \vec{A}}{\partial t} \right) = -\nabla \phi$$

In vacuum, $\nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0}$

\Rightarrow

$$-\nabla^2 \phi - \frac{\partial \nabla \cdot \vec{A}}{\partial t} = \frac{\rho}{\epsilon_0}$$

In order for \vec{E} to remain invariant, ϕ also has to change in gauge transformation in a way.

$$\phi \rightarrow \phi - \frac{\partial \phi}{\partial t}$$

s.t. $\nabla \phi + \frac{\partial \vec{A}}{\partial t}$ remains invariant.

Thus, we can choose an $f(t, \vec{x})$ s.t

$$\nabla \cdot \dot{\vec{A}} = \frac{1}{c^2} = 0$$

In this gauge

$$-\nabla^2 \phi - \frac{\partial}{\partial t} \left(\frac{-1}{c^2} \frac{\partial \phi}{\partial t} \right) = \frac{\rho}{\epsilon_0}$$

$$\Rightarrow \nabla^2 \phi - \frac{1}{c^2} \partial_t^2 \phi = \frac{-\rho}{\epsilon_0}$$

$$\Rightarrow \square \phi \equiv \eta^{\mu\nu} \partial_\mu \partial_\nu \phi = \frac{-\rho}{\epsilon_0}$$

And the vector potential

$$\square \vec{A} = -\mu_0 \vec{J}$$

Whatever we demanded for scalar field and the Spinor field should be true here as well. So, let us start recapitulating the ideas which we learned at the standard electrodynamics level. We know that the working formulae governing electrodynamics are governed by something called the Maxwell's equations. Maxwell's equations have two separate set of information for electric part and magnetic part. In presence of material, charge and currents, these four equations are collectively named as the Maxwell's equations. Now the Maxwell's equations are just collections of four set of equations. They are not core equations themselves because we have two scalar equations. So these are scalar equations which one component each, one equation it is. However, there are two vector equations. So these are three equations in themselves. X component, Y component, Z component should satisfy this equality. And similarly, X component and Y component and Z component should satisfy the other identity which is being set apart. So these set of equations govern the whole of electrodynamics as far as we know. Now, had it been the case we were discussing about electrostatic theory and magnetostatic theory, meaning there was a divide when things were not talking to each other, then magnetic field will not be varying in time, it will be put to 0 and electric field will not be varying in time and that will be put to 0. So that was statics, electrostatics and magnetostatics. When that happens you see that the $\nabla \times E$ is supposed to be 0. If $\nabla \times E$ is supposed to be 0, we know that it can be happening through a gradient class vector. A vector which is made from a gradient of a scalar will always have its curl 0. But the curl of magnetic field is not supposed to be 0 in presence of current. So, therefore, magnetic field will not be a gradient class vector. So, $\nabla \times E$ was always supposed to be 0 in electrostatics. Divergence of \vec{B} was always supposed to be 0 in magneto-status. So, therefore, this is better a gradient class vector. We typically write it as in terms of $-\text{rate of } \phi$ such that $\nabla \times E$ is 0. Similarly, for B to always have A_0 divergence, it should be a curl class vector. This should be a gradient class vector and magnetic field should be a curl class vector because a vector which is made from a curl of another vector will always have its divergence here.

So, therefore, in electrostatic and magnetostatic respectively, magnetic field and electric field are to be obtained from potentials, scalar potential for electric field whose gradient will give me the electric field and vector potential for magnetic field whose curl will give me the magnetic field. Once we go on to the theory of electrodynamics, these equalities break away. No longer electric field and magnetic fields remain in isolated sectors. Electric field and magnetic field start talking to each other. Namely, $\nabla \times E$ is no longer zero. It is equal to the $-\text{of the time derivative of magnetic field}$. And similarly, curl of magnetic field in some sense is also related to the $\nabla \times E$. Remember, the displacement vector d is related to the electric field in terms of the ϵ_0 going to some ϵ relative $\cdot \epsilon_0$, that is the polarization is taken into account. You should go back and look at the nodes of your electrodynamics to see the structure of these equations. But as we see in electrodynamics these isolations were not working, this will no longer be 0 and this will no longer be without free of electric field. The causality in this is that once we have put on a magnetic field on the right hand side of the $\nabla \times E$, then $\nabla \times E$ is no longer 0. And therefore, we cannot say that it is a gradient class vector, which was true for only for electrostatics. Electrostatics right hand side would have been 0 and then we will have demanded that it is a gradient class. Now we do not. However, divergence of B is supposed to be 0 still. Even when electric field and magnetic field talk to each other, they talk through their curls. So magnetic field's divergence still remains the same that it is zero. That means still magnetic field can be a curl class vector. Electric field can no longer be a gradient class vector, but magnetic field can still be a curl class vector. So therefore I can write down the $\nabla \times E$ is $-\partial \vec{B} / \partial t$ of and where \vec{B} is a curl of some quantity. So divergence of $\vec{A} + 1/c^2 \partial \phi / \partial t$. This object could have been anything. But now we know if it is anything which is non-zero, I can do a gauge transformation. I can re-change my A to something, A + something else, re-change my ϕ to $\phi +$ something else. Under this process, this object

will not change. Magnetic field will not change. But this can change by some amount. How much amount? This much amount. And if I choose my this much amount, that exactly cancels its previous value. That means after gauge transformation, I will get up a new A and a new ϕ which are these and these compared to the previous versions, which satisfy that this object is 0. And once this object is 0, we see where it appears, it appears over here, this term whole becomes 0 and I have a Laplacian of A with a $-$ sign $+ 1/c^2$, the double derivative of A with respect to time is equal to $\mu_0 J$. So, you will get using that you will get using that, that $-$ of Laplacian of $A + \frac{1}{c^2} \frac{\partial^2 \vec{A}}{\partial t^2}$ is 0 or is $\mu_0 \vec{J}$. The magnetic

field component now becomes this they become independent of ϕ previously they were dependent on their evolution was dependent on ϕ due to presence of these terms so now I have killed the whole term now it has become independent of ϕ the similar thing you can do for the ϕ itself ϕ is evolution was also entangled with the A 's appearance. So, now what I can do since I have used a gauge where divergence of A is $-$ of $1/c^2 \partial\phi/\partial t$. I can use that fact into this equation and change this divergence of A as $-$ of $\frac{1}{c^2} \frac{\partial^2 \phi}{\partial t^2}$. So, overall this equation also becomes 1 over $-$ of Laplacian of $\phi + \frac{1}{c^2} \frac{\partial^2 \vec{A}}{\partial t^2}$ is equal to $\rho - \mu_0 \vec{J}$.

upon x_0 . Now, this also now totally depends on ϕ . So, I have decoupled those A and ϕ by doing a gauge transformation.

What more? You can see that operators which are appearing hitting on ϕ which is Laplacian with a $-$ sign and double time derivative. Laplacian with a $-$ sign and double time derivative. And if you remember this is our definition of D'Alembertian. So, ultimately with a $-$ sign actually this is $-$ of a D'Alembertian. So, I have the equation that D'Alembertian or box of ϕ is equal to $-\rho$ upon ϵ_0 and $\square A$ is $-\mu_0 \vec{J}$. So, together we have equations like box is something and $\square A$ is also something. And now we will see that this box story becomes more interesting if I try to write things in terms of special relativistic coordinates. So, in order to go and relate things with a special relativistic four vector notations which we are typically familiar with. I see I have four things in hand. One is ϕ scalar potential and three components of vector potential. I can write them as a collection of four objects.

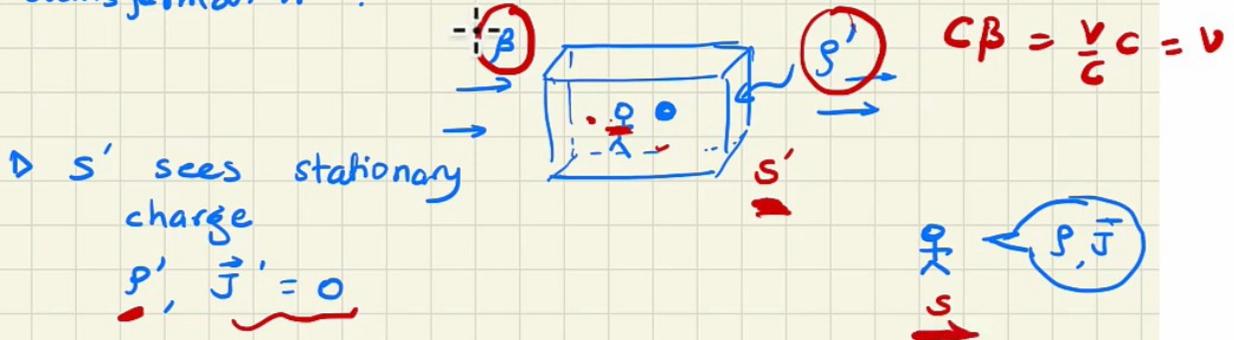
* Collectively we can have set of four quantities

$$\left(\frac{\phi}{c}, \vec{A} \right)$$

$$(c\rho, \vec{J})$$

$$\square A^\mu = -\mu_0 \vec{J}^\mu \quad \checkmark$$

o How do A^μ and J^μ transform under Lorentz transformation?



▷ S' sees stationary charge

$$\rho', \vec{J}' = 0$$

▷ S sees a moving charge; thus a charge density and current.

$$d^3V = \sqrt{1-\beta^2} d^3V' \Rightarrow \rho = \frac{dq}{d^3V} = \frac{dq}{\sqrt{1-\beta^2} d^3V'} = \frac{\rho'}{\sqrt{1-\beta^2}}$$

$$J_x = \rho c \beta = \frac{\rho c}{\sqrt{1-\beta^2}} \beta = \gamma (\sinh \alpha \rho' + \cosh \alpha J_x')$$

↓

$$c \frac{dq}{d^3V} \frac{d^3x}{dt} = c \frac{dq}{d^4x} \frac{d^4x}{dt} \quad \begin{matrix} \text{invariant} \\ \text{transforms like (spatial)} \\ \text{co-ordinate difference (infinitesimal)} \end{matrix}$$

Similarly J_y, J_z .

$$\rho = \frac{dq}{d^3V} = \frac{dq}{d^3V dt} dt = \frac{dq}{d^4x} \frac{dt}{dt} \quad \begin{matrix} \text{invariant} \\ \text{transforms like (temporal)} \\ \text{co-ordinate difference} \end{matrix}$$

★ Collectively we can have set of four quantities $(\frac{\phi}{c}, \vec{A})(c\rho, \vec{J})$

□ $A^\mu = -\mu_0 J^\mu$

◦ How do A^μ and J^μ transform under Lorentz transformation,

► S' sees stationary charges

$\rho', \vec{J} = 0$

► S sees a moving charge; thus a charge density and current.

$$d^3 v \sqrt{1-\beta^2} d^2 v' \Rightarrow \rho = \frac{dQ}{dV} = \frac{dq}{\sqrt{1-\beta^2} dV'} = \frac{\rho'}{\sqrt{1-\beta^2}}$$

$$J_x = \rho c \beta = \frac{\beta c}{\sqrt{1-\beta^2}} = \gamma (\sinh \alpha \rho' + \cosh \alpha J'_x)$$

$$c \frac{dq}{d^3 V} \frac{dl}{dt} = c \underbrace{\frac{dq}{d^4 x}}_{\text{invariant}} \frac{dl}{\underbrace{dt}_f}$$

Similarly $\rho = \frac{dq}{d^3 V} \frac{dt}{dt} = \frac{dq}{d^4 x} dt$

$\underbrace{\frac{dq}{d^4 x}}_{\text{invariant}} dt$ transforms like (temporal)

Whether these four objects are four vectors or Spinors or what, we have to determine that. Because we have seen collection of four objects do not always become vectors. We saw an example in previous set of lectures. Collections of $\psi_1, \psi_2, \psi_3, \psi_4$ was a Spinor. Similarly, arbitrary collection of four numbers do not become, does not become a four vector. For example, I can take t, t^2, x, x^2 . This is not four vector. Any four collection of objects will not be a space time vector. So, we have to determine if I write down this ϕ and a together as a four component object, how does it transform under low-inch transformation. just like we try to obtain how does the spinous size transform under low inch transformation. In addition to that again we have on the right hand side collection of four objects. One scalar charge density and then vectors charge density or current density which we call. So, I have a collection of two four component object. This is four component object this is also four component object. But we do not know whether these four component objects are scalar meaning four independent scalars or four components of vectors or Spinors or any other thing or maybe nothing they will do not transform they are just a random collection of four things. So, for that to happen I will try to recast this equation in a more evident way and try to see what happens. So, in order to do that I just divide the first component by c and multiply the first component of the current density into c . So, let us call it this 4 potential or 4 vector potential and this 4 current density. So, 4 potential and 4 current density we are interested about and we want to know whether this collection which is named thusly is a good vector, Spinor or scalar which transforms appropriately under law of transformation or not. So, I have divided the first component or zeroth component, let us say this whole collection is going to be called as A_μ . A_0, A_1, A_2, A_3 without saying that A_μ is a component of vector.

I am just randomly naming the four components. Similarly, the four components of the current density are going to be called J_μ . So, I divide the first component, first meaning zeroth component of a by c and

multiply the zeroth component of J with c such that the two set of equation which was box of ϕ which was a scalar equation and $\square A$ which was vector equation. We have three component, A_x will satisfy A_x , is equal to J_x , $\square A_y$ will be $\mu_0 J_y$ and $\square A_z$ will $\mu_0 J_z$. So, this is three equations here and one equation here. If I make this multiplication and division through c and use the fact that c^2 is nothing but $\mu_0 \epsilon_0$, then you can see the two box equations together can be written as a single box equation like here, box $\square A_\mu$ is equal to $\mu_0 J_\mu$. Put μ is equal to 0, put 0 here as well. Relate that A_0 was ϕ/c and J_0 was $c\rho$, you will get a $\square\phi/c = -\mu_0 c\rho$. Take this c over here. So, it will become c^2 without c here and c^2 is $1/\mu_0 \epsilon_0$. So, this object will become $-\rho/\epsilon_0$. This is the first equation. And remaining three equations are untouched A_1 is A_1 here, J_1 is J_1 here, A_2 is A_2 here, J_2 is J_2 here, A_3 is A_3 here, and J_3 is J_3 here. So therefore, I do not have to worry. After this just simple multiplication, I have converted the set of equations into one seemingly compact equation and I want to now know whether these things are four vectors, four components of a Spinor, four random scalars or some weird object which we have encountered. In order to know that, let us look at the right hand side first J_μ . Again J_μ has four components ρ and the three surface current density. In order to know how these components transform, Let us look at a case where suppose there is a frame S' in which a person is standing and looking at a stationary charge with respect to this guy. So, therefore, this frame S' , the observer sees that there is a current, there is a charge density, static charge density, which he or she is going to call as ρ' . So this observer says that there is a stationary charge density ρ' and there is a no movement of the charge. All the charge is static. Nothing is moving according to this guy. So therefore, they will declare that there is no current density. J' is zero. Now look at the same story being viewed from another frame S , which is let us say lab frame. And this observer says that the whole box of this observer and the charge is moving. So there is S frame. S' is moving with respect to S with a velocity v or parameter β . β remember is v by c . So $c\beta$ will be the speed of the frame S' with respect to S . So β is just a compact notation of how fast the frame S' is moving with respect to S . Now what is the story according to the person in the frame S ? The person in S frame will see there is a moving charge, a charge which is moving with certain velocity. So they will say that indeed there is a charge and therefore a charge density. Suppose this guy was standing in a box. Suppose this guy was standing in a box. S' was in a box. S also sees there is a box and there is charge inside, so therefore the charge density inside. But they see that there is a velocity to the charge, so there is a current as well. So S sees both charge density as well as surface current density. S' only sees charge density and no surface current density. Now let us see what are their magnitudes. So, first of all, this guy S' will say that according to this him, there is a charge dq upon d^3v' , where d^3v' is the volume of the box around him or her. Now, the same story when viewed from the frame S , they will also say that, okay, there is a charge inside a box, but they will disagree upon on the amount of the volume of the box because the x axis is a boosted thing y and z axis do not change. So, overall length will change with this factor and x and y will not the y and z effect directions will not change. So, overall the volume of the box seen by s' will be d^3v' seen by S will be d^3v and they are related by this $1 - \beta^2$ root factor, 1 over γ factor in more compact notations of special relativity. This is alright because according to boost one arm will be contracted and therefore volume will also be contracted. So, therefore if The S is looking at the volume being contracted. Both of them agree that how much charge is present. If one particle is present, both of them will agree that one particle is present. And charge is a fundamental unit, fundamental property of object. They will all say that net amount of charge is unchanged. Only thing changing is volume description.

So according to S , the same amount of charge is now living in the volume, which is slightly shorter, d^3v which is related to d^3v' by this extra factor. Therefore, the charge density which was declared as dq upon d^3v' by observer s' , s' will say that this much charge, charge remains invariant but this much volume, so ρ' . So, you see the charge density has changed by a factor ρ' upon $1 - \beta^2$. So, s and s' do not agree on the motion of the charge density. How much of current this guy sees? S sees there is a current which is the charge density what it sees and the velocity with which the charge density moves. Remember J is always $\rho \cdot V$. So now we have obtained the expression of ρ what this guy S sees. The v

will be just βc . Now write down the things. ρ is previous ρ' the charge density as seen by S' divided by $1 - \beta^2$ and you see there is a nice structure which emerges out which if you pay close attention is the same structure which we have seen before sin hyperbolic $\alpha \cdot \rho'$ and cos hyperbolic $\alpha \cdot J'$. It so happens that $J' J'_x$ is 0 according to S' . So, therefore this cos hyperbolic term does not arise only the charge density γ and sin hyperbolic α which is precisely this term. This kind of transformation we have seen before sin hyperbolic cos hyperbolic. This was exactly like the spatial vectors were supposed to transform. So, it looks like that the J_x is indeed transforming as if it is component of a spatial vector. One can make this statement much more clear by this realization that remember what was ρ and what was $c\beta$. So, $\rho \cdot c\beta$, $c\beta c$ can be pulled out, β is just $c\beta$ is velocity let us say. So, this c factor can be removed. dL by dt . And while charge density is dq by d^3v . So, overall in the denominator, I have a d^3v and dt . Put together, I have a d^4x in the denominator, which is space-time volume. And if you remember our discussion in spatial relativity, d^4x is invariant under Lorentz transformation. And if you remember our discussion in spatial relativity, d^4x is invariant under Lorentz transformation. dt changes exactly in the opposite amount in which d^3v changes and they exactly cancel out therefore d^4x does not change dq does not change as we have discussed so overall J_x will only change as if dl is changing in frame transformation So now you see indeed J_x transforms like a spatial vectors component. This is dl_x . If I talk about J_x , it will be lens projection along x-axis. If it was J_y , I will get dl_y . And ultimately, there is an invariant piece \cdot a spatial vector. So J_x indeed transforms like a spatial vectors. And similarly, you can argue for J_y, J_z . So, all the J , full J transforms like a spatial Lorentz vector. As the spatial Lorentz vector would transform, the coordinates will transform. The infinitesimal coordinate differences are supposed to transform. They transform like components of a Lorentz vector. This much we had learned before. What about the zeroth component charge? Again, the same story. Charge, write down the charge density dq upon d^3v . I can multiply and divide by dt so in the denominator I will have again have a d^4x and this object again will be invariant and I will be left with the charge density transforms like dt the infinitesimal coordinate difference along the zeroth direction so it also transforms like a space time vectors zeroth component this J is transformed like space time vectors spatial components. Together ρ and J put together transforms like spatial component and temporal component of dt and dx or dl . So, indeed they are up to some multiplication transforming like space time vectors and that multiplication is invariant. So, therefore we have J_μ which is a collection of these four objects transforming like a space time four vector or Lorentz four vector. So, we have identified that the right hand piece transforms like a Lorentz four vector. Can we play some similar kind of game and show that μ is also proportional to space time four vector? We can do that, but there is a easier trick of seeing this through because this, if this equation has to hold, right hand side transforms like a 4-vector, this object transforms like a 4-vector, then the left hand side should also transform overall as a 4-vector, out of which I have a box.

Thus, $(c\rho, \vec{J})$ transforms like spacetime 4-vectors

$J^\mu \equiv (c\rho, \vec{J})$ is a spacetime 4-vector

$$\square A^\mu = -\mu_0 J^\mu \quad A^\mu = \begin{pmatrix} \frac{\phi}{c} \\ \vec{A} \end{pmatrix}$$

\square Invariant

A^μ should also transform like spacetime 4-vector

The building block of EM theory is a spacetime 4-vector.

In empty charge/current less space

$$\rho = 0, \vec{J} = 0 \Rightarrow J^\mu = 0$$

$$\square A^\mu = 0$$

- ★ We now have a vector wave equation
- ★ A 4-component object (like Dirac spinor)
- ★ But transforms via Λ under L.T.
- ★ This wave equation is emergent in a gauge!

Thus $(c\rho, \vec{J})$ transforms like space-time 4-vectors
 $J^\mu \equiv (c\rho, \vec{J})$

$$\square A^\mu = -\mu_0 J^\mu$$

A^μ should also transform like spacetime 4-vector.

The building block of EM theory is a space-time 4 – vector.

In empty charge/ current loss space

$$\rho = 0, \quad \vec{J} = 0 \quad \Rightarrow J^\mu = 0$$

$$\square A^\mu = 0$$

And if you remember our discussion on the special relativistic operators, this box was supposed to be an invariant operator. So, whatever is left with that should transform as a four vector in order to make things consistent. So, I do not know why I have put a c, c can rest over half. This should not have been there, but anyway. So, ultimately left hand side has to be loading four vector because right hand side already is. In the left hand side, I have a box operator which is invariant, which automatically makes a case that A_μ transforms like a space time four vector. So, now we have both the pieces at our hand that this equation which was collectively written for electric field, magnetic field together going into a particular gauge frame. I have wrote down this equation which is a four vector equation. It is not four random scalars, not Spinors four component, but vectors four component. So, this is indeed a space time four vector, a genuine four vector. The building block of the electromagnetic theory is A_μ because A_μ constitutes ϕ divided by c and the three components of a. These together generate electric field and magnetic field for me. So, the building block of the electromagnetic theories component is four vector, electromagnetic four vector, space time four \vec{A}_μ .

What more?

If I go to empty space, some observer is just looking at a plain electromagnetic way of propagating. The electric field and magnetic field are being generated by ϕ and A respectively, and they are satisfying the box equation. But suppose it is just a vacuum, no charge density, no current present, then the right-hand side becomes 0, J_μ becomes 0. If J_μ becomes 0 in one frame, A_0 vector remains 0 in all other floating frames. So therefore, I cannot generate A_0 to non-0 J_μ through non-transformation. Therefore right hand side will be put to zero for all inertial frame, all Lorentz frames. And I have a equation which is $\square A_\mu$ is equal to zero. This is four equations A_0, A_1, A_2, A_3 . That means some equation for ϕ and three equation for a. Now you see this is already a wave equation or the Klein-Gordon equation which we are familiar with so many . So you see that the potential which generates electromagnetic theory is a four vector it has four components, all the components satisfy the Klein-Gordon equation just like the four components of Dirac field were also satisfying the Klein-Gordon equation. So, we can again hope to do the quantization in the way we had done the quantization of a scalar field. The same thing we tried for the spinal field as well, so bad that we ended up changing the commutator structure to anticommutator to make sense of physics. The same thing we tried for the spinal field as well, so bad that we ended up changing the commutator structure to anticommutator to make sense of physics. Let us see what happens for this thing. We still have a Klein-Gordon equation. If I try to quantize it, can I do it in the usual way or should I do it the non-usual way of anticommutator? Till now, amuse are supposed to be real quantities. So, it looks like since I have give, I

do not enforce that they should be complex quantity, I would get through with the usual quantization and indeed that will happen. That this kind of scalar field, this kind of a vector field is very much tied to the scalar fields derivation which we had seen, nothing goes wrong with this. However, we have to be alert to the fact that we have a vector equation at hand It is a four component object just like Dirac's Spinner. But unlike Dirac's Spinner, it transforms under Λ matrix itself, not with $D(\lambda)$. Remember, $D(\lambda)$ were generated by $S_{\mu\nu}$, where $S_{\mu\nu}$ were made from $\gamma_\mu \gamma_\nu$ 1 by 4, while Λ is made from the M matrix, $M_{\mu\nu}$. And these are just 1 0 0 1 kind of matrices all around. Both of them are 4 cross 4 dimensional matrices, but they are not related to any unitary transformation. So, therefore, they are differently different quantum sectors. If they were related by unitary transformation, you would have said, okay, whatever appears as a vector is just a unitary transformed version of the Spinor. They are not. So, therefore, this object is qualitatively distinct from the four component object Spinor what we have seen. And one last bit which is additional in this case, This wave equation does not came out automatically like playing around with the equation, which happened for scalar field and which happened for Spinor. We got the field equation of Klein-Gordon directly. Here we had to do some massaging to the equation. We had to land up on a particular gauge. This wave equation is not always true. This is true in a particular gauge. And that gauge has to be always go along when we try to quantize this theory. We cannot do the quantizing forgetting this gauge. If I want to do harmonic oscillator quantization like Klein-Gordon quantization, I also have to respect the gauge condition because in this gauge only this equation holds. So if I am going to do quantization in this gauge, I should be alert to the condition I had imposed in order to get the wave equation. And the condition I imposed that I will make this term which was appearing in the AS differential equation to 0. That means I have to do quantization of the electromagnetic field with one condition. This condition has to be respected. So this gives up a new set of things apart from the structure of a scalar field. This opens up a new space of something called polarization. So we have to be alert about that as well. And we will see that this condition put together is equivalent to certain scalar fields doing the harmonic oscillator business up to a condition which fixes the number of degrees of freedom of the photon. So, gauge potential has four component, ultimately we will see that using all the gauge conditions and what not we will end up with only two of components of this are meaningful and therefore these are the two polarizations of the photons and that is what we will quantize when we go to the quantum theory. So that we will do in the next class from where we will start to write down a consistent Lagrangian and try to see what kind of phase space structure and Hamiltonian and other things emerge out of it. And then we will see how to impose this gauge condition along with the oscillator structure to get the quantization. So I stop over here for this discussion session. We will move on to the Lagrangian description in the next class.