Foundation of Quantum Theory: Relativistic Approach
Spinor Field quantization 1.4
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Fermionic Fields
Lecture- 25

So, in today's discussion session, we will try to go across the problems what we realized in the last class regarding the quantization of the Dirac field. So, we had learnt previously that everything was working very well till the time we came face to face with the Hamiltonian of the Dirac field. So, appealing to the inherent hiding the hiding harmonic oscillator structure in the Dirac fields. We landed up on the usual quantization which we would have done for a scalar field, thinking that these spinor fields which are collections of many fields ultimately behave as individual oscillators in Fourier domain. And at the end of the day, we obtained that the Hamiltonian was indeed a collection of number excitations of different types of oscillators with their corresponding frequency. It so happened that one type of the oscillators were negative frequency oscillator.
And nothing stops me from doing that because they are free oscillators after all.

$$N\begin{pmatrix} \\ \frac{c\sum_i \sigma_i p_i}{(E+mc^2)}\begin{pmatrix}0\\1\\0\\1\end{pmatrix}\end{pmatrix} e^{\frac{i p\cdot x}{\hbar} - \frac{E t}{\hbar}} \qquad N'\begin{pmatrix} \\ \frac{c\sum_i \sigma_i p_i}{(-E+mc^2)}\begin{pmatrix}0\\1\\0\\1\end{pmatrix}\end{pmatrix} e^{\frac{i}{\hbar}\cdots}$$

$$u^s(p)\, e^{\frac{i\vec{p}\cdot\vec{x}}{\hbar} - \frac{iEt}{\hbar}} \qquad v^s(p)\, e^{\frac{i\vec{p}\cdot\vec{x}}{\hbar} + \frac{iEt}{\hbar}}$$

$$\Psi(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \sum_s \left(\hat{a}^s_p u^s_p e^{\frac{i\vec{p}\cdot\vec{x}}{\hbar} - \frac{iEt}{\hbar}} + \hat{b}^{s+}_p v^s_p e^{-\frac{i\vec{p}\cdot\vec{x}}{\hbar} + \frac{Et}{\hbar}} \right)$$

$$\Psi^+(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \sum_s \left(\hat{a}^{s+}_{\vec{p}} (u^s_p)^+ e^{-\frac{ip\cdot x}{\hbar}} + \hat{b}^s_{\vec{p}} (v^s_p)^+ e^{-\frac{ip\cdot x}{\hbar}} \right)$$

For $[\Psi(\vec{x}), \Psi(\vec{x}')] = \delta^3(\vec{x} - \vec{x}')\, \mathbb{1}_{4\times 4}$

Which leads to :

$$[\hat{a}^s_{\vec{p}}, \hat{a}^{s'+}_{\vec{p}'}] = \delta_{ss'}\, \delta(\vec{p} - \vec{p}')$$

$$[\hat{a}^s_{\vec{p}}, \hat{a}^{s'}_{\vec{p}'}] = 0 = [\hat{a}^{s+}_{\vec{p}}, \hat{a}^{s'+}_{\vec{p}'}]$$

and

$$[\hat{b}^s_p, \hat{b}^{s'+}_{p'}] = \delta_{ss'}\, \delta(\vec{p} - \vec{p}')$$

$$[\hat{b}^s_p, \hat{b}^{s'}_{p'}] = 0 = [\hat{b}^{s+}_p, \hat{b}^{s'+}_{p'}]$$

A quick fix: If instead of $[\psi, \psi^\dagger] = \delta$ we demanded

$$\{\psi(\vec{x}), \psi^\dagger(\vec{x}')\} = \delta(\vec{x} - \vec{x}') \checkmark$$

then

$$\{\hat{a}_p^s, \hat{a}_{p'}^{s'\dagger}\} = \delta_{ss'}\,\delta(\vec{p} - \vec{p}\,') \checkmark$$

$$\{\hat{b}_p^s, \hat{b}_p^{s'\dagger}\} = \delta_{ss'}\,\delta(\vec{p} - \vec{p}\,') \checkmark$$

H will still be the same!

$$H = \int \frac{d^3p}{(2\pi)^3} \sum_s E_p \left(\hat{a}_p^{s\dagger}\hat{a}_p^s - \hat{b}_p^{s\dagger}\hat{b}_p^s\right)$$

$$\bar{u}_s(\vec{p}) \cdot u_{s'}(\vec{p}) = \frac{2\delta_{ss'} mc^2}{(\epsilon + mc^2)} |N'|^2$$

$$v_s^\dagger(\bar{p}) \cdot u_{s'}(-\vec{p}) = 0$$

Further, $$(-i\gamma^i\partial_i + m)\psi = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \sum_s \Big(\hat{a}_p^s (-\gamma^i p_i + m) u_p^s e^{ip\cdot x} + \hat{b}_p^{s\dagger} (\gamma^i p_i + m) v_p^s e^{-ip\cdot x} \Big]$$

Using Dirac equation

$$i\gamma^0\partial_0\psi + i\gamma^i\partial_i\psi - m\psi = 0$$

$$-i\gamma^i\partial_i\psi + m\psi = i\gamma^0\partial_0\psi$$

$$\therefore \quad (-i\gamma^i\partial_i + m)\psi = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_p}} \sum_s \Big(\hat{a}_p^s i\gamma^0\partial_0 u_p^s e^{ip\cdot x} + \hat{b}_p^{s\dagger} i\gamma^0\partial_0 v_p^s e^{-ip\cdot x} \Big)$$

$$\int d^3x\, \psi^\dagger = \gamma^0 \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \sum_s \Big(\hat{a}_p^s \gamma^0 E_p u_p^s e^{ip\cdot x} - \hat{b}_p^{s\dagger} \gamma^0 E_p v_p^s e^{-ip\cdot x} \Big)$$

$$\mathcal{H} = \int d^3x \, \psi^\dagger \gamma^0 (-i\gamma^i \partial_i + m)\psi$$

$$= \int d^3x \int \frac{d^3p}{\sqrt{2E_p}(2\pi)^3} \int \frac{d^3p'}{\sqrt{2E_{p'}}(2\pi)^3} \sum_{ss'} \left(\hat{a}_{p'}^{s'\dagger} u_{p'}^{s'\dagger} e^{-ip'.x} + \hat{b}_{p'}^{s'} v_{p'}^{s'\dagger} e^{+ip'.x} \right) \left(\hat{a}_p^s E_p u_p^s e^{ip.x} - \hat{b}_p^{s\dagger} E_p v_p^s e^{-ip.x} \right)$$

Using $\int d^3\vec{x} \, e^{i(\vec{p} \pm \vec{p}')\cdot\vec{x}} = (2\pi)^3 \delta^3(\vec{p} \pm \vec{p}')$

and the normalization factors we obtain

$$H = \int \frac{d^3p}{(2\pi)^3} \sum_s (\omega_p) \left(\hat{a}_p^{s\dagger} \hat{a}_p^s - \hat{b}_p^{s\dagger} \hat{b}_p^s \right)$$

But since $\hat{a}$ and $\hat{b}$ are different unrelated operators

$$\hat{a}_p^s |0\rangle = 0 = \hat{b}_p^s |0\rangle \quad \forall \; p, s$$

The eigenstates are $|n_s^{(a)}, n_s^{(b)}\rangle$ and

$$\bar{u}_s(\vec{p}) u_{s'}(\vec{p}) = \frac{2\delta_{ss}' mc^2}{(E+mc^2)} |N|^2$$

$$v_s^\dagger(\vec{p}) u_{s'}(-\vec{p}) = 0$$

Further,

$$(-i\gamma\partial_i + m)\psi = \int \frac{d^p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \sum_s \left[\hat{a}_p^s (-\gamma^i p_i + m) u_p^s e^{ipx} + \hat{b}_p^s (-\gamma^i p_i + m) b_p^s e^{-ipx} \right]$$

Using Dirac equation

$$i\gamma^0\partial_0\psi + i\gamma\partial_i\psi - m\psi = 0$$

$$-i\gamma^i\partial_i\psi + m\psi = i\gamma\partial_i\psi$$

$$\therefore (i\gamma^i\partial_0 + m)\psi = \int \frac{d^p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}\sum_s\left(\hat{a}_p^s i\gamma^0\partial_0 u_p^s e^{ipx} + \hat{b}_p^{s\dagger} i\gamma^0\partial_0 v_p^s e^{-ipx}\right)$$

$$= \int \frac{d^p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}\sum_s\left(\hat{a}_p^s\gamma^0 E_p u_p^s e^{ip\dot{x}} - \hat{b}_p^s\gamma^0 E_p v_p^s e^{-ip\dot{x}}\right)$$

$$H = \int d^3x\, \psi^\dagger\gamma^0(-i\gamma^i\partial_i + m)\psi$$

$$= \int d^3x \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}\int \frac{d^3p'}{(2\pi)^3}\frac{1}{\sqrt{2E_p'}}\sum_s\left((\hat{a}_p^{s'\dagger}{}' u_p^{s'\dagger}{}' e^{-ip'\dot{x}} + \hat{b}_p^s{}' v_p^s e^{-ip\dot{x}})(\hat{a}_p^s E_p u_p^s e^{ip\dot{x}} - \hat{b}_p^s E_p v_p^s e^{-ip\dot{x}})\right)$$

Using $\int d^3\vec{x}\, e^{i(\vec{p}\pm\vec{p}')\vec{x}} = (2\pi)^3\delta^3(\vec{p}\pm\vec{p}')$ and the normalization factors we obtain

$$H = \int \frac{d^3p}{(2\pi)^3}\sum_s(\omega_p)(\vec{a}_p^{s\dagger}\vec{a}_p^s - \vec{b}_p^{s\dagger}\vec{b}_p^s)$$

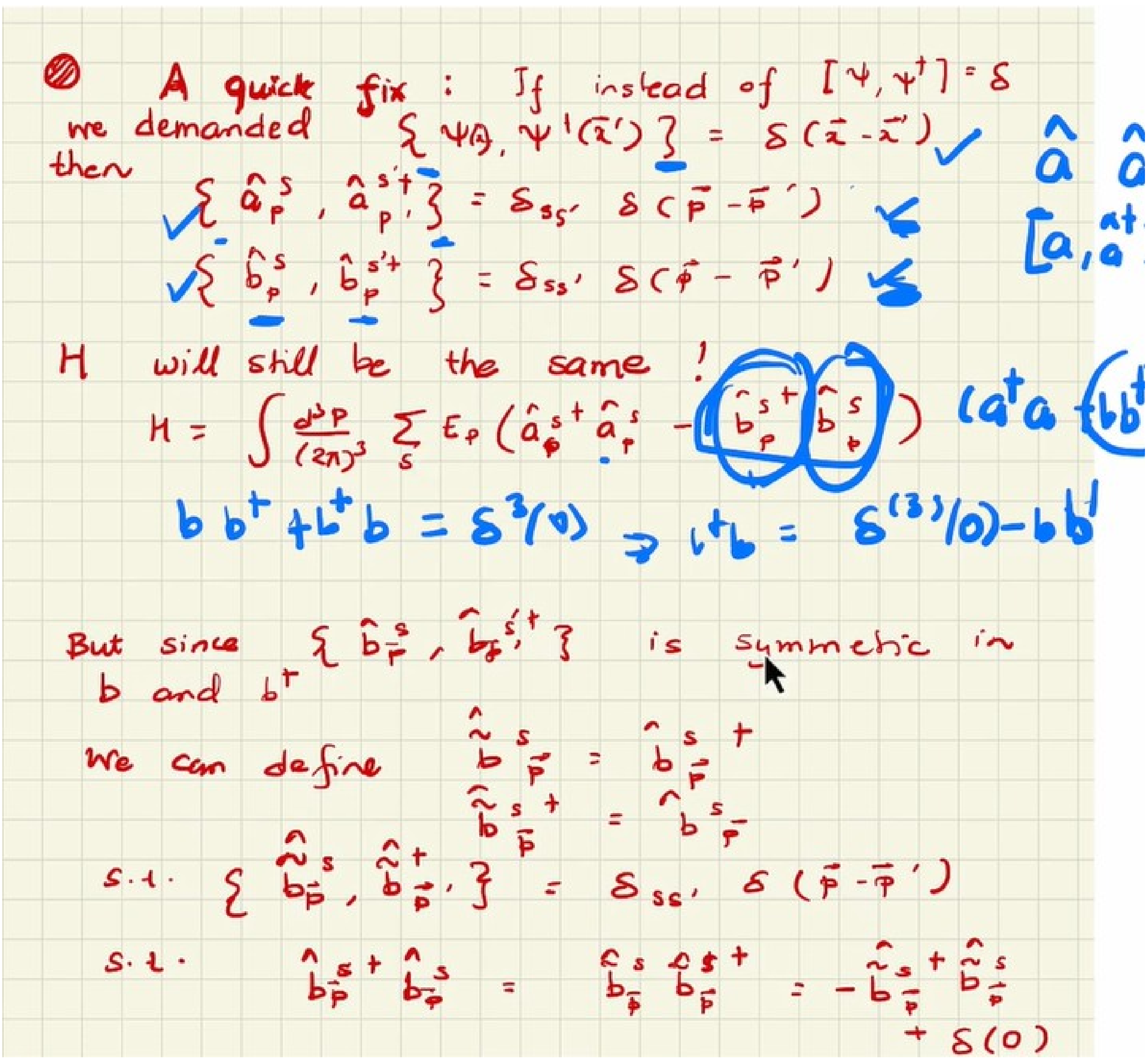
But since $\hat{a}$ and $\hat{b}$ are different unrelated operators

$$\hat{a}_p^s|0\rangle = 0 = \hat{b}_p^s|0\rangle\ \forall\ p, s$$

The eigenstates are $|n_s^{(a)}, n_b^{(s)}\rangle$ and if $n_b > n_a$ we have -ve energy states!

⇒Unbounded from below!!

Suppose I want to get a good consistent Hamiltonian and good consistent field theoretic structure. Is it necessary to cling to the oscillator story or we can do something else? Why do we have to insist that we have to quantize things via oscillators? There is no priory region because in scalar field we realize that it is a harmonic oscillator kind and therefore we appeal to it let it be harmonic oscillator kind of quantization which should work here. Which is fine, which could work here but it mathematically it works but physically it gives you meaningful and meaningless answers for spinors. So, therefore one can try to have a different quantization scheme despite it satisfying the oscillator equations of motions. So, oscillators equations of motions are classical equations of motions. There is no way of saying that once oscillator at the classical level is given, you know how to quantize it. You can do it under different schemes of things as well. And therefore, we are in some sense open to explore other possibilities of quantizing complex valued oscillators, for example. So, that is what we will try to do. At the spot where we inserted the oscillator like feature was this commutator. What if we have landed upon a species which does not have any classical analog? Then I do not know whether we should start the story in the classical space and then go to quantize it. Maybe there is an inherent different quantization which is at work, which does not appeal to the Poisson bracket structure. And then all these things which we have done goes through only at the place I would rethink about to impose this commutation relations which are coming from our considerations of classical oscillators quantization. Maybe we have ended up on systems which is not classical because ultimately it is giving me nonsense solutions of negative energies. So maybe it is not a classical solutions being quantized. It is inherently a quantum system with no classical analog. In that game, if we adopt to, we are open to play with this. This commutator set which we had provided, we are open to play with this and tweak this to say that this system behaves under quantization like these commutators which will fix the game for us. So let us first spot where was the problem. The problem which we ended up in getting the Hamiltonian was in the term over here. Okay, this $\hat{b}_s^\dagger\hat{b}_s$. I can try to do something to it so that means I can use the structure which is $\hat{b}_s\hat{b}_s^\dagger$ if I take there is a p always I am not writing it is supposed to be a delta function of sS' so sS' S and S's are the same then it is one and then there is a delta of $p - p'$ now if I take this $\hat{b}_s^\dagger\hat{b}_s$.It appears in the commutator like this $\hat{b}_s\hat{b}_s^\dagger - \hat{b}_s^\dagger\hat{b}_{s\,s}$ delta of 0 because p becomes p'. Therefore, this $\hat{b}_s^\dagger\hat{b}_s$ is $\hat{b}_s\,\hat{b}_s^\dagger -$ delta of 0. All right.



$$\therefore \quad H = \int \frac{d^3\bar{p}}{(2\pi)^3} \sum_s E_p \left(\hat{a}_{\bar{p}}^{s\dagger} \hat{a}_{\bar{p}}^{s} + \hat{b}_{\bar{p}}^{s\dagger} \hat{b}_{\bar{p}}^{s} - \delta^3(0) \right)$$

So, therefore, I can try to convert this into $\hat{b}_s \hat{b}_s^\dagger$, but that does not solve for me anything. I will get a – delta 0 which will add to this, but still a $\hat{b}_s \hat{b}_s^\dagger$ operator which will be coming with a negative sign. So this is not going to help me if I keep sticking to this kind of voiceover, this kind of commutator structure. a quick fix without realizing, without motivating it from a classical origin, I just want to fix the sign of the Hamiltonian. a quick remedy to that, I argue, can be obtained by just taking the commutator and converting it into anti–commutators. So if I do not demand $\{\psi(\mathrm{x}), \psi^\dagger(\mathrm{x})\}$s commutator to be delta function, I will demand their anti–commutator to become delta function.
This is a radical choice I am making as of now, but I will see soon it fixes my problem of negative Hamiltonian. So, everywhere the commutators appear, I am just going to force without motivating from classical physics that let it be a new rule which is coming with spinors that anti–commutators have to satisfy the oscillator kind of structure rather than the commutator. So, this is unlike the classical oscillator we have seen. This is still some oscillator because it satisfies oscillators equations of motion in the classical sector, but its quantization is different. Its mode function satisfies in Fourier domain, mode function satisfy in Fourier domain the oscillator equations of motion, but its quantization, raising and lowering operators rule I am going to change, demanding that it is unlike any classical oscillator we have seen. And this is fine with the mathematical description, nothing stops me from doing that. So, let us go ahead and do this. So, once we open ourselves to that possibility, then something interesting happens. First of all, we have just changed the commutation relations between ψ and $\psi^\dagger$ and what not. Therefore, the derivation of Hamiltonian is untouched. In deriving the Hamiltonian, we never used the fact that it has to be commutator or anti–commutator because Hamiltonian lives in the phase space classically as well. So, I will still get the same expression for the Hamiltonian before quantization, let us say. So, before quantization of the system, the Hamiltonian was this. This will keep functioning. Now, no discussion or no description of quantization was inserted into deriving this equation. So, that will remain fine. Second step, I will ask that side and side diggers are still like that. Only thing I am changing is these commutation relations.So, let us go ahead and try to see where did we use the commutator for the first time.So, again writing these operators, this was Dirac equation was used. So, that is also intact and maintaining the Dirac equation. We are just trying to isolate the case where is the commutator structure which is hiding for the first time we saw it. So, you will see that at in these derivations at no place actually, we needed to get the commutator till the derivation of Hamiltonian structure. So, you will see that at in these derivations at no place actually, we needed to get the commutator till the derivation of Hamiltonian structure. p going to p' this all this exponential multiplied together with the d3 x integral is converted into delta function and one of the integration you have to do without the use of commutator we ended up getting this.

So therefore the Hamiltonian story The Hamiltonian story will still remain the same even if the commutator has been replaced by anti–commutator. So Hamiltonian remains the same. But now there is something interesting which we can do. Previously we were trying to replace $\hat{b}_s \hat{b}_s^\dagger$ with $\hat{b}_s^\dagger \hat{b}_s$ with $\hat{b}_s \hat{b}_s^\dagger$ and we ended up getting a $\delta^3(0)$. When I wrote this it became $\hat{b}_s \hat{b}_s^\dagger - \delta^3(0)$. So negative sign survives. Not anymore because when I am going to do the anti–commutator derivation this time, now you will see that I would have a $\hat{b}_s \hat{b}_s^\dagger + \hat{b}_s^\dagger \hat{b}_s$ that would be equal to delta of 3 of 0 because I am using anti–commutator now this equation, this second equation rather. So, that means $b^\dagger$ b is equal to delta $\delta^3(0)$ $\hat{b}_s \hat{b}_s^\dagger$. Okay, why we are doing this mathematical algebra will be clear in a minute. So, let us do that. So, therefore, if I replace this $\hat{b}_s^\dagger \hat{b}_s$ with $\delta^3(0) - \hat{b}_s \hat{b}_s^\dagger$, I will get $\hat{a}_s \hat{a}_s^\dagger + \hat{b}_s \hat{b}_s^\dagger - \delta^3(0)$. Okay, that is what it will emerge out if I do this, make use of this.So what this is not a number operator right number operator should be $\hat{b}_s^\dagger \hat{b}_s$ not $\hat{b}_s \hat{b}_s^\dagger$ so how to make this $\hat{b}_s \hat{b}_s^\dagger$ so previously we realized that a was a lowering operator $a^\dagger$ was a raising operator and their commutator was anti–symmetric in the replacement of $\hat{a}$ and $\hat{a}^\dagger$ if i replace a and $a^\dagger$ across i was supposed to in get a – sign if i flip positions of a and $a^\dagger$ but not in anti–commutators. Anti–commutators are symmetric under flip.

So you do not know which one of them, if I flip the right hand side remains the same. So I do not know which one is raising, which one is lowering because both of the actions should be symmetric. So therefore I can keep calling back this thing as a lowering operator $\hat{b}_s^\dagger$ and $\hat{b}_s$ as a raising operator and still the anti–commutator structure will remain the same. So I am using two facts. I am converting the commutator which appears for usual classical mechanically motivated system to anti–commutator. And then I am saying that since $\hat{b}_s$ and $\hat{b}_s^\dagger$ appear symmetrically, I can replace the role of $\hat{b}_s$ by $\hat{b}_s^\dagger$ and $\hat{b}_s^\dagger$'s role by $\hat{b}_s$. So therefore, I will keep calling a new set of variables. Let us say $\hat{\tilde{b}}_s$ which is equivalent to $\hat{b}_s^\dagger$ and $\hat{\tilde{b}}_s^\dagger$ which is equivalent to $\hat{b}_s$.So if in terms of those variables, if I write, you will see all the algebra goes through. Still the $\hat{\tilde{b}}_s$ and $\hat{\tilde{b}}_s^\dagger$ satisfy the same anti–commutator relation which was satisfied by $\hat{b}_s$ and $\hat{b}_s^\dagger$.

I have just flipped their roles. If I do that and go through these steps which I was just discussing, here the $\hat{b}_s \hat{b}_s^\dagger$ which was coming with a + sign will now become $\hat{b}_s^\dagger \hat{b}_s$. So, now you see I have both the number excitation operator of a type oscillators and $\hat{\tilde{b}}_s$ type oscillators with positive signs. At the post of a there is a $\delta^3(0)$ with a negative sign that means the lower limit of the oscillators are set to $\delta^3(0)$ which is this is the term which was coming with a + sign remember with a + sign in the scalar fields. For spinor fields, we have generated a Hamiltonian, which is now goes only upwards. It does not go downwards, but it has a cost of a – $\delta^3(0)$. So, if I do not know what to deal with, how to deal with this $\delta^3(0)$, as much as I did not know how to deal with the + sign $\delta^3(0)$ in the Hamiltonian of the scalar field as well. So, all these things, + thing and – thing, assigned they are problematic or aspects in terms of how to deal with them but now hamiltonian has become one way only increases it does not decrease down from its minimum value so therefore it does not remain unbounded from below if i think of a composite system containing scalar fields and spinors, both total Hamiltonian of the whole system will contain the + delta three zero and – delta three zero. And maybe they cancel, maybe they do not. So who knows? So therefore, I'm not going to worry about the constant things which do not change fro m state to state. This is not an operator. This is just a constant thing. Overall, this is the integration times this times a delta 30, which is a number, which will not change from state to state, either for a scalar field or for a spinor field. This piece is just a DC shift.

- Anti commutators have no classical Poisson bracket analog !

$\Rightarrow$

☆ Since
$$\{ \hat{a}^{s}_{\vec{p}}, \hat{a}^{s'\dagger}_{\vec{p}'} \} = \delta_{ss'} \, \delta(\vec{p} - \vec{p}')$$
$$\{ \hat{a}^{s\dagger}_{\vec{p}}, \hat{a}^{s'\dagger}_{\vec{p}'} \} = 0 = \{ \hat{a}^{s}_{\vec{p}}, \hat{a}^{s'}_{\vec{p}'} \}$$

$$\hat{a}^{s\dagger}_{\vec{p}} |0\rangle = |s, \vec{p}\rangle$$

$$\hat{a}^{s\dagger}_{\vec{p}} \, \hat{a}^{s'\dagger}_{\vec{p}'} |0\rangle = - \hat{a}^{s'\dagger}_{\vec{p}'} \, \hat{a}^{s\dagger}_{\vec{p}} |0\rangle$$

$$|s, \vec{p}\,; s', \vec{p}\,'\rangle = - |s', \vec{p}\,'; s, \vec{p}\rangle$$

* Antisymmetric particles

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$$\hat{a}^{\dagger}_{s,\vec{p}} \, \hat{a}^{\dagger}_{s,\vec{p}} |0\rangle = - \hat{a}^{\dagger}_{s,\vec{p}} \, \hat{a}^{\dagger}_{s,\vec{p}} |0\rangle$$

$$\Rightarrow \hat{a}^{\dagger}_{s,\vec{p}} \, \hat{a}^{\dagger}_{s,\vec{p}} |0\rangle = 0$$

Two particles can not occupy the same state: $\rightarrow$ fermions !

● Anti commutators have no classical Poisson bracket analog!

⇒

⋆ Since

$$\{ \ \hat{a}^s_{\vec{p}}, \hat{a}^{s\dagger}_{\vec{p}}{}' \ \} = \ \delta_{ss'} \delta(\vec{p}-\vec{p}\,')$$

$$\{ \ \hat{a}^s_{\vec{p}}, \hat{a}^s_{\vec{p}}{}' \ \} = 0 = \{ \ \hat{a}^{s\dagger}_{\vec{p}}, \hat{a}^{s'\dagger}_{\vec{p}} \ \}$$

and

$$\{ \ \hat{b}^s_{\vec{p}}, \hat{b}^{s\dagger}_{\vec{p}}{}' \ \} = \ \delta_{ss'} \delta(\vec{p}-\vec{p}\,')$$

$$\{ \ \hat{b}^s_{\vec{p}}, \hat{b}^s_{\vec{p}}{}' \ \} = 0 = \{ \ \hat{b}^{s\dagger}_{\vec{p}}, \hat{b}^{s'\dagger}_{\vec{p}} \ \}$$

⋆ Antisymmetric particles

$$\hat{a}^{\dagger}_{s,\vec{p}} \hat{a}^{\dagger}_{s,\vec{p}} |0\rangle = -\hat{a}^{\dagger}_{s,\vec{p}} \hat{a}^{\dagger}_{s,\vec{p}} |0\rangle$$

$$\Rightarrow \quad \hat{a}^{\dagger}_{s,\vec{p}} \hat{a}^{\dagger}_{s,\vec{p}} |0\rangle = 0$$

And if we want my physics to be invariant to insensitive to the decision, we can just simply ignore this. This goes by the name of normal ordering. That means I will write the Hamiltonian always such that I will write the dagger operators on the left and the non dagger operator on the right that means I will throw away these things so when I am getting something like $aa^\dagger + bb^\dagger$ its normal ordered version will be $a^\dagger$ a + $\hat{b}_s^\dagger$ $\hat{b}_s$ I will not use commutators to flip this I will just say normal orders means that bring all the $a^\dagger$s on the left and $\hat{b}_s^\dagger$s on the left so similarly we can say. The normal ordered Hamiltonians are physical Hamiltonians which are insensitive to the DC shift. DC shift is a positive kind for scalar field and negative kind for the spinor field. So I am now ready to do business for this new set of oscillators which are not like classical set of oscillators. They come with anti–commutators and normal ordered versions of the Hamiltonian is very similar to the normal ordered version of the Hamiltonian for scalar field. So apart from a divergent piece both the scalar field and spinal field had come up out with a negative sign there is similar physics going on for spinors and scalar field but for the fact one thing is scaling with commutator one thing is scaling with anti–commutator and that is the new thing which we have to live with now. We have a system testifying anti–commutator structure which has no classical Poisson bracket analog. This did not get motivated from first a Poisson bracket and then a commutator. If there was any way a Poisson bracket that could not have been converted to anti–commutator because Poisson bracket is anti–symmetric under replacement of objects, anti–commutator is not. So therefore it is not coming from a Poisson bracket. It is not that it is a special oscillator of a particular kind in the classical space. It is oscillator, but not in the classical space. So, if we choose to make this choice, mathematics allows for this. There is nothing problematic anywhere we have got. We have just broken our correspondence to the classical oscillators. That is all.

So, therefore, we are saying that if such things exist, that means there are non–trivial complex oscillators in the nature whose possibility arises out of plain mathematics and Lorentz symmetries. And this come with spin half integer objects. And these are the fermions as we now know. The fermions are supposed to satisfy this anti–commutator structure as in a commutator structure. We will see more of this fact when we deal about the causality in a couple of weeks lectures from now that this anti–commutator is really required not only from fixing the sign of Hamiltonian but also from the space–time causality as well. Okay, so apart from these causal requirements and positivity of the Hamiltonian,

There is another interesting aspect which comes about from these spinor fields, which is as follows. So see, what we have done, we had converted all the commutators into anticommutators. So still the commutability which was previously gets replaced by anticommutability, but the relations remain the same. a and $a^\dagger$s of the same spin talk to each other, different spins do not talk to each other. Similarly, for $\hat{b}_s$ and $\hat{b}_s^\dagger$s, they should be of the same spin and same momentum. So, that is reflected by this delta function in spin and delta function in momentum. This should be true for a as well as b. Further, different $a^\dagger$s irrespective of same spin or different spin, same momentum, different momentum should be 0. Anti–commutator of same, annihilation operator of same two $\hat{a}$s with same S_p or different $S'_{p'}$. It is also zero so a and a do not talk to each other $\hat{a}$ and $\hat{a}^\dagger$ do not talk to each other and a and $a^\dagger$ talk to each other through delta function in the nt commutator Similarly, is the structure for $\hat{b}_s$ sets, $\hat{b}_s$ and $\hat{b}_s^\dagger$ will have this non–trivial anticommutation relation, while $\hat{b}_s$ will talk to, $\hat{b}_s$ and $\hat{b}_s$ anticommutation should be 0 for irrespective of spin being same or different, p being same or different. Similarly, $\hat{b}_s^\dagger$ and $\hat{b}_s^\dagger$ anticommutator should be 0 as well for different or same spins and different and same momentum irrespective of that. So, these sets are respected by all the oscillators a and $a^\dagger$s, $\hat{b}_s$ and $\hat{b}_s^\dagger$s. So, remember we had converted things into $\tilde{\hat{b}}_s$ and $\tilde{\hat{b}}_s^\dagger$ in order to cast them in terms of the usual structure of a raising operator. But after that, we can restore calling them to $\hat{b}_s$ and $\hat{b}_s^\dagger$, meaning initially I start with a $\hat{b}_s$ and convert it along the way to $\tilde{\hat{b}}_s$. Otherwise, I could have started from $\tilde{\hat{b}}_s$ and call the flip quantity as $\hat{b}_s$. So depending upon how many symbols you are willing to write, you can keep calling the final Hamiltonian as $\hat{a}^\dagger\hat{a}$ + $\tilde{\hat{b}}_s$ $\tilde{\hat{b}}_s^\dagger$ or keep calling $\hat{a}^\dagger\hat{a}$, $\hat{b}_s^\dagger$ $\hat{b}_s$. It doesn't matter. So now both of them do come with same sign and that is what it matters. So okay. So now out of this a very interesting thing comes about. So now let us say the vacuum. This is the vacuum of all the oscillators. A type oscillators and $\hat{b}_s$ type oscillators as well. The collective thing of all the oscillators. 0 is product of all the A type thing, meaning the positive frequency thing on all p and S. So there is a product over the S spin as well. And similarly, a product over negative E, negative frequency as well. 0, p and S. Okay so this is the whole oscillate all the oscillators either a type or $\hat{b}_s$ type or irrespective of which spin type we are talking about they are all in the ground state now I add $a^\dagger$ operator with a momentum p and spin S it will generate a particle with spin S and part momentum p compare it with the scalar field oscillator $\hat{a}_p^\dagger$ when I hit it with $\hat{a}_p$. I would generate a particle with momenta p this time I am generating a particle with two quantum numbers momentum as well as spin. So, this is the new feature which has come out of this new type of non–classical oscillators with complex field structure. They have spin as well. There was an untold spin for spinors as well, which we now know is zero. Spinors are spin zero objects. So, there was no need to write a spin. But here, I have a spin of two kinds, + half or – half. So, the dagger operator will of that kind will tell me which particle is generated and with how much spin either half spin or – half spin. So, now think of a state which is
$|S',p'\rangle$ that means from the vacuum one† operator of S' spin and p' momenta has acted upon that and I want to generate another particle with $|S',p'\rangle$ $|S',p'^\dagger\rangle$, let us say, $|S, p^\dagger\rangle$ kind of thing. There is a particle already present with spin S' and momentum p'. I want to generate another particle in the state with a different momenta p and different spin S.So I will act this operator into $|S',p'\rangle$
nd the resultant thing will be this. Okay, so that will become two particle state. One particle with Sp momenta and already there was a $|S',p'\rangle$ which was present. So this will become this. However, I know that this is $A^\dagger$ and $A^\dagger$s of a similar spin or different spin irrespective of that, they anticommute to 0. That means $\hat{a}^s{}_p{}^\dagger$, $\hat{a}^s{}_{p'}{}^\dagger$ is negative of the flipped version of that because they have to anticommute to 0. So that means I will have this state which is the flipped version of that negative of that, negative of the first one. First particle has $S'p'$ quantum numbers and the second particle has sp or the other way if you call this as a first particle this has a momenta $S, \vec{p}$ and this is the second particle and the other one is reversed so they come up with a negative sign irrespective of the unlike the scalar field if I have generated two particles they would have generated a state with something called $k\,|\,p\rangle$ first p and then

k. They come up with a – sign because of this anti–commutator structure. Therefore, now we are talking about particles which get a negative sign under their flip. So these are anti–symmetric particles. As we know, they are fermions from the picture. Fermions are known to follow this property. They are anti–symmetric under space exchange or momenta exchange. p exchange completely to negative side. is also equivalent to the two particle flip between $\vec{p}$ and $\vec{p}\,'$ is also equivalent to space flipping because Fourier transform will generate x for u and x' for u from Fourier transform of p and p' respectively. So you do Fourier transformation with respect to e^{ipx}, $e^{ip'x'}$. You will see that it will generate a particle at x and particle at x'. And similarly, it will generate a particle of x' here and particle at x in this case. They are negative of each other. So therefore, we are talking about anti–symmetric particles. So these spinors are not only unusual oscillators with anti–commutators, they have a strong sine factor that when you flip them, they pick up a negative sign in the spatial part. Secondly, what happens if I try to excite two particles in that?

That means I make S and S' the same. That means this object and these operators are the same and they are negative of each other this and this become the same therefore the negative also have the same operator and therefore this quantity is negative of itself and we know only one quantity can that do that which is negative of itself which is zero that means the action of two operators on the vacuum is zero so this is again a very strong statement that not only you cannot flip them without a sign difference you have you will get a sign difference they are anti–symmetric and by the virtue of them being anti–symmetric you cannot have two particles with the same quantum number if you go to fourier domain the same spin and same location cannot be occupied same spin and same momentum cannot be occupied so two particles cannot have all the same quantum numbers you have to change something you have to make these two operators slightly different either change S or change p you cannot have both the numbers same. So therefore, we are talking about anti–symmetric particles and particles which can be populated in a given p fock basis only once. You cannot populate a fock basis with the same spin twice. And this is a very crucial aspect of fermions. So thus, we learned that the new oscillators come up with a new structure of anti–commutation, anti–symmetric properties and single state population. And this will become very handy when we talk, when we deal with their quantum expectation value, quantum properties, the stress energy tensor, Hamiltonian momentum, what not, which we are not going to do in this set of discussion. You can take it as exercise, but keep in the mind that never you will be able to excite anything to twice excitation in the same quantum number. So this is the new species under quantization which has come up with new features which was not present classically because we know it has no correspondence to classical oscillators and therefore it is a purely quantum particle. So I stop the discussion on spinor things in this class, in this set. Next class onwards, we will start dealing with a vector field, which is the electromagnetic field. But you can see, if I keep demanding the structure of Lorentz symmetry, I can have a scalar, I can have a spinor, I can have a vector, I can have a tensor, and the list goes on. So this generates various sort of particles. The spectrum of particles becomes very rich under Lorentz transformation. So just for completeness of the discussion, I will just discuss a couple of lectures on electromagnetic field which is a vector field and then start dealing with coupling atoms to any of these kind of fields to see what happens when atoms start talking to these kind of fields the quantum fields okay so I stop over here for this class and in the next class we will move on to the vector field quantization.