Foundation of Quantum Theory: Relativistic Approach

Spinor Field quantization 1.3

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Dirac Field Quantization

Lecture- 24

Usual Quantization ✓

Recall : The Dirac equation has solution

$$\begin{pmatrix} \phi_\pm \\ \chi_\pm \end{pmatrix} e^{\frac{i\vec{p}\cdot\vec{x}}{\hbar} - \frac{iE_\pm t}{\hbar}} \qquad E_\pm = \pm\sqrt{p^2c^2 + m^2c^4}$$

We can write the four solution as

$$N\begin{pmatrix} 1 \\ 0 \\ \frac{\sum_i c\sigma_i p_i}{(E+mc^2)}\begin{pmatrix}1\\0\end{pmatrix} \end{pmatrix} e^{\frac{i\vec{p}\cdot\vec{x}}{\hbar} - \frac{Et}{\hbar}} \qquad N'\begin{pmatrix} 1 \\ 0 \\ \frac{\sum_i c\sigma_i p_i}{(-E+mc^2)}\begin{pmatrix}1\\0\end{pmatrix} \end{pmatrix} e^{\frac{i\vec{p}\cdot\vec{x}}{\hbar} + \frac{Et}{\hbar}}$$

$$N\begin{pmatrix} 0 \\ 1 \\ \frac{c\sum_i \sigma_i p_i}{(E+mc^2)}\begin{pmatrix}0\\1\end{pmatrix} \end{pmatrix} e^{\frac{i\vec{p}\cdot\vec{x}}{\hbar} - \frac{Et}{\hbar}} \qquad N'\begin{pmatrix} 0 \\ 1 \\ \frac{c\sum_i \sigma_i p_i}{(-E+mc^2)}\begin{pmatrix}0\\1\end{pmatrix} \end{pmatrix} e^{\frac{i\vec{p}\cdot\vec{x}}{\hbar} + \frac{Et}{\hbar}}$$

$$u^s(p)\, e^{\frac{i\vec{p}\cdot\vec{x}}{\hbar} - \frac{iEt}{\hbar}} \qquad v^s(p)\, e^{\frac{i\vec{p}\cdot\vec{x}}{\hbar} + \frac{iEt}{\hbar}}$$

$$\psi(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \sum_s \left(\hat{a}^s_p u^s_p e^{\frac{i\vec{p}\cdot\vec{x}}{\hbar} - \frac{iEt}{\hbar}} + \hat{b}^{s\dagger}_p v^s_p e^{-\frac{i\vec{p}\cdot\vec{x}}{\hbar} + \frac{Et}{\hbar}} \right)$$

$$\psi^\dagger(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E}} \sum_s \left(\hat{a}^{s\dagger}_p (u^s_p)^\dagger e^{-\frac{ip\cdot x}{\hbar}} + \hat{b}^s_p (v^s_p)^\dagger e^{-\frac{ip\cdot x}{\hbar}} \right)$$

For $[\psi(\vec{x}), \psi(\vec{x}')] = \delta^3(\vec{x} - \vec{x}')\, \mathbb{1}_{4\times 4}$

O Usual quantization

Recall: The Dirac equation has a solution

$$\begin{pmatrix}\phi_{\pm.}\\ \psi_{\pm.}\end{pmatrix} e^{\frac{i\vec{P}\dot{\vec{x}}}{\hbar}-iE\pm\frac{t}{\hbar}} \qquad E_{\pm}=\pm\sqrt{p^2c^2+m^2c^4}$$

We can write the four solution as

$$N\begin{pmatrix}1\\0\\ \sum_i \frac{\sigma_i p_i}{(E+mc^2)}\begin{pmatrix}0\\1\end{pmatrix}\end{pmatrix} e^{\frac{i\vec{P}\dot{\vec{x}}}{\hbar}-iE\pm\frac{t}{\hbar}} N'\begin{pmatrix}1\\0\\ \sum_i \frac{\sigma_i p_i}{(-E+mc^2)}\begin{pmatrix}0\\1\end{pmatrix}\end{pmatrix} e^{\frac{i\vec{P}\dot{\vec{x}}}{\hbar}+iE\pm\frac{t}{\hbar}}$$

$$\begin{pmatrix}1\\0\\ \sum_i \frac{\sigma_i p_i}{(E+mc^2)}\begin{pmatrix}0\\1\end{pmatrix}\end{pmatrix} e^{\frac{i\vec{P}\dot{\vec{x}}}{\hbar}-iE\pm\frac{t}{\hbar}}=u^s(p)e^{\frac{i\vec{P}\dot{\vec{x}}}{\hbar}-iE\pm\frac{t}{\hbar}}$$

$$\begin{pmatrix}1\\0\\ \sum_i \frac{\sigma_i p_i}{(-E+mc^2)}\begin{pmatrix}0\\1\end{pmatrix}\end{pmatrix} e^{\frac{i\vec{P}\dot{\vec{x}}}{\hbar}+iE\pm\frac{t}{\hbar}}=v^{(s)}(P)e^{\frac{i\vec{P}\dot{\vec{x}}}{\hbar}+iE\pm\frac{t}{\hbar}}$$

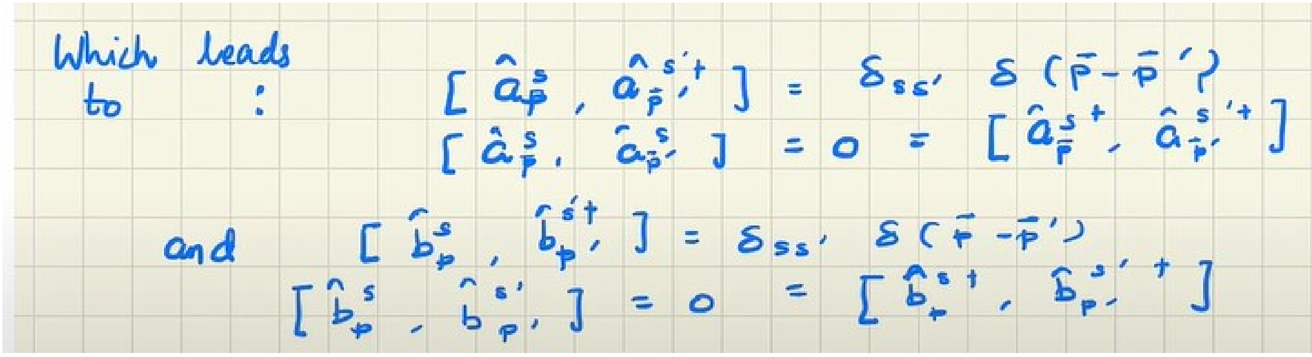
$$\psi(x)=\int\frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E}}\sum_\sigma\left(\hat{a}_p^s u_p^s e^{\frac{i\vec{p}\dot{\vec{x}}}{\hbar}-\frac{iEt}{\hbar}}+\hat{b}_p^{s\dagger}v_p^s e^{\frac{-i\vec{p}\dot{\vec{x}}}{\hbar}+\frac{iEt}{\hbar}}\right)$$

$$\psi^\dagger(x)=\int\frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E}}\sum_\sigma\left(\hat{a}_p^{\dagger s}(u_p^s)^\dagger e^{\frac{i\vec{p}\dot{\vec{x}}}{\hbar}-\frac{iEt}{\hbar}}+\hat{b}_p^s v_p^{s\dagger} e^{\frac{-i\vec{p}\dot{\vec{x}}}{\hbar}+\frac{iEt}{\hbar}}\right)$$

$$For[\psi(\bar{x}),\psi(\vec{x}')]=\delta^3(\vec{x}-\vec{x}')1_{4\times4}$$

So, in the previous discussion session, we learned that the equations of motion for Dirac field, which was the Dirac's equations of motion, can be converted into a harmonic oscillator kind of equation of motion. Rather than a harmonic oscillator, we obtained the Klein-Gordon equation, this equation which we obtained originating from the Dirac equation of motion. What we did? We acted upon on the Dirac's equations of motion with *i* times, $-i\gamma^\mu\partial_\mu$, and ultimately the Dirac equation therefore became the Klein-Gordon equation which we know how to quantize, courtesy the scalar fields quantization which we have learned in the previous sessions. In this case, only thing which we have is that this is not a real scalar field first of all. It is a 4 cross 1 column vector which comes with a 4 component thing, ψ_1, ψ_2, ψ_3, ψ_4. Remember, the ψ here is a representative of a collection of a 4 cross 1 vector which has 4 complex quantities at its disposal. So, we have this structure which we obtain for Klein-Gordon equation. So, now we want to know that the usual way of quantizing the scalar field in the class, in the discussion session which we had done, we discussed about the quantization of the real scalar field, but anyway a complex scalar field also can be quantized along the similar lines, which we have not done for in the discussion session, but you will see that it is not much different from the real scalar field. In the real scalar field, remember ,the coefficient of e to the e^{ikx} and $e^{-ik\dot{x}}$ where complex Hermitian conjugate of each other in order to get you a real ϕ. So, this d^3k gave rise to a real ϕ for this and that to become Hermitian conjugate to each other. You will do obtain ,a real ϕ but suppose you drop the requirement of having a real ϕ then there is no reason that these quantities and these quantities should be hermitian conjugate to each other in principle I can put a different b of k over here and then get my business done initially remember we had started doing the business like that and then we imposed the reality condition So, this time I am talking about Dirac's spinor and which involves four of the fields, none of them are necessarily real. So, I drop the demand that $\hat{a}_k\, e^{ikx}$ and $\hat{a}_k^{\dagger} e^{-ikx}$ should be the structure of its Fourier domain. So, let us go forward and try to see what kind of a structure we can impose and therefore, what do we learn from the standard quantization procedure. So okay so we do first the usual quantization and surprisingly or unfortunately for us we will see that this usual quantization scheme which was very well working for a harmonic oscillator structure and the scalar field despite appealing it leads us to some unphysical conclusions which we will see and therefore we have to tweak the usual quantization scheme in order to get a meaningful physics out. So let us see what is the unphysicality inherent in the usual quantization procedure. So again we are looking for a solution of this equation first classically I know these are the solutions while we when we were discussing the Dirac equation of motion in the relativistic quantum mechanics domain there also the same story emerged that the the particles are supposed to satisfy the Dirac equation and hence the klein gordon equation as well this time as well we are the same structure I know Despite it being a 4 cross 1 dimensional vector, its space-time dependent part satisfies the same equation which a scalar field does.

Because all the derivatives are spatial derivatives or temporal derivatives. So, therefore, its space-time dependence cannot be different from the space-time dependence of the scalar field. So, therefore, the plane wave structure should be inherent in that as well. So, I know the space time dependency should be through $e^{ipx/\hbar - iE+}$ – because energy comes in two signatures plus or –. This was true for even scalar field. So, this e, this e was not the energy but let us say frequency, $E/\hbar$ was the frequency, positive frequency and negative frequency which was appearing even for scalar field. What is the total energy would be known from the Hamiltonian only. In the quantum field theoretic picture but plane wave structure comes with positive frequency and negative frequency and $e^{ip\dot{x}}$ $e^{-ip\dot{x}}$ both of them are captured by a generic p vector because p vector can be positive or negative so this is the solution of the Dirac equation okay. ,Now, so this part which is the finer part of this, this is the new feature which has arrived because this is not a scalar field. Scalar field part was this *much*. ,Since it is not a scalar field, a new part in terms of a 4 cross 1 dimensional column vector has appeared. And we know again in order to satisfy the Dirac equation, this column vector has to maintain a particular structure. For instance, it could be coming with definite energy, positive or negative. So it has positive energy or positive frequency solution and negative frequency solutions. As well as we realized when we were discussing about the spinal structure that it has extra degree of freedom or extra operator which quantifies that which is the spin or the helicity of that. So either it can come with one helicity spin half or spin – half with the same energy or spin half or spin – half with negative energy or frequency. So we list down both of them. , So these two, this one and this one are positive energy or positive frequency solutions. So – E $/\hbar$ appears in the exponential.



Which leads to:

$$[\hat{a}^s_{\vec{p}}, \hat{a}^{s\dagger}_{\vec{p}}{}'] = \delta ss'\delta(\vec{p}-\vec{p}\,')$$
$$[\hat{a}^s_{\vec{p}}, \hat{a}^s_{\vec{p}}{}'] = 0 = [\hat{a}^{s\dagger}_{\vec{p}}, \hat{a}^{s'\dagger}_{\vec{p}}]$$

and

$$[\hat{b}^s_{\vec{p}}, \hat{b}^{s\dagger}_{\vec{p}}{}'] = \delta ss'\delta(\vec{p}-\vec{p}\,')$$
$$[\hat{b}^s_{\vec{p}}, \hat{b}^s_{\vec{p}}{}'] = 0 = [\hat{b}^{s\dagger}_{\vec{p}}, \hat{b}^{s'\dagger}_{\vec{p}}]$$

So these are positive frequency solutions because if you work this $i\partial\partial_t$ you will get a positive numbers out as a eigen function and the spinor part was remember was something called a ϕ_0 ,and a ψ_0 and there was a relation between ϕ_0 and ψ_0 which was related by $\sigma_i p_i$ summation over all the poly matrices and all the momentum components in the helicity operator and divided by E_+/mc^2 . So this was the structure. Go back to your Dirac equations note when we were trying to solve the Dirac equation. There this solution emerged out. In order to satisfy the Dirac equation as well as the Klein-Gordon equation, ϕ_0 and ψ_0 in the doublet ,The doublet ϕ_0 and ψ_0 cannot be arbitrary. They have to be related like this. And further if I demand that ϕ_0 and ψ_0 individually have to be eigen functions for eigen states for the helicity operator. It can come with 1010 or 0101. 1010 structure was for plus $\hbar/2$ helicity operator and this one is – $\hbar/2$. Or call it loosely call the spin value. The spin half, the spin – half or as helicity Sz plus half Sz – half with positive energy. For the negative energy exponential comes with a wrong sign plus $E/\hbar$t, this time it will also be plus $E/\hbar t$, this is plus $\hbar/2$ in the spin and this is – $\hbar/2$ in the spin. The factor which will changes in the denominator of these two things as well, e goes to – e. So in the negative energy solution part, negative frequency solution part, you will get this $E+mc^2$ or E $-E+mc^2$ depending upon which sector you are talking about, positive frequency or negative frequency. So keep a note that while I am calling it positive energy, just looking at the symbol E, energy ultimately will get decided by the Hamiltonian. This is field theory. So the parameter appearing over here is not ,has been not been shown to be the eigen function of or the eigen value of the Hamiltonian operator. So, I am loosely calling it energy in the spirit of the Schrodinger equation structure which we used to call it. So, this is really positive frequency and negative frequency solutions which we are talking about. So, therefore, if I go below ,So I am going to collect all the positive frequency solutions as symbol U. S will tell me about what spin value I am going to talk about. Spin can take two values, plus half or – half. Similarly, there are two negative frequency solutions, Vs, $Vs_{+\,1/2}$ and $Vs_{-1/2}$,. Their space time dependence part are plane waves, one with $+iEt$ and one with $-iEt$. So these are the four set of solution, two in the spin half sector, positive energy, spin half and spin – half and two in the negative frequency sector, spin half and

– half. And in the spirit of harmonic oscillator structure with the solution, now I can write down all the possible mode function. Remember these things do satisfy the Klein-Gordon equation upstairs. The Klein-Gordon equations have now four solutions. ,positive frequency solution, positive spin solution, positive frequency solution, negative spin solution, negative frequency solution, negative spin solution, negative frequency solution, positive spin solution. So all these four are your mode functions. So these are the mode functions I am going to write, positive frequency and negative frequency. And I am going to associate operators just like harmonic oscillators mode function. Remember the same discussion. The mode functions of harmonic oscillators came with a and $a^{\dagger}$ with u star. I had to add $a^{\dagger}$ because I wanted the position operator to be Hermitian. This time I am not bounded to obtain that because the field which we are describing is not necessarily a real field. So in principle I can write any other operator. So let me write it as a $\hat{b}s^{\dagger}$. There is no relation of priory between a and b.There are four different operators for positive frequencies and different s. And similarly, there are, sorry, so there are two different operators for positive frequency with two different spins. And similarly, for negative frequencies, there are two different spins and two different operators. So, overall, I have a $\hat{a}_p^{s(1/2)}$, $\hat{a}_p^{s(1/2)}$ – half, $\hat{b}_p^{s(1/2)}$, $\hat{b}_p^{s(1/2)}$.

This is the structure I have obtained you can check that this is still a solution of a Klein Gordon equation or the harmonic oscillator in the fourier space okay so far so good I can obtain the side† ,the side† which is Hermitian conjugate of this by just taking the conjugate of this operator. So all this a will become† b which was already I had written in $a^{\dagger}$ form this will become just b.I had written it in the spirit of the harmonic oscillator one thing comes with a and one thing comes with $a^{\dagger}$ you could have started with a and b as well but just making connection more clear with the harmonic oscillator if b becomes a it becomes a real scale field so† structure I have put in by hand there is no necessity that it has to be† over here and not the b itself but if you make this choice then your side † becomes † here and no† here, ,the course that this u_s^p remember this u_s^p is not a numbers now it is a column vector one of the column vectors this or that depending upon spin half this is this column vector or spin – half this is this column vector so when i take the† it will become a row vector so therefore it is not a star but $a^{\dagger}$ so therefore I will get $a,\ u_s^{p\dagger}$ and this thing was b$,\ v_s^p$ which becomes $a^{\dagger}$ as well so therefore I have a more robust or more mathematically rich structure for the Dirac field because in addition to the oscillator operators, I have mode functions which are also vectors, row vectors or column vector depending upon whether you are talking about ψ or ψ divergence.

Okay, so now we know that ψ and $\psi^{\dagger}$ which we have obtained are not just independent quantities anymore in the phase space but $i\psi^{\dagger}$ is supposed to be the momenta conjugate to ψ. So ψ conjugate momenta is pi which should be $i\psi^{\dagger}$ and I know that the $\psi(x)$ and $\phi(x')$ should satisfy $i\delta(x - x')$. So therefore, ,There should be a non-trivial commutation relation between ψ and $\psi^{\dagger}$. So, ψ and $\psi^{\dagger}$, so here should be $a^{\dagger}$, should therefore be δ functions up to identity 4 cross 4. Why this identity 4 cross 4?

Because all the times we are going to talk about this fields not as a collection of a single object operator, but a field coming with a structure commensurate with the 4 cross 1 vectors which are appearing in the mode functions as well. So therefore, this is the structure which we are going to look for. Okay, so this will be ensured. So if I write my ψ like this and if I write my $\psi^{\dagger}$ like this and I demand ψ, $\psi^{\dagger}$ should be δ function and use the fact that $d^3pe^{ip(\dot{x}-x')}$ should be a δ function, should be a 2 $\phi^3\delta^3(x - x')$. So, you see when I write ψ and $\psi^{\dagger}$ side by side, I will get two integrals, one from d^3p from the ψ and another d^3p' integral let us say from $\psi^{\dagger}$. And collection of these two d^3p should at the end of the day ,First it should get converted into a single integral over d^3p and then after the end of the day I should get a δ function out of it. You can see ,maintaining what is the up and what is the up† what should you should get and so on so forth you will see that it would be realized that this commutation relations between ψ and $\psi^{\dagger}$ will be realized only when a and $a^{\dagger}$ satisfy the usual commutation relation that for different spin S and $S^{\dagger}$ they should not be talking to each other spin numbers one has its own oscillator structure, spin numbers two has its own oscillator structure. So there are like two different oscillators. So one oscillator's a operator does not talk to another operator's $A^{\dagger}$, it talks to only itself. So first spin should be the same and then the δ function should come about. If the spins are different, that means the oscillators are different. So they should not be commuting, they should not be non-commuting, they should be commuting rather. So, spin δ function ensures that we are talking about the same. And similarly for space time dependent P also, they should be the same. Otherwise, we are not talking about the same oscillator and therefore the operators would commute. Similarly, different $a's$ *s*hould also commute depending whether or not P and P' are same or not and S and S' are same or not. Two different A's are supposed to always commute, two different a diagrams are always supposed to commute and the same story goes for the second set of oscillators of b.So, B's are the oscillators with let us say negative frequency, but anyway that is all right ,and they have also the same instruction in order to get the good commutation relations and using the fact that the good commutation relation will come only from the converting the double integral over p into a single integral over p with this function in the hand. I should impose this kind of commutator structure on $\hat{a}$ and $\hat{b}$, which is usual commutator relation between oscillators. So, this is nothing mysterious. Only thing is that we started with an oscillator kind of equation. It has more than one component. Therefore, we are getting more than one operator set. But all of the sets are talking about oscillators, since ψ talks about not one operator, but really eight operators. Operators in the sense of four complex entries. So therefore this set of operators are also different and you will have this $\hat{a}'s$ for plus half – half, $a^{\dagger}$ plus half – half, $\hat{b}_s$ plus half – half and $\hat{b}_s^{\dagger}$ plus half – half. So this is the structure which emerges out. This is mathematically slightly more rich but ,nothing conceptually very different apart from the fact that we are not talking about a real scalar field but a complex oscillator kind of thing further you need to impose when you do this commutator kind of a structure you will see that this u_s^p and this $u_s^{p\dagger}$ will hit each other so these are vectors co-vectors hitting each other or column vector row vector hitting each other. And therefore, we should know what happens when a column vector hits a row vector. So, there should be, if I look for just the structure we had, this was the u_s set. The u_s set was here and the v_s set was over here. They both come with their individual normalization thing. I do not know how to normalize this vector here. I have not told how to normalize this vector here. So, some normalization function should be there. It so happens for it should be independent of spin.

Further $\quad u_s^\dagger(\vec{p})\cdot u_{s'}(\vec{p}) = \dfrac{2\delta_{ss'}E}{(E+mc^2)}|N|^2$

$v_s^\dagger(\vec{p})\cdot v_{s'}(\vec{p}) = \dfrac{-2\delta_{ss'}E\,|N'|^2}{(mc^2-E)}$

We can normalize these

$$\bar{u}_s(\vec{p})\cdot u_{s'}(\vec{p}) = \frac{2\delta_{ss'}mc^2}{(E+mc^2)}|N|^2$$

$$v_s^\dagger(\vec{p})\cdot u_{s'}(-\vec{p}) = 0$$

Further, $\quad (-i\gamma^i\partial_i+m)\psi = \int\dfrac{d^3p}{(2\pi)^3}\dfrac{1}{\sqrt{2E_p}}\sum_s\left(\hat{a}_p^s(-\gamma^i p_i+m)u_p^s e^{ip\cdot x} + \hat{b}_p^{s\dagger}(\gamma^i p_i+m)v_p^s e^{-ip\cdot x}\right]$

Using Dirac equation

$$i\gamma^0\partial_0\psi + i\gamma^i\partial_i\psi - m\psi = 0$$

$$-i\gamma^i\partial_i\psi + m\psi = i\gamma^0\partial_0\psi$$

$$\therefore\ (-i\gamma^i\partial_i+m)\psi = \int\frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_p}}\sum_s\left(\hat{a}_p^s\, i\gamma^0\partial_0\, u_p^s e^{ip\cdot x} + \hat{b}_p^{s\dagger}\, i\gamma^0\partial_0\, v_p^s e^{-ip\cdot x}\right)$$

$$= \int\frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}\sum_s\left(\hat{a}_p^s\gamma^0 E_p u_p^s e^{ip\cdot x} - \hat{b}_p^{s\dagger}\gamma E_p v_p^s e^{-ip\cdot x}\right)$$

Further

$$u_s^\dagger(\vec{p})u_{s'}(\vec{p})=\frac{2\delta_{ss}'E}{(E+mc^2)}|N|^2$$

$$v_s^\dagger(\vec{p})v_s'(\vec{p})=\frac{-2\delta_{ss}'E}{(E-mc^2)}|N|^2$$

$$\bar{u}_s(\vec{p})u_{s'}(\vec{p})=\frac{2\delta_{ss}'mc^2}{(E+mc^2)}|N|^2$$

$$v_s^\dagger(\vec{p})u_{s'}(-\vec{p})=0$$

Further,

$$(-i\gamma\partial_i+m)\psi=\int\frac{d^p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}\sum_s\left[\hat{a}_p^s(-\gamma^i p_i+m)u_p^s e^{ipx}+\hat{b}_p^s(-\gamma^i p_i+m)b_p^s e^{-ipx}\right]$$

Using Dirac equation

$$i\gamma^0\partial_0\psi+i\gamma\partial_i\psi-m\psi=0$$

$$-i\gamma^i\partial_i\psi+m\psi=i\gamma\partial_i\psi$$

$$\therefore (i\gamma^i \partial_0 + m)\psi = \int \frac{d^p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_p}} \sum_s \left(\hat{a}_p^s i\gamma^0 \partial_0 u_p^s e^{ipx} + \hat{b}_p^{s\dagger} i\gamma^0 \partial_0 v_p^s e^{-ipx} \right)$$

$$= \int \frac{d^3 p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \sum_s \left(\hat{a}_p^s \gamma^0 E_p u_p^s e^{ip\dot{x}} - \hat{b}_p^s \gamma^0 E_p v_p^s e^{-ip\dot{x}} \right)$$

This is easy to verify that the spin half and spin – half do not come with different normalization and similarly for the negative frequency sector as well. But there is no a priori region that for negative frequency and positive frequency as well you should have the same normalization. So, I am maintaining one normalization in the positive frequency sector and another normalization in negative frequency sector. And in order to obtain this set of relation there should be a good normalization which should be used and that good normalization is also useful for getting a more clear story later on as well so this is an exercise for you what kind of n one should choose in order to get This in along with this commutation relation getting this commutator between ψ and $\psi^\dagger$ what should be the n this should be fairly easy exercise impose this in your commutators and find out what are the n's and n's okay that would also become a clear in a minute when we do another computation with us and vs so now. ,We have this relations in hand. We have commutator structure in hand. So let us go ahead and operate things on this. So first I will want to obtain this operator which is all the spatial derivatives coming with a γ factor acting on ψ. Why do I need this operator? For that let us go back to the Hamiltonian which we had obtained for the Dirac field. Remember in the Dirac fields Hamiltonian derivation the temporal derivative got exactly cancelled out and i we were left with only the spatial derivative with appropriate γ_i is multiplied to them so you see Hamiltonian is made up from there is a γ_0 terms in the first term and the second term both so I can pull that out so i would have a $\psi^\dagger$ and γ_0 and I should have a – $i\gamma_i\partial_i$ and then m times ic is equal to one unit I am using so this operator's action on ψ should be known in order to get the Hamiltonian so that is what I am going to do I should be do taking this operator hitting it on the ψ and try to see what do i get so when i take this and hit on the ψ there are γ part which is matrix and the derivatives part which are action on space time. ,Space only, this is not time, this is spatial derivative. So spatial derivatives are only in the e^{ipx} part and the vector part of the matrix is only in the us. This is a column vector. So this operator and these operators together, they do not see anything else. They go right across the integration up to us. One part is behaving like vector to be hit upon by γ_i's and one part is behaving like a function to be hit upon by δx, δx_i. Similarly for the second part as well the operator will go inside and hit that as well okay so far so good but there is another uh another easier way a convenient way of obtaining this action of this operator you can do this exercise also you can find out what happens after the end of the day but a easier way is this use the Dirac equation I know the Dirac equation is $i\gamma^\mu\partial_\mu\psi$ not $\gamma_i\partial_i$ which is required over here. So, I will write it like this. So, therefore I know the part which I am interested about ,is negative of – $i\gamma_0\partial_0\psi$ so that means the part I am interested over here is just this $i\gamma_0\partial_0\psi$ so all this action of all this operator is equivalent to action of $i\gamma_0$ times the temporal derivative so why not do that why to take this operator three derivatives and three matrix multiplation better one derivative and one matrix multiplation. So $\delta(0)$ will hit only the exponential part because the temporal dependence is only there. And similarly in the second part also the derivative will hit only the exponential part. This was a positive frequency thing. So it will give me an E_p out. And this was a negative frequency thing. So it will give me a – E_p out. So there will be a sign. – sign will come between the two terms. Previously, there were no sign. They were added with a plus. After the derivative action or equivalently after the action of this whole operator, I will get a negative sign in between. And a γ_0, which I am still to act. But wait, this is the structure. But I want to know the Hamiltonian. Hamiltonian involved, apart from the operator which we have obtained, ,Another γ_0 should multiply the whole operator and then a $\psi^\dagger$ should be multiplied and d^3x integration should be done. So that is what I am aiming to do.

I have obtained the action of this operator part which is this. I will multiply a γ_0 to the whole thing. I will multiply a $\psi^\dagger$ to the whole thing and I will do the d^3x integration. So this is what I am supposed to do and that will give rise to the Hamiltonian. So this operator ,which gives rise to a negative sign difference between these two times the d^3p integrals which is this. And the $\psi^\dagger$ which is this and this d^3p integral and γ_0 when I multiply a γ_0 you see in this term both the terms we are having a γ_0 here and a γ_0 here. When I multiply a γ_0 it becomes a γ_0 square which is identity. So I get rid of γ_0 in the Hamiltonian immediately.

$$\mathcal{H} = \int d^3x \, \psi^\dagger \gamma^0 (-i\gamma^i \partial_i + m)\psi$$

$$= \int d^3x \int \frac{d^3p}{\sqrt{2E_p}(2\pi)^3} \int \frac{d^3p'}{\sqrt{2E_{p'}}(2\pi)^3} \sum_{s s'} \left(\hat{a}^{s'\dagger}_{p'} u^{s'\dagger}_{p'} e^{-ip'\cdot x} + \hat{b}^{s'}_{p'} v^{s'\dagger}_{p'} e^{+ip'\cdot x} \right) \left(\hat{a}^s_p E_p u^s_p e^{ip\cdot x} - \hat{b}^{s\dagger}_p E_p v^s_p e^{-ip\cdot x} \right)$$

Using $\int d^3\vec{x} \, e^{i(\vec{p} \pm \vec{p}')\cdot\vec{x}} = (2\pi)^3 \delta^3(\vec{p} \pm \vec{p}')$

and the normalization factors we obtain

$$H = \int \frac{d^3p}{(2\pi)^3} \sum_s (\omega_p) \left(\hat{a}^{s\dagger}_p \hat{a}^s_p - \hat{b}^{s\dagger}_p \hat{b}^s_p \right)$$

But since $\hat{a}$ and $\hat{b}$ are different unrelated operators

$\hat{a}^s_p |0\rangle = 0 = \hat{b}^s_p |0\rangle \quad \forall \; p, s$

The eignestates are $|n^{(a)}_s, n^{(b)}_s\rangle$ and if $n_b > n_a$ we have -ve energy states!

$\Rightarrow$ Unbounded from below!!

$$H=\int d^3x\,\psi^\dagger\gamma^0(-i\gamma^i\partial_i+m)\psi$$

$$=\int d^3x\int\frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}\int\frac{d^3p'}{(2\pi)^3}\frac{1}{\sqrt{2E_p'}}\sum_s\left((\hat{a}_p^{s'\dagger}{}'u_p^{s'\dagger}{}'e^{-ip'\dot{x}}+\hat{b}_p^s{}'v_p^se^{-ip\dot{x}})(\hat{a}_p^sE_pu_p^se^{ip\dot{x}}-\hat{b}_p^sE_pv_p^se^{-ip\dot{x}})\right)$$

Using $\int d^3\vec{x}\,e^{i(\vec{p}\pm\vec{p}\,')\vec{x}}=(2\pi)^3\delta^3(\vec{p}\pm\vec{p}\,')$ and the normalization factors we obtain

$$H=\int\frac{d^3p}{(2\pi)^3}\sum_s(\omega_p)(a_{\vec{p}}^{\vec{s}\dagger}a_{\vec{p}}^{\vec{s}}-b_{\vec{p}}^{\vec{s}\dagger}b_{\vec{p}}^{\vec{s}})$$

But since $\hat{a}$ and $\hat{b}$ are different unrelated operators

$$\hat{a}_p^s|0\rangle=0=\hat{b}_p^s|0\rangle\,\forall\,p,s$$

The eigenstates are $|n_s^{(a)},n_b^{(s)}\rangle$ and if $n_b>n_a$ we have -ve energy states!
⇒Unbounded from below!!

All I am left to do is just multiply the $\psi^\dagger$ and take out the γ node. So, the $\psi^\dagger$ multiplation will give me this and a 1 d^3p integrals and the action of this whole operator was this term with a sign difference with this d^3p another integral and at the end of the day I have to do the d^3x integral to obtain the full Hamiltonian. Now you see the d^3x integral will just see for the spatial dependency. Spatial dependency is coming only in the exponentials. Okay. And then you can use the fact you can have a ,special integrals of the exponential give rise to $\delta^+(p+p')$ or $\delta(p-p)'$ depending upon whether you combine this with this you combine this with that or this the second term with the first of the next term and the second term of the second of the next term so you will get various combinations of exponential and all of them doing this special integral will give rise to one of the δ functions use that put it in the δ function and do one of the momentum integral. So δ function will get rid of one of the momentum integral and then you will have just simple combinations of $\hat{a}_p$ multiplied to this, $\hat{a}_p$ multiplied to that, $\hat{b}_p$, multiplied to this and $\hat{b}_p$ multiplied to that. You will see the cross term will vanish under this.
There will be no term which this $\hat{a}$ and $\hat{b}$, multiplied versus this and this , multiplied. Once you make use of this δ function set integration all this cross term will get cancelled all it will survive will be this term surviving with this and this term surviving with that. So as a simple exercise, you will get this structure, $\hat{a}_s^\dagger\hat{a}_s$ and $\hat{b}_s^\dagger\hat{b}_s$. S takes plus half – half value and P is the integral you will be left with. One of the integration of the momentum has been taken care by δ function, but you will be left with one of the integrations at the hand. Now let us stare at the Hamiltonian, what we have. The Hamiltonian what we have is almost like the harmonic oscillator Hamiltonian, which this believed as a numbers operator and this is the frequency they come up with. However, this time we have a collection of two set of oscillator, positive frequency and negative frequency. This is set of positive frequency, $\hat{a}$'s are the positive frequency operators and $\hat{b}$'s are the negative frequency operators. So, they do come with positive frequency and negative frequency and ultimately what is happening is that we are counting all the positive frequency oscillators and all the negative frequency oscillator and combining them with their frequency weightage to get the Hamiltonian. Compare with the structure which we had for scalar field. We had just $\hat{a}_p^\dagger\hat{a}$ and a plus half of ,$\delta(0)$. That $\delta(0)$ has gone away and this is being replaced by the negative frequency oscillators. But they are oscillators after all because thus the spinal field has more set of fields inside this is not one field but a collection of various fields so therefore the Hamiltonian you are going to get is also collection of various oscillators it so happens that some of the

oscillators come with negative frequency because we did not demand the hermeticity or anything for the field the cost we have to pay is that you have to add up the negative frequency.
Remember, why did we not demand the hermeticity?
Because we saw that they have to be complex numbers, they have to be operator in 4 cross 1 space and so on. All these things when we built our first order equations of motion story, there was inbuilt into that this can be realized only with the complex fields. It cannot be realized with the scalar fields which are the real quantities. So, in order to have a first order equation, the cost you have to pay is to come up in the Hamiltonian with a negative frequency part. But they are oscillators and they do not talk to each other. Commutators between a and b are zeros. So this is one set of oscillators and these are second set of oscillators. And we can assume that there is a ground state again or the vacuum state let us say which is annihilated by both this ground state is annihilated so this is like two oscillators joint ground state will be oscillator first ground state tensor oscillator two's ground state and therefore this combined ground state will be annihilated by either of the operators the same thing is happening here the true ground state which is ground state of all p's. So, remember in the scalar field part the ground state was supposed to be ground state of all momentum vacuum. This time it has not only the momentum vacuum, this has spin vacuum as well as negative frequency vacuums. The full vacuum is vacuum for all oscillator of a positive frequency side times vacuum of all ,negative frequency oscillators as well. So therefore, the full vacuum will be annihilated by both A's and B's. But the story is problematic even after that. Though I have a vacuum state, now I can raise as many A's positive frequency things and as many $\hat{b}$'s negative frequency thing independently. There is no reason that , $\hat{a}$ and $\hat{b}$ should be correlated I can think of a different numbers operator in the positive frequency sector and different operator number excitation in the negative sector this is like two oscillators ground state I can have either zero or one or I can have a one or zero I can have a two here or a zero here ,just like different p's oscillator structure did not command what other oscillators excitation should be similarly positive frequency sector does not command what has to be excitation in the negative energy sector how many particles I could should keep in oscillators of one kind does not determine how many oscillators how many particles of those oscillators I should keep in the negative frequency side so therefore these and these are independent excitations. And therefore, there is a problematic possibility that I can excite the negative frequency particles more than the positive frequency particles and I will lead to a Hamiltonian which is negative. And if I keep on increasing the negative frequency oscillator sector more and more I will get more and more negative Hamiltonian and therefore negative energy states this Hamiltonian becomes unbounded from below because in principle I can excite infinitely many particles in the negative energy sector and no particle in the positive energy sector positive frequency sector or finite particles in positive frequency sector and yet will be Hamiltonian will become – infinity. So at the end of the day doing all the field theory everything I am back to square one where my Hamiltonian has become unbounded from below. So the all clever tricks which we played to convert things into operators in the hope of getting rid of negative frequency solutions does not pay off. It did quantization for scalar field wonderfully. But for spinor field, we are led to the same falsehood, same unphysicality which we were dealing with in the case of quantum mechanics. So quantum field theory has not helped us as far as spinor fields, spinor particles are concerned. How to handle this fact, whether these spinor things really exist or not? Or there is something in which we can do in the quantization to save this story will be discussed in the next class. Or there is something in which we can do in the quantization to save this story will be discussed in the next class. The oscillator quantization does not work for these particles. These are some new particles. They are not behaving the same way scalar particles would have behaved.
So, I stop here.