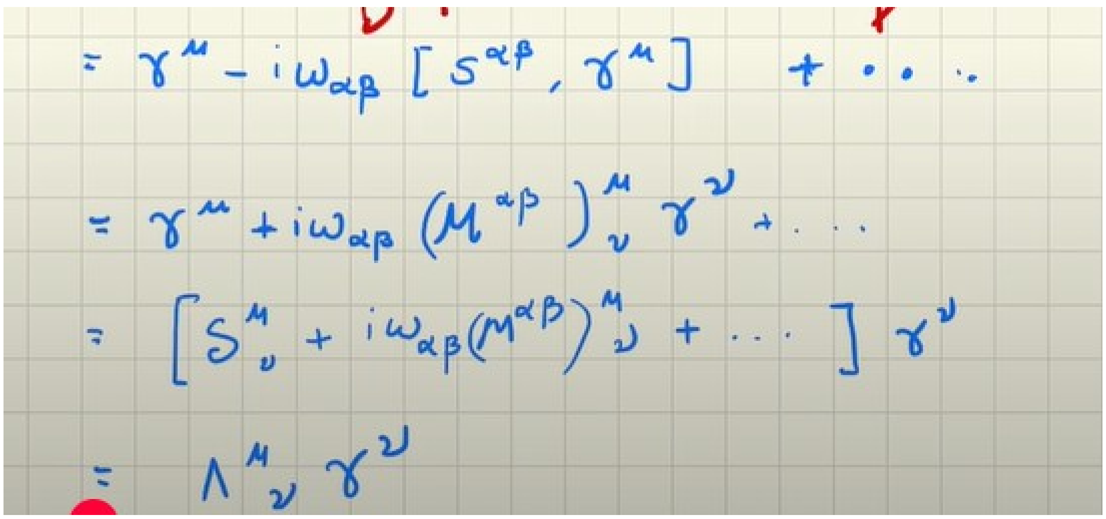
Foundation of Quantum Theory: Relativistic Approach
Spinor Field quantization 2
Prof . Kinjalk Lochan
Department of Physical Sciences
IISER Mohali
Dirac Field Equation
Lecture- 23

So in the previous class we had seen that there was a necessity of changing from the Lagrangian form $\psi^\dagger\psi$ to $\bar{\psi}\psi$ because this object is not invariant under noise transformation while this object is this happened precisely because the transformation generated by matrices *D(λ)* for low range transformation are not unitary.



Defining $\bar{\psi} = \psi^\dagger \gamma^0$

$\bar{\psi}\psi \to \psi^\dagger D'(\Lambda)\gamma^0 D(\Lambda)\psi$

$= \psi^\dagger D'(\Lambda)\gamma^0\gamma^0 D(\Lambda)\psi$

$= \psi^\dagger \gamma^{0\gamma} D(\Lambda)\underbrace{\gamma^0\gamma^0}_{0}\psi$

$= \psi^\dagger\gamma^0\psi = \bar{\psi}\psi$

=

$\bar{\psi} r^\mu \partial_\mu \psi = \psi^\dagger \gamma^0 D(\Lambda)^\dagger \gamma^\mu (\Lambda^{-1})^\nu_\mu \partial_\nu (D(\Lambda)\psi)$

$= \psi^\dagger\gamma^0 D(\Lambda)^{-1}\gamma^0\gamma^\mu(\Lambda^{-1})^\nu_\mu D(\Lambda)\partial_\nu\psi$

$$D(\Lambda)^{-1} r^\mu D(\Lambda) = \Lambda^\mu_\nu \gamma^\nu$$

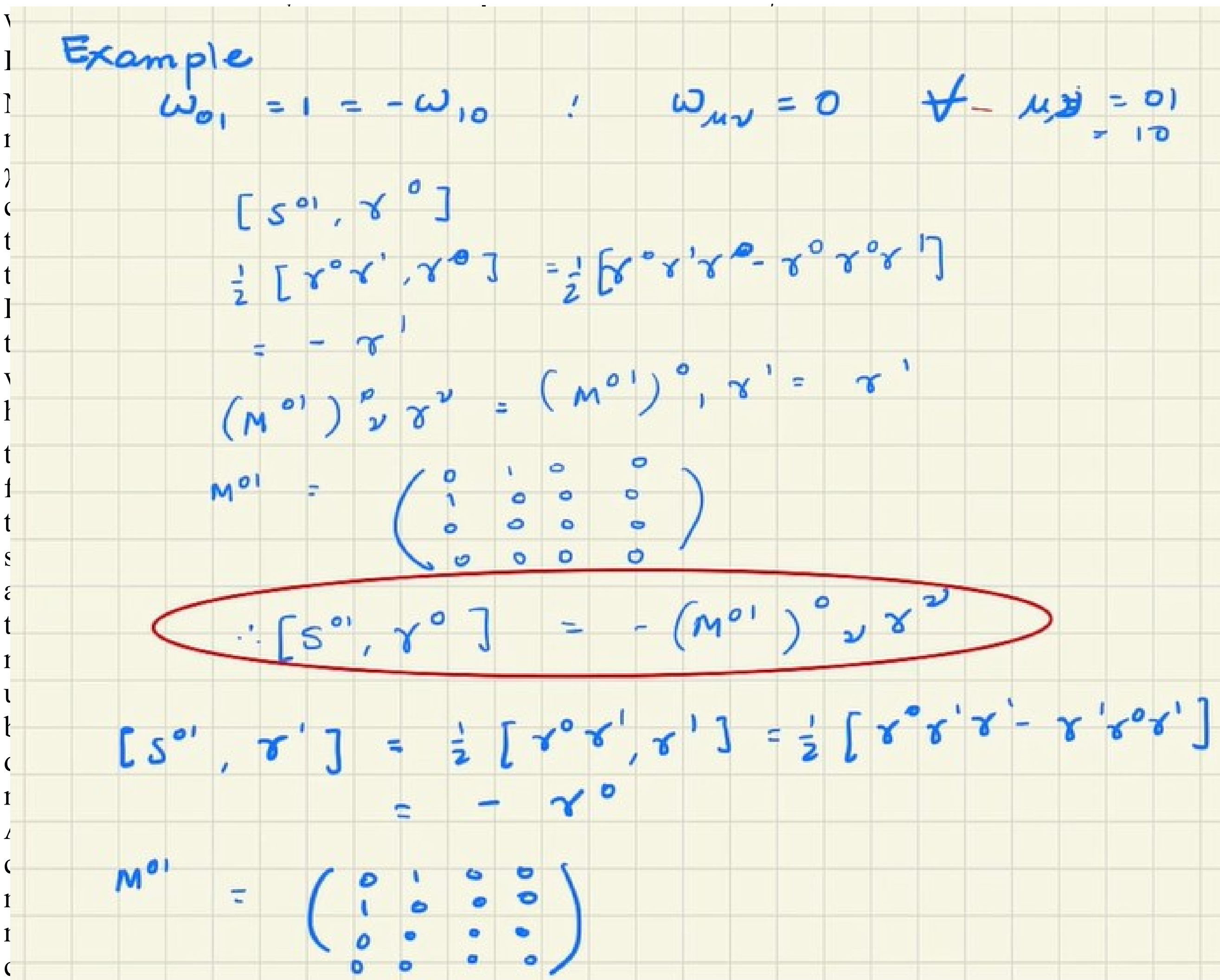
For the vector representation

$\Lambda = ,\quad \Lambda = \exp(i\omega M^{\mu\nu})$

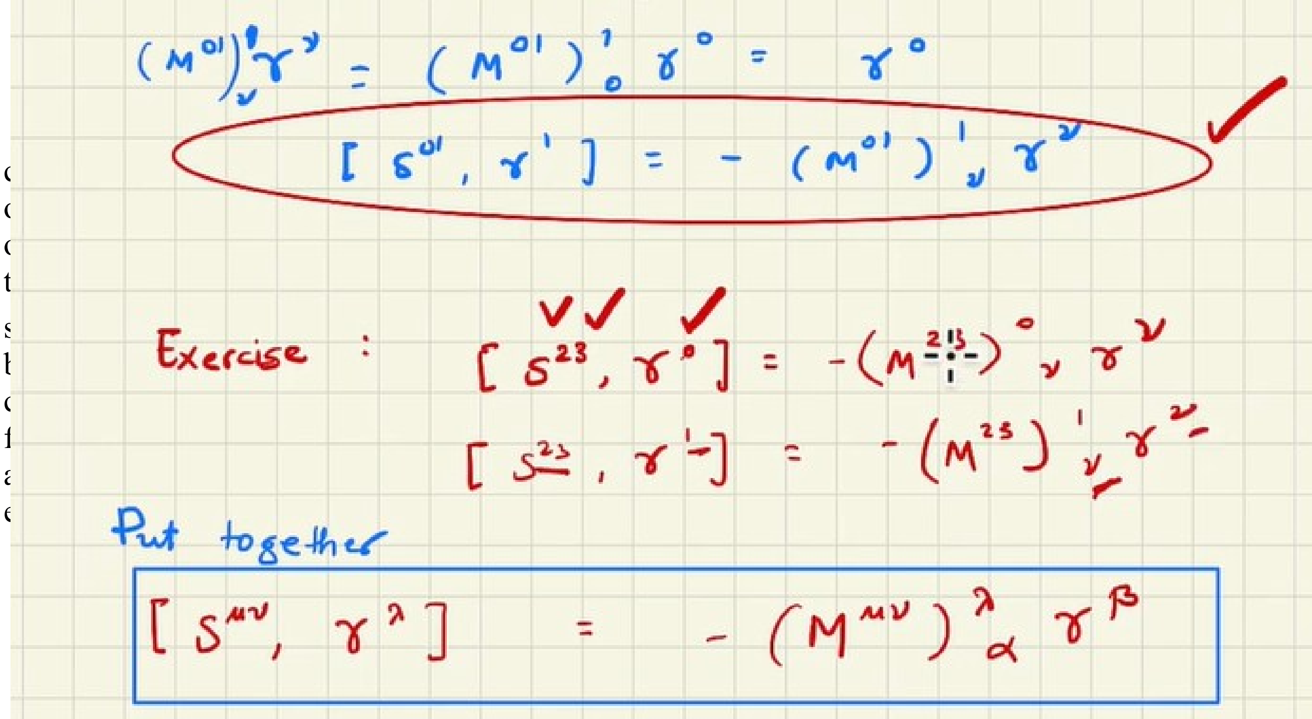
$$M^{\mu\nu}=\left[\begin{pmatrix}0&1&0&0\\1&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}\begin{pmatrix}0&0&1&0\\0&0&0&0\\1&0&0&0\\0&0&0&0\end{pmatrix}\begin{pmatrix}0&0&0&1\\0&0&0&0\\0&0&0&0\\1&0&0&0\end{pmatrix};\begin{pmatrix}0&0&0&0\\0&0&1&0\\0&-1&0&0\\0&0&0&0\end{pmatrix}\begin{pmatrix}0&0&0&0\\0&0&1&0\\0&-1&0&0\\0&0&0&0\end{pmatrix}\begin{pmatrix}0&0&0&0\\0&0&0&1\\0&0&0&0\\0&-1&0&0\end{pmatrix}\right]$$

$$(1-\omega_{\alpha\beta}S^{\alpha\,\beta}=\ldots)\gamma^{\mu}(1+\omega_{\rho б}S^{\rho б}+\ldots)$$

That means $D^{\dagger}D(\lambda)$ are not unitary, are not identity. So therefore, I cannot use $\bar{\psi}\,\psi$ or $\psi^{\dagger}\psi$ as a invariant object in the Lagrangian. Remember the Lagrangian which we proposed initially was something like $\psi^{\dagger}i\hbar\gamma^{\mu}\partial_{\mu}\psi - mc\psi$. So, this kind of Lagrangian we thought would do our job of giving rise to the Dirac equation, which it does. Variation with respect to $\psi^{\dagger}$ indeed gives rise to the Dirac equation, but unfortunately the piece $\psi^{\dagger}\psi$ which is appearing with a mass over here is not Lorentz invariant. So we realized that we should not be using $\psi^{\dagger}\psi$ but we should use $\bar{\psi}\,\psi$ rather. So we realized that we should not be using $\psi^{\dagger}\psi$ but we should use $\bar{\psi}\,\psi$ rather. So if that is the proposition we should check whether this kinetic term is invariant now under low-inch transformation or not. And indeed we will see that it is invariant under low-inch transformation if ,the thing which is appearing inside so we try to see how does it transform inside there are things which appeared like that so you see d† $\gamma_0\,\gamma^{\mu}\,\gamma$ inverse of$_{\mu\mu}$ so these all these things will appear if I do try to do low inch transformation ψ will transform with a $d\partial_{\mu}$



ultimately is proportional to γ matrices back and the matrix which is multiplying which is now a generator of Lorentz transformation for vectors. S were the generators of Lorentz transformation for Spinors. M is the generators of Lorentz transformation for vectors. These are the M's. The six M's. Three rotations and three boosts. You see three boosts here and three rotations are in the red. So you can see this is the claim I am making. The commutator between S and γ gives me a result proportional to γ and M. M is the generator of the vector Lorentz transformations. If this happens then you can see that again the whole expansion adopts a form of a exponential. ,These are the different terms appearing in the expansion of the exponential and ultimately it will become exponential ,$i\ \omega\ \alpha\beta$ this time $M_{\alpha\beta}$ will appear because of the commutators being converted into m you see in the expansion commutator



Example

$\omega_{01} = 1 = -\omega_{10}$; For all $\mu\nu = 01$

$\omega_{\mu\nu} = 0$ $= 10$

$$[S^{01},\gamma^0]=\frac{1}{2}[\gamma^0\gamma^{1,}\gamma^0]=\frac{1}{2}[\gamma^0\gamma^1,\gamma^0-\gamma^0\gamma^0,\gamma^1]=-\gamma^1$$

$$M^{01}=\begin{pmatrix}0&1&0&0\\1&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}$$

∴

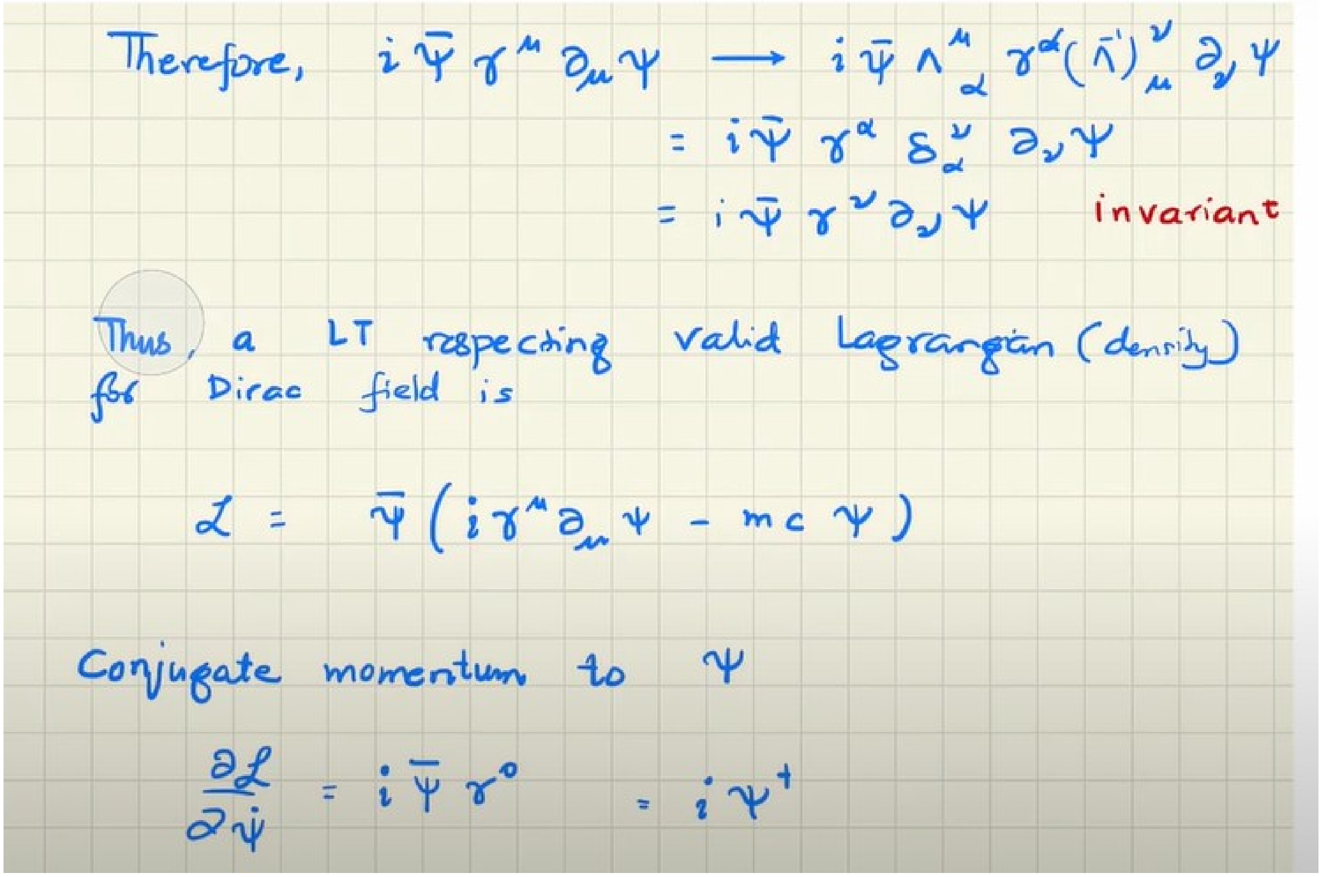
$$[S^{01},\gamma^1]=\frac{1}{2}[\gamma^0\gamma^{1,}\gamma^1]=\frac{1}{2}[\gamma^0\gamma^1,\gamma^0-\gamma^0\gamma^0,\gamma^1]=-\gamma^0$$

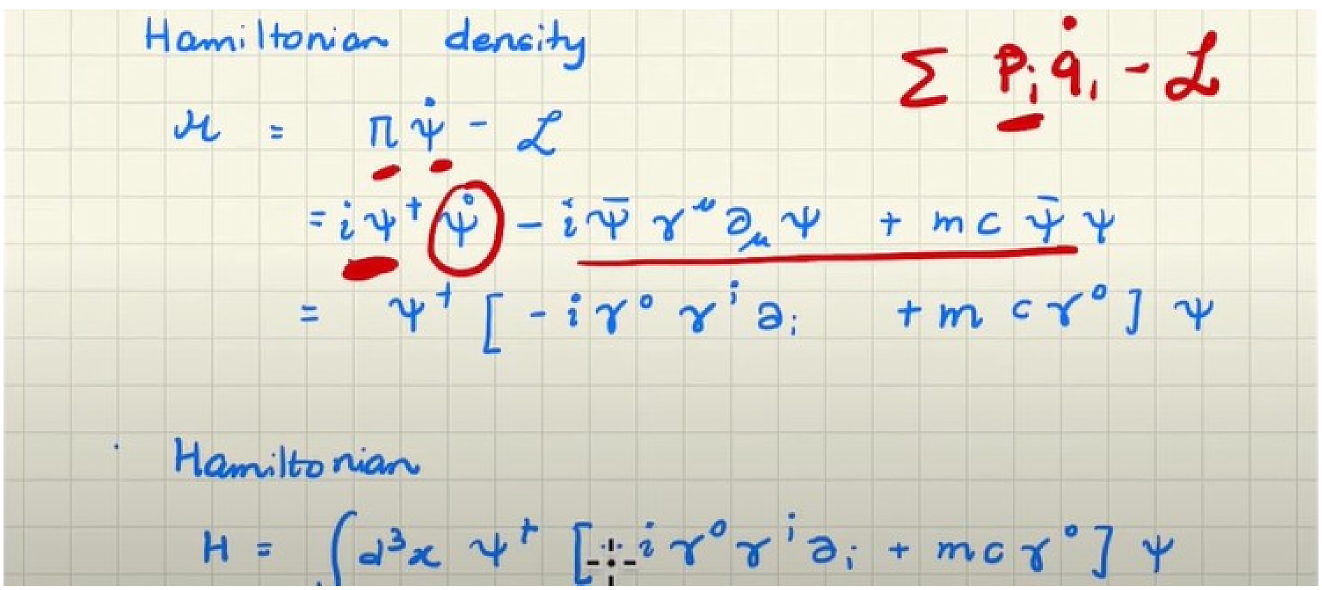
$$M^{01}=\begin{pmatrix}0&1&0&0\\1&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}$$

So, this will become true, therefore this will become true, this will also become true and hence the piece which is the kinetic term will become invariant. So, that is what we are going to see with the help of certain examples. So let us work it out and see whether we get the claimed identity or not. So the thing which we are trying to chase is whether $S^{\alpha\beta}\gamma^{\mu}$ commutator indeed becomes multiplation of $M_{\alpha\beta}$ and γ^{μ}. Where the row newth row this decides the row and the column this is newth column rather and new is being summed over along the row you sum new 0 0 1 1 2 2 3 3 and so on. So, let us try to see whether this emerges out or not. So, I am going to think of a low range transformation which is ,perfectly boost like. So, I am going to do a Lorentz transformation which is just a boost along the x-axis. That means all other parameters are 0, only boost along x-axis parameter is non-zero. I am going to choose that parameter ω_{01} which is 1 and being anti-symmetric it is also equal to – of ω 1, 0 and all other parameters are 0. Boost along y-axis is not being done, boost along z-axis is not being done, rotation along any of the x, y, z-axis is not being done. So, $\omega_{\mu\nu}$ is equal to 0 for all $_{\mu\nu}$ which are not equal to 0 1. Only ω_{01} and ω_{01} are non-zero all other $\omega_{\mu\nu}$ are 0. So, therefore, the B matrix which transforms the Spinors which was being written as exponential of $i\omega_{\mu\nu}$ $S^{\mu\nu.}$ This will become equal to exponential of $i\omega_{01}$ which is 1 and S^{01} that is all. Now, if that is the case ultimately the claim identity which we are going to look at should have a S^{01} γ^{μ} should be equal to $M^{01,\mu,\nu}$ and γ^{ν} . This is what should emerge out under this Lorentz transformation. So that means I have to choose four$_{\mu}$'s differently. I can choose μ is equal to 0, 1, 2, 3 and correspondingly right hand side will have 0, 1 or 2 or 3. So let us do one example for$_{\mu}$ is equal to 0. That means I want to evaluate what is the commutator between S and . Result says, looking at the result, it looks like the commutator should emerge out to be M01, 0 new. This should be the claimed result. Let us verify whether we do get this result back or not. So, let us write down the S01 matrix. S01 matrix is half of γ_0 γ_1 because that is the generator. γ_0 is just appearing over here and then let us do this multiplation, opening up the commutator. If I open up the commutator, I will get two terms γ_0, γ_1, γ_0, which is the first term and γ_0, γ_0, γ_1, which is the second term. Now two γ_0s are appearing over here, which will make γ_0 square and we all know γ_0 square is identity. So therefore, the second term will just become – of γ_1, – of γ_1. Look at the first term which is γ_0, γ_1, γ_0, γ_0, γ_1, γ_0. And we also know that γ_1, γ_0 over here is negative of – of γ_0, γ_1. So this is equal to – of γ_0, ,γ_0 γ_1 and again γ_0 γ_0 is identity. So, this first term is also – of γ_1. So, half of – of twice of γ_1 is equivalent to – of γ_1. This is the left hand side. Let us check what do we get on the right hand side.

Right hand side if this identity is true I should get , So, sorry I am supposed to get a – over here. Look at this previous result. Here when I wrote down the commutator, it became m up to a negative sign change. So, the claim result is that commutator is negative of *M*. So, that is what we are trying to prove. Now, let us find out this object over here and see whether it is negative of – γ_1 or not. This is the row which is 0, this is the column which is row 0 and new gets summed over. The first entry, second entry, third entry and fourth entry. So M^{01} along this row has only one non-zero element which is M^{01} .So when I sum over new, new will take value 00112233. Out of these four entries only M^{01} is surviving, M^{01} 01 is surviving because M^{01} matrix is this. So, in this summation only one term is going to survive with value 1 and this will become γ_1. So, indeed it is negative of the commutators. So, S01 γ_0 is indeed negative of $-M^{01}\ 0_\nu\ \gamma^\nu$. So, at least we have verified for this pair 0, 1 and μ is equal to 0, it works out. You can do another example, you can take $S^{01}\ \gamma_1$ here, you will again find out that it is indeed coming out to be true that $S^{01}\gamma_1$ is indeed negative of – n01, 1 ν and summation over γ^ν . And actually you can keep doing this exercises for different values of α, β and μ. If you take α is equal to 2, β is equal to 3,μ is equal to 0, you will get M_{23} 0ν and summation over ν. And similarly if you take 231, you will get 231 ν and summation over$_\nu$. ,You can verify for all of the terms you are indeed getting this structure. So the claim which we are going to prove indeed turns out to be verified and we have not proven but verified this that put together we have a general structure that you take any of the ,$S^{\mu\nu}$ and you take any of the γ matrix. You will get the$_{\mu\nu}$ will identify which m we are talking about. The component of the γ will tell me the upper index of that m and the lower index will be summed over. So indeed this structure emerges out which we assumed if that emerges out that means the reverse map which we had talked about will work fine that if the claim identity is true then this commutator over here with a – sign will be replaced by a positive of generator m for vectors and higher commutators square of commutators will be squares of M's and what not ultimately will become exponential of ω times m which is the vector transformation matrix. So, therefore d inverse γ times d gives you ,gives you back Λ times γ which was the claimed result which we are trying to obtain. And once that happens we can now see that the kinetic term is also invariant under Lorentz transformation. So, now we have a good Lagrangian.

So, as we can see kinetic term the D inverse γ^μ and B which was appearing now has been written in terms of ,this Λ the $\Lambda_\mu{}^\alpha\gamma^\alpha$ which is this $\Lambda_M\ \alpha\ \gamma^\alpha$and already there was a $\Lambda^{-1}{}_{\mu\mu}$ which was coming from the transformation of the partial derivative this and this they are just numbers they are not matrices these are component of matrices they can combine and give you a Kronecker $\delta_\mu\alpha$ because they are the matrices are inverses of each other that means components multiply to each other such that the row of one combines with the column of other and so on gives you a identity matrix that means the component becomes the Kronecker δ unless they are the same they will give you 0 if they are the same they will give you 1.





Therefore $\quad i\hbar\bar{\psi}\gamma^{\mu}\partial_{\mu}\psi \rightarrow i\bar{\psi}(\Lambda)^{\mu}_{\alpha}\gamma^{\alpha}(\Lambda)^{\nu}_{\mu}\partial_{\nu}\psi$

$= \quad i\bar{\psi}\gamma^{\alpha}\delta^{\nu}_{\alpha}\partial_{\nu}\psi$

$= \quad i\bar{\psi}\alpha\gamma^{\nu}\partial_{\nu}\psi$ **invariant**

$(i\hbar\gamma^{\mu}\partial_{\mu} - mc)\psi = 0 \quad \ldots (1)$

Thus s LT respecting valid Langrangian (density) for Dirac field is

$$L=\bar{\psi}\left(-i\hbar\gamma^{\mu}\partial_{\mu}\psi-mc\psi\right)$$

s.t. Variation w.r.t. $\psi^{\dagger}$,gives – (1) while variation w.r.t ψ yields

$$-i\hbar\gamma^{\mu}\partial_{\mu}-mc\psi=0$$

Conjugate momentum to $\boldsymbol{\psi}$

$$\frac{\partial L}{\partial\dot{\psi}}=i\bar{\psi}\gamma^{0}=i\psi^{\dagger}$$

Hamiltonian density

$$H=\Pi\dot{\psi}-L$$

$$=\quad i\psi^{\dagger}\psi^{0}-i\bar{\psi}r^{\mu}\partial_{\mu}\psi+mc\bar{\psi}\psi$$

$$=\quad \psi^{\dagger}\psi^{0}-i\bar{\psi}r^{\mu}\partial_{\mu}\psi\psi+mc\bar{\psi}\psi$$

$$=\quad \psi^{\dagger}[-\gamma^{0}\gamma^{I}\partial_{i}+mc\gamma^{0}]\psi$$

Hamiltonian

$$H=\int d^{3}x\,\psi^{\dagger}[i\gamma^{0}\gamma^{i}\partial_{i}+mc\gamma^{0}]\psi$$

This is the inverse transformation matrix which we had written. Once the Kronecker δ appears, it will make the new appearing over here to the value α. So ultimately or you can say that it will make α appearing here taking the value new. So ultimately the end result will be this index here and this index here will become same. That is the Einstein summation coefficient. So these two are being summed over and the left hand side was also the same structure. So it is invariant. This has become new new, but new is being summed over, right? Similarly, here new was written, but new was summed over. So, there are four summation term and there are four summation term, which are exactly the same. So, this new here is a fictitious dummy index, which is summed over. You could have written this after the end of the day, new as well. So, therefore, the kinetic term is also invariant, the mass term which we had written is also invariant now and therefore, the whole Lagrangian which now we can write as $\bar{\psi}$ the kinetic term and the mass term together is an invariant Lagrangian under low-inch transformation. Okay, so once we have a valid Lagrangian at our hand, again we will do the similar exercise which we had done for the scalar field, first we will try to go to the phase space of this ,So, remember in this case we have complex field not real field, ψ is a complex, ψ had first of all it is a column matrix, it has ψ_1, ψ_2, ψ_3, ψ_4, which we had collected in terms of ϕ_0 and ψ_0 if you remember, but anyway. So therefore, $\bar{\psi}$ appearing over here in hides a $\psi^{\dagger}$ which is not equal to ψ. Because it is not a real scalar field, it is a complexed Spinor field. So it has real entries and imaginary entries in the ψ. So either you can treat collection of ψ as a collection of four complex fields $\begin{pmatrix}\psi_1\\ \psi_2\\ \psi_3\\ \psi_4\end{pmatrix}$,that has a four real parts and four imaginary parts or equivalently you can think of the whole thing as a four complex ψ and four complex side $\dagger$ so this is what one can do the one can do the business with that you will say that real part on imaginary part are independent fields or you can say that ψ and side † are in independent fields both of them are equivalent. Now, if I try to do this, we will take this Lagrangian and first I will try to find out the momenta corresponding to the field ψ. ,Remember there are 8 fields ultimately, real part of ψ_1 , real part of ψ_2, real part of ψ_2 , real part of ψ_4 , imaginary part of ψ_1 , imaginary part of ψ_2 , imaginary part of ψ_3 , imaginary part of ψ_4 . Equivalent description are 4 complex field ψ and four complex field $\psi^{\dagger}$ okay so this is how we can do the business so if we can define either the real part as a four collection and imaginary part of four correction or ψ and $\psi^{\dagger}$ as two separate fields so first i take the obtain the momentum corresponding to ψ Momentum corresponding to ψ, that means any component of ψ you take, its momentum corresponding to that would be $\dot{\psi}$. Okay, $\partial\partial\dot{\psi}$, this object. That means it has four components, take any of them. $\dot{\psi}$, it will try to look in the Lagrangian and find out where is ψ_1 hiding. So, this ψ and this $\bar{\psi}$ together will generatenumbers . $\bar{\psi}$ will be $\psi_1^{\dagger}$, $\psi_2{}^{\dagger}$. This will become a star after taking $\dagger$ and row will become a column. A column will become a row rather. So, $\psi_3{}^{\star}$, $\psi_4{}^{\star}$and then there will be a γ_0 which is under $\dagger$ get back to itself. And then this $i\gamma^{\mu}$ derivatives of ψ. Derivatives of $i\gamma_0$ derivatives of ψ will again will be $\dot{\psi}_1$, $\dot{\psi}_2$, $\dot{\psi}_3$, $\dot{\psi}_4$. So, you will see after all the multiplation, there will be term which will be $\psi_1{}^{\star}$going to $\dot{\psi}_1$, $\psi_1{}^{\star}$ with multiplation of these things in between, giving you identity actually, it will not do anything, i can be taken out, γ_0 square is independent. So, therefore, $\psi_2{}^{\star}$ will get multiplied with $\dot{\psi}_2$, $\dot{\psi}_3{}^{*}$ will get multiplied to $\dot{\psi}_3$ and so on so forth. So that means whenever I take $\partial L/\partial\dot{\psi}_1$, I will get $i\psi_1{}^{\star}$. Similarly for $\dot{\psi}_2$, I will get $i\psi_2{}^{\star}$, $\psi\dot{\psi}_3$ I will get $i\psi_3{}^{\star}$ and so on. Collectively I can write the, if I write then down the $\dot{\psi}$. The end result would be $\psi^{\dagger}$ because $\psi^{\dagger}$ is component $\psi_1{}^{\star}$, $\psi_2{}^{\star}$, $\psi_3{}^{\star}$, $\psi_4{}^{\star}$. So put 1 here you will pick this one. Put 2 here you will pick the second one. So all the stars appearing over here can collectively be written as $a^{\dagger}$. So therefore I will have a momenta corresponding to ψ will be obtained as $i\psi^{\dagger}$. So ψ and $\psi^{\dagger}$ were supposed to be independent fields as we discussed. However, they happen to be related in the phase space. In the phase space of the theory, they are conjugate to each other. This is momenta of ψ. And in this way I have written nowhere the derivatives of ψ stars are appearing, only the derivatives of ψ are appearing. $\psi^{\dagger\star}$ is not appearing anywhere. So, if I compute the momenta corresponding to $\partial L/\partial\psi^{\dagger\star}$, I am going to get 0. There is no momenta corresponding to this in the way we have written the Lagrangian. There is another cliched way of writing ,the same thing as a product of the derivative action hitting this term and a total derivative term. So, those things I am not going to discuss in this course because this is not relevant for us. But the way we are writing, the way we are writing the Lagrangian which is consistent with low-range transformation, I am going to get the momenta corresponding to ψ. is $\psi^{\dagger}$ with i. However, the momenta corresponding to $\psi^{\dagger}$ itself is 0, because there is no $\psi^{\dagger\star}$ appearing in the Lagrangian the way we have written. So therefore, the total Hamiltonian density which we should be obtaining in phase space should be summation over all the P momenta Pi's and all the dynamical variables dot – L. In this case only one set of momenta survives which are the ,$\psi^{\dagger}$s times i which are the derivatives of $\dot{\psi}$. $\psi^{\dagger\dagger}$ does not appear, its momenta does not appear, so therefore that pair will not appear in this summation. So, ultimately only $\dot{\psi}$its momenta will be there and the Lagrangian which can be written like that. So, you see the Lagrangian over here has become a nice structure in which ,Only thing of interest you can see that the $\dot{\psi}$appearing from the first term gets exactly cancelled from the first term of the Lagrangian. So, $\dot{\psi}$terms gets wiped out under this computation. Only spatial derivative terms survive. So, this has four terms γ_0, ∂_0, γ_1, ∂_1, γ_2, ∂_2, γ_3, ∂_3 ,out of this γ_0 ∂_0 term exactly cancels this. Because this $\psi\bar{\psi}$ also has a γ_0, so γ_0 γ_0 becomes identity and these two terms exactly cancel each other for$_{\mu}$ is equal to 0. So only thing which survives ,Not only the three things which survive are the three spatial derivatives. The temporal derivative gets exactly cancelled out, spatial derivatives survive with respective γ and the mass term survives with a positive sign. Because Lagrangian was taken out with a – sign, in the Lagrangian already the mass term was coming with a – sign, so ultimately it becomes plus.

So overall the Hamiltonian density is size†. Spatial derivatives with appropriate γ_0 and γ_i multiplation with a – i factor and the mass term with γ_0 factor and then a ψ there. So, you see this is a column matrix, this is a row matrix, inside there are certain matrices. So, ultimately you are going to get a numbers . The Hamiltonian density has to be a number , Lagrangian density has to be a number . The Hamiltonian density has to be a number , Lagrangian density has to be a number . And once I integrate the Hamiltonian density, ,over all space I will get the total Hamiltonian just like we did for scalar field as well. So, starting with the consistent Lagrangian for chlorine transformation, I am able to generate or write down rather not generate, write down the Hamiltonian for a Dirac system. This does not look like very much like a harmonic oscillator thing which we obtained for the scalar field, but we will soon see that indeed it is hiding a harmonic oscillator structure and therefore we can use the harmonic oscillator quantization techniques which we had learnt previously as well. So, in order to see that let us do certain algebraic manipulation and massaging to the equation and then we will see how the oscillator structure emerges out naturally. ,So, look at the again the Dirac equation.

$$\left(i\gamma^{\mu}\partial_{\mu}\psi - m\psi\right) = 0.$$

Hitting it with

$$-i\gamma^{\alpha}\partial_{\alpha} = i\left[\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\partial_0 + \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}\partial_i\right]$$

$$-i\left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\partial_0 + \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix}\partial_i\right]\begin{pmatrix} \phi_0 \\ \chi_0 \end{pmatrix} e^{\frac{-iEt + \vec{p}\cdot\vec{x}}{\hbar}}$$

$$-i\gamma^{\alpha}\partial_{\alpha}\left(i\gamma^{\mu}\partial_{\mu}\psi\right) + m\, i\gamma^{\alpha}\partial_{\alpha}\psi = 0$$

$$+\gamma^{\alpha}\gamma^{\mu}\partial_{\alpha}\partial_{\mu}\psi + m^2\psi = 0$$

$$\frac{1}{2}\{\gamma^{\alpha},\gamma^{\mu}\}\partial_{\alpha}\partial_{\mu}\psi + m^2\psi = 0$$

Using the Clifford algebra $\{\gamma^{\mu},\gamma^{\nu}\} = -2\eta^{\mu\nu}$

$$\Rightarrow -\eta^{\alpha\mu}\partial_{\alpha}\partial_{\mu}\psi + m^2\psi = 0$$

Thus solution of Dirac eqn. also satisfy the Klein Gordon (Harmonic oscillator) eqn.

▷ We can apply the usual (oscillator) quantization

(I γ^ μ ∂_ μ ψ – m ψ) = 0

Hitting it with

$$-\gamma^{\alpha}\partial_{\alpha}=i\left[\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\partial_0\begin{pmatrix}0 & \sigma^{i}\\ -\sigma^{-i} & 0\end{pmatrix}\partial_0\right]$$

$$-i\left[\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\partial_0\begin{pmatrix}0 & \sigma^{i}\\ -\sigma^{-i} & 0\end{pmatrix}\partial_i\right]\begin{pmatrix}\phi_0\\ \chi_0\end{pmatrix}e^{\frac{iEt+\vec{p}\dot{\vec{x}}}{\hbar}}$$

$$-i\gamma^{\alpha}\partial_{\alpha}(i\gamma^{\mu}\partial_{\mu}\psi)+m\,i\gamma^{\alpha}\partial_{\alpha}\psi=0$$

$$+\gamma^{\alpha}\gamma^{\mu}\partial_{\alpha}\partial_{\mu}\psi+m^{2}\psi=0$$

$$\frac{1}{2}\gamma^{\alpha},\gamma^{\mu}\partial_{\alpha}\partial_{\mu}\psi+m^{2}\psi=0$$

Using the Clifford algebra { $\gamma^{\alpha},\gamma^{\mu}$ } = $-2\eta^{\mu\nu}$

$$\Rightarrow\eta^{\alpha\mu}\partial_{\alpha}\partial_{\mu}\psi+m^{2}\psi=0$$

The variation of the Lagrangian with respect to $\psi^{\dagger}$ or $\bar{\psi}$ is going to give me this Dirac equation. Here now on I am putting c is equal to 1 and $\hbar$ is equal to 1 because I have to maintain the c and $\hbar$s at so many places. So this is the natural units in which things are written. You know that whenever I am writing m here, it should have been mc and somewhere $\hbar$ should also be appearing because of the derivative version of the momenta which I had written. But anyway, right now I am just putting them to one just for convenience. Looking at their mass dimension, we will be able to settle where c is appearing, where $\hbar$ and where their combinations are appearing. So this gives us a nice structure to look at. μ is getting summed over in the Dirac equation. Remember in the Hamiltonian, the 0th part has been cancelled. The summation is only over i-th part. But in the equations of motion, all the 4μs are appearing. So, let us look at the equations of motion. What I do, I take the whole equation which is a matrix equation multiplying a 4 cross 1 matrix. So, this γ^{μ} is a 4 cross 1 matrix, this ψ is a, this γ^{μ} is rather 4 cross 4 matrix, this ψ is a 4 cross 1 matrix.This m is hiding a identity in between. So, ultimately it is again 4 cross 4 multiplying of 4 cross 1. So, overall structure of this equation is some matrix which is a combination of all the derivatives and γ^{μ} put together acting on ψ and some m multiplied with identity acting on ψ. So, ultimately I will get a 4 cross 4 dimensional matrix

multiplying a 4 cross 1 dimensional vector and that should be 0. This is the equation telling you. What I do? I multiply the whole equation with another matrix. Which another matrix? I take this – $i\gamma^\alpha \partial_\alpha$. Again α is being summed over. That means I will take $\gamma_0 \partial_0 + \gamma_1 \partial_1 \gamma_2 \partial_2 \gamma_3 \partial_3$. So I take this matrix which is in its expanded form i ,which is coming from here, γ_0 the derivative with respect to time, γ_1 which will be $\sigma_1, \sigma_1, -\sigma_1$ here, $\delta(1)$, γ_2 which will be $\sigma_2, -\sigma_2$ and γ_2 or $\delta(2,)$ γ_3 which will be $\sigma_3, -\sigma_3$ and δ^3. Collectively I am writing $\sigma_i, -\sigma_i$ and δ_i. Again i has to be summed over. This is just a notational version of the full expansion. So, that matrix I take and multiply this matrix equation of motion. So, I take the whole thing the equation of motion multiply it through this matrix. Ultimately this solution which I am going to multiply it through will be some will be matrix multiplation this matrix multiplying this equation this whole thing ,This whole thing equation of motion is some 4 cross 1 dimensional matrix at the end of the day. Because 4 cross 4 multiplying a 4 cross 1 should give me a 4 cross 1 which should be a 0 matrix. But ultimately this left multiplation is some 4 cross 4 multiplying a 4 cross 1 already and to that I am going to multiply this whole object which is this. So this is the mathematical formal structure. So let us see it cleanly what do I get. So I have this equation at hand, I multiply this equation with this object and let us see what are the different terms I am going to get. The first term becomes this. An extra multiplation with $\gamma^\alpha\partial_\alpha$ and the second term also becomes m times multiplation with $-i\gamma^\alpha \partial_\alpha$. So – term which was already appearing in the equation of motion becomes a plus over here and ultimately I have these two terms, which is fine. This *I can* be pulled out, multiply this $-i$ becomes plus 1 and then I have a γ^α and the derivative ,The derivative will go across because γ^μ matrices, none of the γ matrices have any space time dependence. So the derivatives will just go through and I can collect all the derivatives on one side and all the γ matrices on the other side. But in this order matters. First I have a γ^α and then γ^μ. I cannot write it as γ^μ, γ^α whichever way I want because these are matrices. Two matrices A and B cannot be written B and A unless they commute. And I know γ^α, $\gamma\beta$ do not commute. So I will be attentive to the fact and I will write the order right. That γ^α first appears and then γ^μ appears. The derivatives can be written in any way because they are not matrices, they are just operational partial derivatives and partial derivatives commute with each other. So, that is fine. The second thing which is appearing over here, see $i\gamma^\alpha \partial_\alpha \psi$ should be equal to m of ψ already ,because of the equations of motion $i\gamma^\mu \partial_\mu\psi$ where$_\mu$ is summed over is equal to m ψ. So, therefore, the term appearing over here which is already i γ some index δ some index ψ and that index is being summed over should be equal to m ψ. So, using this equation of motion the second term will become m square ψ. While the first term I have a $\gamma^\alpha \gamma^\mu$ and the two partial derivatives. Now you can see that what I can do this α is also summed over$_\mu$ is also summed over α are going to take value 0, 1, 2, 3$_\mu$ is also going to take value 0, 1, 2, 3. So that means I can write it in a nice way that it is $\gamma^\alpha \gamma^\mu \partial_\alpha \partial_\mu$,plus one half of this the same thing again $\gamma^\alpha \gamma^\mu\partial_\alpha \partial_\mu$ the same thing I can write right where α and the$_\mu$s are being summed over now in the second term I can do a relabeling so α is being summed over $\gamma^\alpha \gamma^\mu$ and $\partial_\alpha \partial_\mu$ is there. So what I can see, both α and μ are being summed over. So it does not matter, this has to take value 0, 1, 2, 3 and μ also has to take value 0, 1, 2, 3. So it does not matter whatever I call what. So I can call for α, replace α with$_\mu$ and μ with α. ,So this will become$_\mu$ and this will become α. This will become$_\mu$ and this will become α. However, the partial derivatives as we discussed commute with each other. So I can write it back as $\partial_\alpha \partial_\mu$. First I converted$_\mu$ into α then the roles of$_\mu$ and α get flipped off. But this first two γ matrices cannot be brought back in the previous form because they do not commute but partial derivatives commute. So I can write down $\partial_\alpha \partial_\mu \partial_\mu \partial_\alpha$ which was appearing again back as $\partial_\alpha \partial_\mu$. So therefore I will get a structure half times $\gamma^\alpha \gamma^\mu$. And then $\gamma^\mu \gamma^\alpha$ both coming with $\partial_\alpha\partial_\mu$. Which is the anti commutator of $\gamma^\alpha \gamma^\mu$. So this see this thing with clarity. First what I did I just wrote it into half of twice of its pieces. And in one of the pieces I made use of the fact that this partial derivatives commute. Therefore, I can write the whole term as this. You can verify the whole term appearing over here is the same as this. Okay, you can just open it up $\gamma^\alpha\gamma^\mu$ plus $\gamma^\mu\gamma^\alpha$ and do the summation. So that means here α and μ summation is implied. And here also α and μ summation is implied. Do these things, obtain what are the terms you are getting.
You will see that both the terms are the same. So therefore I can write the term which is appearing over

here is as $\gamma^\alpha\gamma^\mu$ anticommutator and $\partial_\alpha\, \partial_\mu$ of the ψ. Now I also know that the anticommutator of $\gamma^\mu\, \gamma^\nu$, that means $\gamma^\alpha\gamma^\mu$ is – twice of the Minkowski metric $\eta^{\mu\nu}$ that means this can be written as a ,– of $\eta^{\alpha\mu}$ and α and μ are being summed over. So, that means summation over α_μ is implied. And now you see this is nothing but the Klein-Gordon equation. So, therefore the Dirac equation which we obtained as a solution for Dirac equation of motion also satisfies the Klein-Gordon equation. This is not a mystery because ,The second order Klein-Gordon equation is the unique solution which is Lorentz invariant. So therefore, its second derivative version should better satisfy the Klein-Gordon equation as well. So therefore, the solution which we have obtained for the Dirac equation is a solution to a harmonic oscillator structure as well. So ψ is like a harmonic oscillator again as well. So therefore we can employ, right now it is not a harmonic oscillator, it is Klein-Gordon, but going to Fourier basis it will become harmonic oscillator equation for different k basis as we had done for harmonic, the scalar field. So therefore this can be converted back into harmonic oscillator like structure and therefore the usual quantization scheme which we adopt for a scalar field which is quantization of oscillators in Fourier space can be adopted here as well. So, I know the solution of harmonic oscillators. I can write down those as the operators and then quantize the full field and that is what we can do for the Spinor field as well. In the next class, we will try to see how to do this for Spinor field. The usual way looks very *much* settled that we can do it the way scalar field was done because both of them satisfy the same equation and we know it has our richer matrix structure scalar field was just a numbers this time it is we have 4 plus 1 matrix that means it has different components all the components satisfy the Klein-Gordon equation so therefore it is a collection of more oscillators Spinors is collection of more oscillator so we can quantize it like we quantize an oscillator and that is what we will do in the next case.