Foundation of Quantum Theory: Relativistic Approach
Quantum Field Theory 2.3
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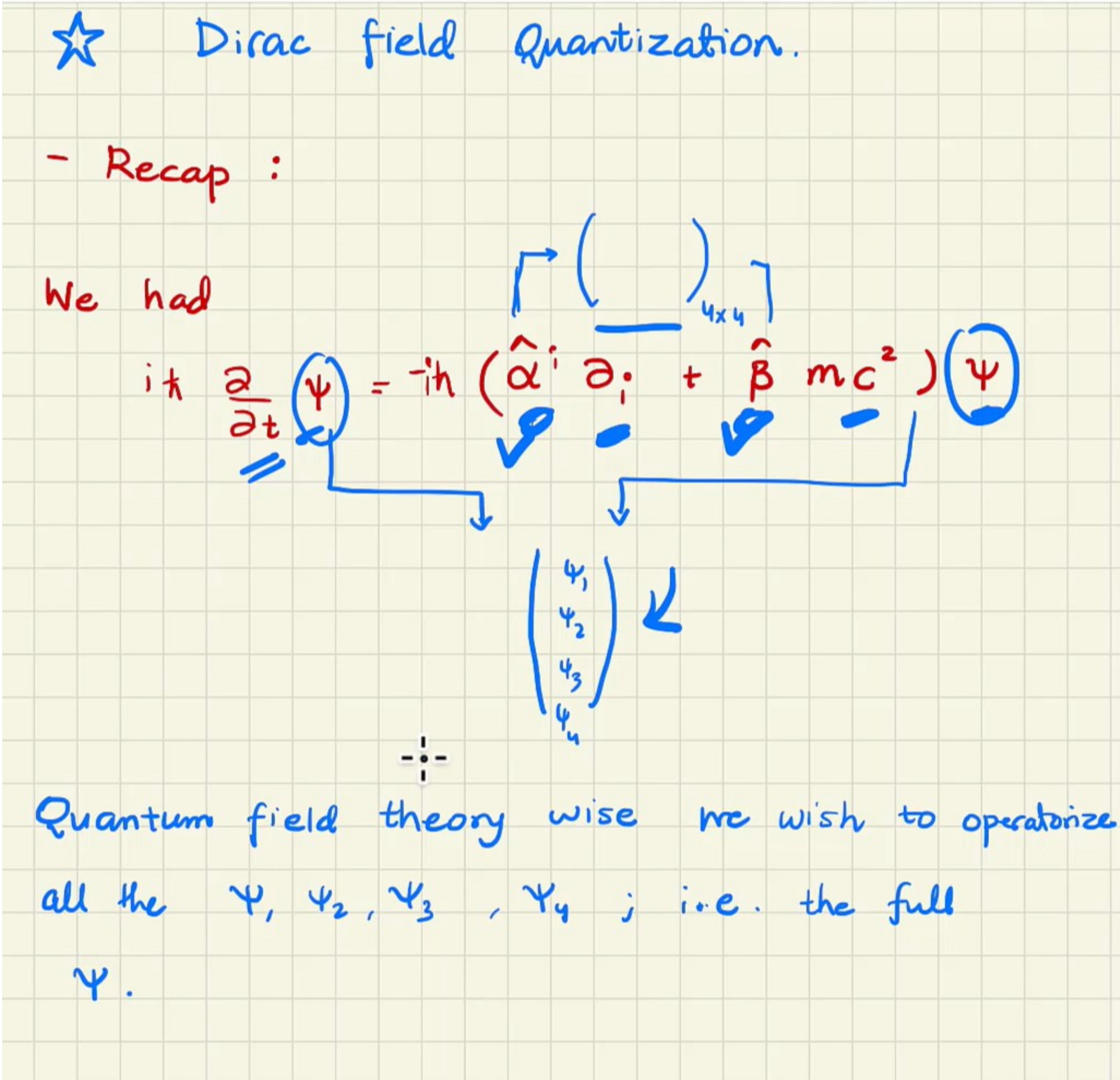
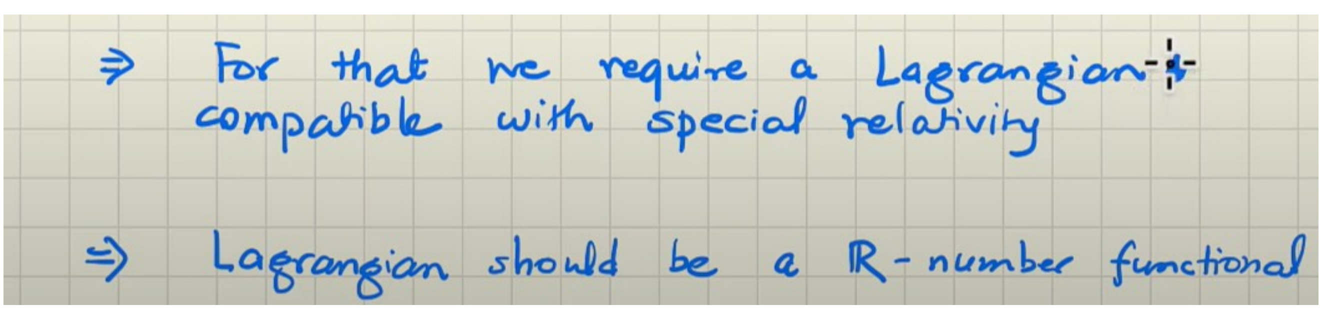
Dirac Fields 1
Lecture- 21

So, today we will discuss about the quantization of the Dirac field and as we remember that Dirac field was something or Dirac particles was something which we landed up while demanding for relativistically consistent quantum theories. Today we will try to go a quantum field theory version of that, but in this case we have to be slightly careful corresponding to the case when we were doing the scalar field. Today we will try to go a quantum field theory version of that, but in this case we have to be slightly careful corresponding to the case when we were doing the scalar field because scalar field was a trivial case and the scalar field did not transform at all under low-inch transformation. However, we have given the Dirac particles a name Spiner and here in this discussion it will become what does it mean to be a Spiner and what properties are spiners carry and how do they transform under low-inch transformation. Clearly it is not a scalar, so it is going to transform, but how do these things exactly transform is a discussion which we should do slightly carefully and then we will move on to once we are comfortable with the knowing the structure of how does it transform into writing a consistent Lagrangian and obtaining the operators corresponding to that. Things remember we wanted a first order differential equation consistent with the relativistic transformation or special relativity and we ended up getting this kind of Dirac equation structure okay where I have the first derivative of some Dirac particles wave function and that turned out to be equivalent to the spatial derivatives and the total mass energy content multiplied with coefficients αs and βs which turned out to be the case that they can no longer be the number or the usual vectors they better have to be matrices 4 cross 4 dimensional matrices if you recall the minimum they can start with is a 4 cross 4 dimensional way 4 for 4 dimensional matrices therefore this equations to make sense this ψ appearing here and here was a 4 cross 1 dimensional column vector.





So this equation's structure is consistent with special relativity at the cost of making this αs and βs appearing over here into a 4 cross 4 dimensional matrices and this ψ appearing over here as a column vector of 4 objects. This might remind you something very similar to space time vectors a_0, a_1, a_2, a_3 but we will see in a minute that these are closely related to those quantities, but are slightly different compared to the space-time vectors. So, these four components do not transform as the four component of space-time exactly one to one. They are slightly related, but they have a somewhat different structure. At more mathematical group theoretical level, they are closed by objects. They are isomorϕc to each other in some sense, but we will not go into that details. We will just alert you that these quantities transform non-previally. They neither transform like a genuine space-time vector just directly or they do not transform as a scalar as well. And once we will try to find out what is this thing ψ 1, ψ 2, ψ 3, ψ 4 transform, how does this object transform? We will want to make a quantum field theoretic version and therefore want to convert all these quantities ψ 1, ψ 2, ψ 3, ψ 4 into quantum operators. For that to happen, remember, we need to get its phase space, its conjugate momenta, its Hamiltonian and so on. And therefore, we will need a Lagrangian corresponding to that. And Lagrangian, whose equation of motion is this, just like for Klein-Gordon equations of motion, we had written down a Lagrangian, which got me the correct differential equation or correct equations of motion. Similarly, I want a Lagrangian for Dirac particles as well or Dirac field as well whose variation will give me this equations of motion. And to no surprise I would want that Lagrangian to also have the compatibility with spatial relativity. I should write down the Lagrangian which not only gives me the correct equation of motion under variation, but it respects all the symmetry of spatial relativity as well.

Again to reiterate I want to extremize the action for that to happen action and therefore the Lagrangian should be a real number functional. That means at the end of the day, once I am done with all integrations, it should give me a real number, not a matrix, not a complex number. Right now the equation of motion is a matrix equation. So Lagrangian what should I write should not be a matrix Lagrangian. I should have rather a scalar Lagrangian. Remember Lagrangians actions are to be real number, real scalar, real number based scalar functions because extremization greater than smaller than is known for us at least trivially for real number cases. So therefore, I want a Lagrangian which is a real number functional which also has all the symmetry of special relativity in built it in. And from there, I should get this equations of motion and afterwards when I have this complete set, I would try to write down the operators corresponding to the phase space variables. The Poisson bracket will be converted into quantum operators, that is the job which we for scalar field as well. Here also we are going to do this. The middle aspect of this is going to be slightly problematic that I have to write down a Lagrangian which gives rise to this equation of motion is not difficult that I can write down, I can just straight away give you. But we will see that writing a special relativity compatible Lagrangian is slightly non-trivial in this case compared to the scalar field case. Let us see why. So, first the Dirac equation which we are going after this, I can bring the whole thing into the left hand side and therefore, the whole quantity can be written like this after amultiplication of the β completely. So, let me reiterate what I have done. I will multiply the whole equation with β, β here and β here and a β here. This β times β will become β^2 and if you remember β^2 was identity so i will left with a β here and β times α over here and identity over here and then i will bring the whole left hand side whole right hand side into the left hand.

We can write the equation as

$$i\hbar\hat{\beta}\frac{\partial\psi}{\partial t} + i\hbar\hat{\beta}\hat{\alpha}^i\partial_i\psi - mc\psi = 0$$

Defining

$$\hat{\beta} = \gamma^0$$

$$\hat{\beta}\hat{\alpha}^i = \gamma^i$$

$$\Rightarrow i\hbar\left(\gamma^0\frac{\partial\psi}{\partial t} + \gamma^i\partial_i\psi\right) - mc\psi = 0$$

$$\Rightarrow i\hbar\gamma^\mu\partial_\mu\psi - mc\psi = 0$$

$$(\gamma^i)^2 = \hat{\beta}\hat{\alpha}^i\hat{\beta}\hat{\alpha}^i = -\beta\hat{\alpha}^i\hat{\alpha}^i\beta$$
$$= -\beta^2 = -\mathbb{1}$$

$$(\gamma^0)^2 = \hat{\beta}^2 = \mathbb{1}$$

$$\gamma^i\gamma^j = \hat{\beta}\hat{\alpha}^i\hat{\beta}\hat{\alpha}^j = -\hat{\beta}^2\hat{\alpha}^i\hat{\alpha}^j$$
$$= +\hat{\beta}^2\hat{\alpha}^j\hat{\alpha}^i$$
$$= -\hat{\beta}\hat{\alpha}^j\hat{\beta}\hat{\alpha}^i = -\gamma^j\gamma^i$$

$(i\hbar\gamma^\mu\partial_\mu - mc)\psi = 0$ is a matrix eqn. ✓

How does ψ transform under Lorentz transformation?

Scalar : $\phi \longrightarrow \phi$ does not transform

(1×1 dim matrix)

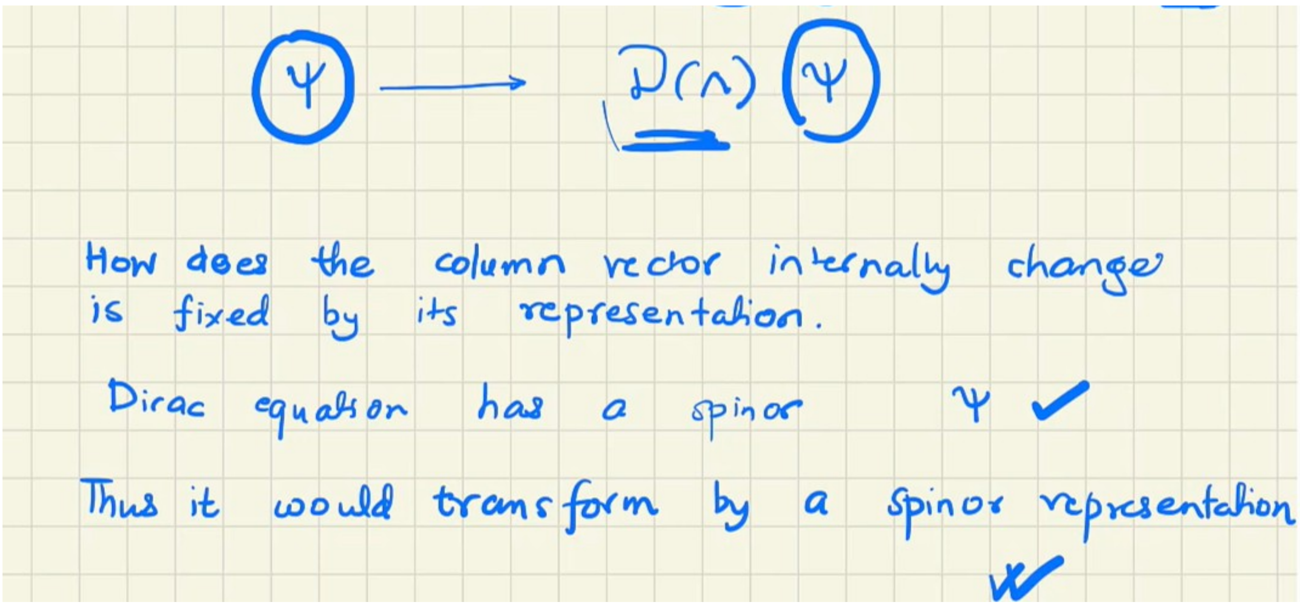
Vector : $A^\mu \longrightarrow \Lambda^\mu{}_\nu A^\nu$

(4×4 dim matrix)

Tensor $B^{\mu\nu} \longrightarrow \Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta B^{\alpha\beta}$

(16×16 dim matrix)

A general species is supposed to transform via some matrix (representation) under LT.



We can write the equation as

$$i\hbar\hat{\beta}\frac{\partial\psi}{\partial t}+i\hbar\hat{\alpha}\,\partial_i\psi=mc\psi$$

Defining hat $\beta=\gamma^0$

$$\hat{\beta}\hat{\alpha}=\gamma^i$$

$$\Rightarrow\quad i\hbar\left(\gamma^o\frac{\partial\psi}{\partial t}+\gamma^i\partial_i\psi\right)-mc\psi=0$$

$$\Rightarrow\quad i\hbar\gamma^\mu\partial_\mu\psi-mc\psi=0$$

$$(\gamma^i)^2=\hat{\beta}\hat{\alpha}^i\hat{\beta}\hat{\alpha}^i=-\beta\hat{\alpha}^i\hat{\alpha}\beta=-1=-\beta^2$$

(γ^0)^2 = hat β^2 = *1*

$$\gamma^i\gamma^j=\hat{\beta}\hat{\alpha}^i+\hat{\beta}\hat{\alpha}^j=\hat{\beta}^2\hat{\alpha}^i\hat{\alpha}^j$$

$$=+\quad\hat{\beta}\hat{\alpha}^j\hat{\alpha}^i$$

$$=\quad-\hat{\beta}\hat{\alpha}^j\hat{\beta}\hat{\alpha}^i=-\gamma^j\gamma^i$$

$(i\hbar\gamma^\mu-mc)\psi=0$ Is a matrix eqn.

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Vector: $A^\mu\rightarrow\Lambda^\mu_\nu A^\nu$
Tensor: $B^{\mu\nu}\rightarrow\Lambda^\mu_\alpha\Lambda^\nu_\beta B^{\alpha\beta}$

A general specied is supposed to transform vaia some matrix (representation) under LT.

$$\Psi\rightarrow D(\Lambda)\Psi$$

If I do so, I will have an equation like this at hand, $i\hbar\, \beta$ times the temporal derivative plus $i\hbar\, \beta$ times α i which has come from the left – mc times identity is here, but I am not writing that. That identity was coming from β^2. So I have this structure. What I am going to do further, I am going to call this, this is one matrix and these are three matrices. β α 1, β α 2, β α 3. There are three matrices. And this is just a single matrix. So what I am going to do, the collection of this β and the three matrices here as I am just relabeling them, calling this the first one as γ_0 .

This is the matrix. And the remaining three also as three different matrices, γ_1, γ_2, γ_3. This is just for convention and notation wise convenience we are going to write up. All the time I do not want to write β times α 1, β times α 2, rather γ_1, γ_2, γ_3. So, with this nomenclature, this is nothing physical meaning or any importance of any relevance. I am just making a convenient writing choice. The equation over here can be written as $\gamma_0 \partial\, \psi$ upon ∂T plus γ_i coming with the different special derivatives. The convenience is that collectively these derivative terms can be written as a $\gamma_\mu \partial_\mu \psi$. Remember$_\mu$ and$_\mu$ are appearing the diagonal einstein convention holds it will be γ_0 $\partial_0 \partial_0$ γ_1 ∂_1 γ_2 $\partial_2 \gamma$ ∂_3 so all these four terms will be generated so with this choice i have a neat convenience at hand that whole this big term can be just written as a γ_μ $\partial \partial_\mu$ or $\partial_\mu \psi$ okay and then – mc ψ so i have written the Dirac equation in a convenient way as this, $i\hbar$ γ_μ $\partial_\mu \psi$ – mc ψ is equal to 0. Remember again this is a 4 cross 1 dimensional vector, this is a 4 cross 1 dimensional vector, γ_μ is a 4 cross 4 and there is identity hiding somewhere here which is also 4 cross 4. We previously knew what are the properties of αs and βs, from there we can derive the properties of γ_μs as well. The first one would be γ_0, which is same as β. So, all its properties of β will be translated into γ_0. For example, β^2 will be was identity. Therefore, γ_0^2 will also be identity. What about γ_i, γ_1^2, γ_2^2, γ_3^2? You do the exercise. You take γ_1, multiply it with the γ_1. So, you will have this structure.

And remember α and β used to anti-commute if you remember they had some anti-commutation rule so you can flip them once you flip them you will bring either α i's along it or if you if you flip this pair you will bring α α α in between and β β on the right hand side or you do the other way you flip this version you will get a β β in between an α α on that side either case you will end up getting either – α i^2 or – β^2 which is equal to – identity. And remember α and β used to anti-commute if you remember they had some anti-commutation rule so you can flip them once you flip them you will bring either α i's along it or if you if you flip this pair you will bring α α α in between and β β on the right hand side or you do the other way you flip this version you will get a β β in between an α α on that side either case you will end up getting either – α i^2 or – β^2 which is equal to – identity. Okay, so the three matrices which we have redefined have this structure. Okay, so the three matrices which we have redefined have this structure. γ_0 the square is identity. The three γ_1, γ_2, γ_3's individual^{2}s are equal to – identity. Further, what is γ_i times γ_j? Again write out γ_i times γ_j. Use the flipping relations once more. That you will flip the middle pair this time with a – sign. So, this β will come here and you will get a – β^2. This is a elementary exercise you can do and do with some two minutes. You will be able to see that γ_i γ_j is equal to – γ_j γ_i. That means they anticommute that γ_i γ_j is zero. If i and J are different if i and J are the same then you will get a – identity twice of – identity and γ_i γ_0 lastly again write down what is γ_i write down what is γ_0 flip it once and you will get – γ_0 γ_i again they anti-commute with each other. So, you have a γ_0^2 which is identity, $\gamma_{1,2,3}$ individual^2 which is – identity and all of them anticommute with each other. Collectively, I can write down the all set of all these relations which we have written, this one three relations here, one relation here, γ these relations here and these relations here. Collectively, they can all be summarized as this equation. γ_μ γ_ν anticommutator is – twice of $\eta_{\mu\nu}$ identity. $\eta_{\mu\nu}$ is – 1 1 1. So, you see all the relations which we obtained can be obtained from various components of this. If I put γ_0 $\gamma_{0\mu}$ is equal to 0_μ is equal to 0 i will have a 0 0 here η^{00} is – 1 you will get a twice identity and therefore twice of γ_0^2 is twice of identity that means γ_0^2 is identity and so on all other relations can be similarly derived from this now with this after identifying the properties of hand. I have this matrix equation as the Dirac equation $i\hbar\, \gamma_{\mu}\, \partial_{\mu}$ – mc ψ is equal to 0 which is a matrix equation. Our job is to find out how the ψ transforms under low-end transformations and what happens to this equation under low-end transformation. Previously we had argued that under low-end transformation the the equations of motion should remain the same which we would want in this case as well so therefore the Lagrangian should also carry that property and in order to have a consistent Lagrangian i should know how different components transform for instance for scalar field i know that it does not transform at all so either I can say that it transformed with identity matrix which is just one cross one dimensional identity matrix it does not transform at all for a vector on the other hand we know any vector is given to me it will transform with amultiplication of this λ matrix low-range transformation matrix remember that would be a 4 cross 4 dimensional matrices and this will act upon the 4 dimensional vector and give me a new 4 dimensional vector a'_{μ} so this is how the vectors are supposed to transform they are supposed to transform with 4 cross 4 dimensional matrices. If I think of a tensor, b_{μ} nu,$_{\mu}$ can take value 0, 1, 2, 3.$_{\nu}$ can take a value 0, 1, 2, 3.Overall, b can take 16 value, 0, 0, 0, 1, 0, 2, 0, 3. And similarly, up to b, 3, 0, 3, 1, 3, 2, 3, 3. There are 16 component object. That transforms as two λs. Remember, we had defined tensors like how many upper indexes are there. There are 16 component objects so together ultimately it is generating a 16 cross 16 dimensional matrix overall there are 256 objects in this α can take value 4 value$_{\mu}$ can take 4 value$_{\mu}$ can take 4 value and β can take 4 values overall we have 256 such different objects these constitute components of a 16 cross 16 dimensional matrix which will act upon a 16 dimensional vector and give you a new 16 dimensional vector at hand so you see different scalars different tensors and so on despite the way we have written it classified what is happening that they are transforming via some matrix. In this case a trivial matrix which is identity in this case a 4 cross 4 dimensional matrix in this case a 16 cross 16 dimensional matrix if i take three rank tensor then there will be further objects which will come about and therefore this is how the things will keep evolving so therefore depending upon their structure their species whether it is scalar or a vector or a tensor a general species transforms via some matrix. And these matrices are called the representations of low-A transformations which matrix will transform what object is a story of the representation theory of low-end transformation. In this course we are not going to discuss in elaboration what is the representation theory and what are its something called irreducible representations of low-end transformation low-end group. Those are elaborate discussions of some other course most likely in quantum field theory course which is available you can get this. I will just touch upon the basic structures and guide you towards seeing that the spinors also have this structure that they are neither of scalar nor the vector nor the tensor but somewhere in between that they are also some n cross n dimensional matrices transform them but not as simple as these okay so we will see in a minute. So, the bottom line is that whatever species we think of a scalar, vector, tensor, spinor, all of them have a common property that under low end transformation, they transform via some matrix. And that matrix is called a representation, this set of matrix is called a representation of a low end transformation. So, we have already named the Dirac equation or Dirac four component object as a spinor. We will do a brief discussion to just look upon on the spinor representation of the Lorentz group. That means for spinors what matrices appear that make the spinor vector, the spinor object not the spinor vector. Vector would be a bad name now. This is a matrix, column matrix 4 cross 1 that is a spinor. And spinors transform with matrix which is of the Dirac spinor transforms with a matrix which is 4 cross 4 dimensional matrix. Vectors also transform with 4 core of 4 dimensional matrices. But the spinor and the vectors are not one to one in the sense are not the same thing exactly. They are closely related as I discussed in the beginning which we will not touch upon in this discussion. So, just to do a quick survey what is a representation and then I will give you the essential features what is there for the Lorentz transformation and what is their representation. Remember we are going to look for a representation of a Lorentz group.

What is a representation ?

- Writing action of Lorentz transformation via action of matrix on a vector

- Which matrix and what vector ?

- Any n×n dim matrix and any n dim vector space which respects the soul of the group (Lie Algebra).

- A continuous group is identified by its generators and relations amongst them

Rotation and angular momentum

Orbital angular momentum (in Hilbert space)

$$J^i = \sum_{jk} \varepsilon^{ijk} \hat{x}^j \hat{P}^k$$

- 3×3 dimensional representation (spin-1)

$$\hbar\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & i \\ 0 & -i & 0 \end{pmatrix} \quad \hbar\begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix} \quad \hbar\begin{pmatrix} 0 & i & 0 \\ -i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

- 2×2 dimensional space (spin $\frac{1}{2}$)

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$$J^i = \frac{\hbar}{2} \sigma^i$$

$$\frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \quad \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

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Orbital angulae momentum (in Hilber space)

$$J^i = \sum_{jk} \varepsilon ijk\, \hat{x}^j \hat{P}^k$$

● 3 × 3 dimensional respresentaion (Spin -1)

$$\hbar\begin{pmatrix}0&0&0\\0&0&i\\0&-i&0\end{pmatrix}\hbar\begin{pmatrix}0&0&i\\0&0&0\\-i&0&0\end{pmatrix}\hbar\begin{pmatrix}0&i&0\\-i&0&0\\0\mathrm{i}&0&0\end{pmatrix}$$

● 2 × 2 dimensional space (spin $\frac{1}{2}$)

$$J^i = \frac{\hbar}{2}\delta^i$$

$$\frac{\hbar}{2}\begin{pmatrix}0&1\\0&1\end{pmatrix}\frac{\hbar}{2}\begin{pmatrix}0&-i\\i&0\end{pmatrix}\frac{\hbar}{2}\begin{pmatrix}1&0\\0&-1\end{pmatrix}$$

Before that we should just be clear about what do we mean by a representation. So, representation is basically writing the action of Lorentz transformation through an action of matrix on some vector. For example, here this was a 4 cross 4 dimensional matrix which was acting upon a 4 cross 1 dimensional vector. Here it was 16 cross 16 dimensional matrix which was acting upon 16 cross 1 dimensional vector. So, ultimately a representation is identifying certain matrix acting upon certain vector which is essentially capturing the features of low-inch transformation. Then the question will arise whSo, ultimately a representation is dentifying certain matrix acting upon certain vector which is essentially capturing the features of low-inch transformation.at matrix will do that and what vector it should act upon. So, the basic statement is that any n cross n dimensional matrix and any n cross 1 dimensional vector which respects the essential features of the Lorentz group. This is a cliched statement now, the essential feature or the soul of the group. What is the most important thing about the group through which it is identified is called the Lie algebra. This is again a terminology, you should not be too $_\mu$ch of worried about that what it is. We will just see what does it mean to be some group. So this is some identification feature of a group should be identified that this is how you identify a group. Since a group As we are talking about representation, I am not demanding about its dimension to be fixed. I am going to look for any dimensional matrix. So that does not mean that a group will be identified with a size of a matrix. Size of a matrix is a changing thing. In different representation, the size of matrix will change. But still there should be some essential feature which remains the same across all other different dimensional matrices. And that we have seen already, we will just see in case of simple rotations. We will first see how to identify rotations group and what is the soul of the rotation group. That soul is typically called in linguistics of mathematics, the Lie algebra of a group. Any continuous group is identified by the relations between its generators and that relation is called the Lie algebra. If I change my dimension of the matrix, The size of the matrix will change, the component of the matrices will change, but what will not change is the relation between the generators of that group in any dimensional matrices. We will see an example to make things clear. For example, we have seen in quantum mechanics that rotations are generated by angular momentum, just like translations are generated by spatial momentum. And we know in Hilbert space how to write down the angular momentum operator. Angular momentum operator is the cross product between the position and the momentum r cross p which I have written in terms of ϵ_{ijk}. I hope you are familiar with Levy-Sivita tensor, which is a totally anti-symmetric tensor, which 0, 1, 2, 3 is 1 and their cyclic permutations is 1. And any odd, any single permutation, ϵ_{123} is 1. But if you just take, just flip one pair, ϵ_{213} 1 and 2 have been flipped, then it will pick up a – sign. If you maintain the cyclicity, that means ϵ_{123} 1, 2, 3 becomes

3, 1, 2. That means I have just given it a rotation. It will maintain the order, then it will be 1. Otherwise, the cyclicity is broken. You have flipped one pair, then it will pick up a – sign. And for any repeated object, 1, 1, 2 or 1, 2, 2, all those things are 0. So, this is a completely anti-symmetric Levy-Civita tensor, which is a convenient way of writing a cross product. Two vectors cross product can always be written as x cross y vectors can be written as ϵ, let us say I am talking about ith component of this, the resultant cross product would be ith component of that, that will be ϵ_{ijk} xJ yk and $\sum_{jk}$ over . This is how cross product is written for any vectors. So, that is what I am making use of on writing the angular momentum operator in quantum mechanics as cross product between x and p. This is an angular momentum operator in Hilbert space of wave functions. This generates rotation. This was a rotation in Hilbert space. In position space, what generates rotation? The rotation matrices, if you remember, was cos θ, sin θ. cos θ sin θ 0, – sin θ cos θ 0, 0, 0, 1. This is the rotation about the z direction, z axis, z axis. Similarly, there can be Y axis, X axis. There are three rotations possible. And generators of this will be infinitesimal form of these rotations. So, any rotation matrix you take, take its derivative with respect to θ and put θ is equal to 0 after derivative. That will identify for you that what is the generator of that group. There is a $i$and h barmultiplication which typically is done in order to make it quantum mechanical operator and such that the rotations and other things are written as quantum operator. But you can simply do that $i\hbar$multiplication times this is the generator of a group. So, take rotation about X axis do this operation you will get this, take rotation about y axis you will get this, take rotation about z axis you will get this. And these are the generators therefore of the position space rotation. This was in Hilbert space rotation, this angular momentum and this is in position space. Similarly, in spin space you might have done already, poly matrices are supposed to generate rotations in spin space, J_1, J_2, J_3. The three components of J_1, J_2, J_3 from here, the three component J_1, J_2, J_3 here and the three component J_1, J_2, J_3 here. All of them are angular momentum operators they are all talking about rotation group you see their structure is different sometimes it is three dimensional three cross three dimensional matrix sometimes it is two plus two dimensional matrix sometimes it is some linear operators on wave functions but they are all talking about the same group so what is common across this not the matrix dimension that is not a common feature not the matrix component that is also not a common feature. What is a common feature such that these operators here, these matrices here and these matrices here can be claimed that they are all from the same group is the statement that the three component of this, the three component of this and the three component of this, they all satisfy the same relation which is given over here, the box. You take J_1, J_2, find out its commutator, you will get that it is equal to J_3 with $i\hbar$ as we have discussed. So, that means despite their appearance, their matrix, their size and matrix component they look very different from each other, but at their heart they are satisfying the same relation amongst each other which any recreation group generator would do. So, they are different matrices of different size.

* Recall spin operators are classified by j, i.e. $(2j+1)\times(2j+1)$ dimensional matrices

Common features of all these set of operators

$$[J^i, J^j] = \sum_k i\hbar\, \varepsilon^{ijk} J^k \qquad R(\vec{\theta}) = \exp\left(-i\,\frac{\vec{\theta}}{\hbar}\cdot\vec{J}\right)$$

Similarly there is an identification algebra for the generators of Lorentz transformation

Three rotations: (J)

$$\hbar\begin{pmatrix}0&0&0&0\\0&0&1&0\\0&-1&0&0\\0&0&0&0\end{pmatrix} \quad \hbar\begin{pmatrix}0&0&0&0\\0&0&0&1\\0&0&0&0\\-1&0&0&0\end{pmatrix} \quad \hbar\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&1\\0&0&-1&0\end{pmatrix}$$

Three Boosts (Y)

$$\hbar\begin{pmatrix}0&1&0&0\\1&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix} \quad \hbar\begin{pmatrix}0&0&1&0\\0&0&0&0\\1&0&0&0\\0&0&0&0\end{pmatrix} \quad \hbar\begin{pmatrix}0&0&0&1\\0&0&0&0\\0&0&0&0\\1&0&0&0\end{pmatrix}$$

$$\left.\frac{\partial \Lambda}{\partial \beta_i}\right|_{\beta_i=0}$$

$$[J^i, J^j] = i\hbar \sum_k \varepsilon^{ijk} J^k$$

$$[Y^i, Y^j] = i\hbar \sum_k \varepsilon^{ijk} J^k$$

$$[J^i, Y^j] = -i\hbar \sum_k \varepsilon^{ijk} Y^k$$

$$D(\Lambda) = \exp\left(-i\,(\theta_m J^m + i\beta_m Y^m)\right)$$

★ Recall spin operators are classified by J, ie, *(2j+1)* × *(2j +1)* dimensional matrices.

Common features of all these set of operators.

◯ Similarly there is an identification alegbra for the generators of Lorentz transformation

Three rotation: $\hbar\begin{pmatrix}0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & -1 & 0 & 0\\ 0 & 0 & 0 & 1\end{pmatrix}\hbar\begin{pmatrix}0 & 0 & 0 & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & 0 & 0\\ -1 & 0 & 0 & 0\end{pmatrix}\hbar\begin{pmatrix}0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & -1 & 0\end{pmatrix}$

Three boosters
(Y)

$$\hbar\begin{pmatrix}0 & 1 & 0 & 0\\ 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{pmatrix}\begin{pmatrix}0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0\\ 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{pmatrix}\hbar\begin{pmatrix}0 & 0 & 0 & 1\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 1 & 0 & 0 & 0\end{pmatrix}$$

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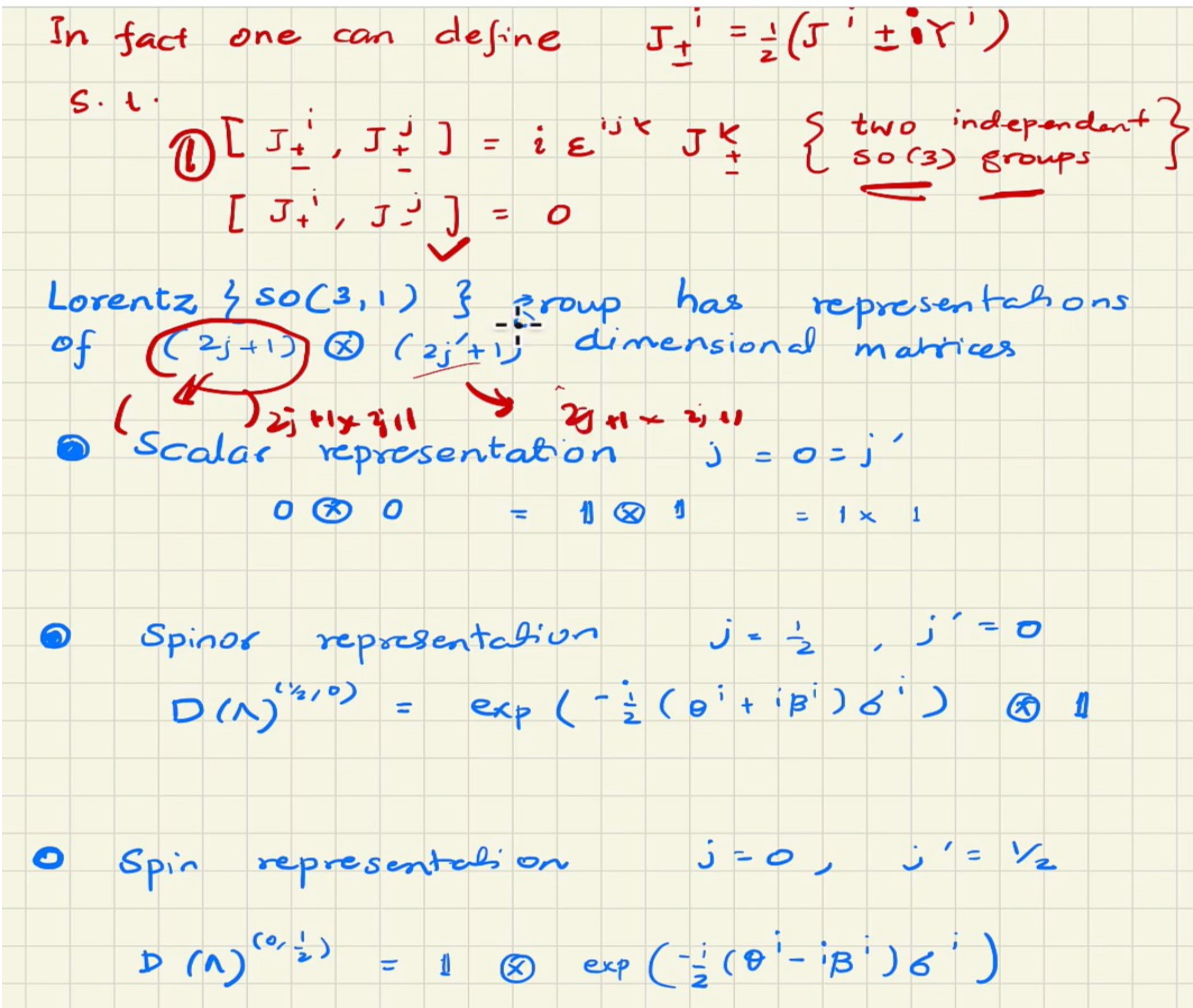
$$[J^i, Y^j] = -i\hbar\sum_k \varepsilon^{ijk} Y^k$$

$$D(\Lambda) = \exp\left(-i(\theta_m J^m + i\beta_m Y^m)\right)$$

So, this is called spin *1/2* dimensional representation, this is called spin one dimensional representation, this is spin zero representation in some sense, because it is not about any finer object, this is orbital angular momentum. So in general i am talking about $2J_{+1}$ cross $2J_{+\ 1}$ cross dimensional matrices this will be therefore 3 cross 3 this will be spin 1 will be 3 cross 3 spin 2 spin *1/2* will be 2 cross 2 and so on their sizes will change their matrix dimension will change their components will change but what will not change is this relation this will always remain true and the second relation that in any space this is this rotation is generated by these matrices through this exponential operation. Take the *J* operator, dot it with the θ, θ_1, θ_2, θ_3 . Three dimensions, how $_\mu$ch angle you want to rotate by along this axis, that axis and that axis, y axis and z axis. These two relations will be always true across any dimension or any representation.So I will not care about what is the matrix size, dimension. I will not care about what is the component. I will care whether they satisfy this equation and whether they satisfy this equation or not. If they do, I will call that all of them are rotation belonging to the rotation group, just their representations are looking different.So, group is an abstract object in different spaces, it manifests itself as a different matrices. But at their heart, the generators always satisfy this relation.You can verify that here we have written the three things, here we have written the three things, here we have written the three things. Despite their shape and form being very different, they satisfy

the same relation. So, this is called the Lie algebra of rotation group. First we identified How many generators are there? Three generators corresponding to three choices I can make, three rotations I could have done. Then I take the rotation matrix, do this derivative with respect to the parameter of rotation and go for infinitesimal, that means θ tending to 0. Once I do that, I will end up getting the generators of that group and then I will uncover what is the relation between the generators. You will realize that this relation is always working for any representation. of that particular group. So just like rotation is identified with the fundamental algebra of generators it has, we can expect since Lorentz group is also a continuous group, it should also have some similarly identified or identification algebra amongst its generators. So just like rotation is identified with the fundamental algebra of generators it has, we can expect since Lorentz group is also a continuous group, it should also have some similarly identified or identification algebra amongst its generators. How many generators are there for the low range group? Remember low range group is a group of spatial rotations and boosts. So three spatial rotations are possible along X axis, Y axis, Z axis. And three boosts are there along X axis, Y axis, Z axis. So three rotations and three boosts together are the total operations low range group can do. So there could be six generators. Again i will do the same thing λ is the this time is the matrix which will be identify the operations a simple rotation operation for example will be temporal axis is unchanged suppose i am doing a rotation along the z axis then the z axis will also be unchanged and i will have just cos θ sinθ_0 – sine θ cos θ. This will be a rotation operator in the space time along the z axis.
Again, do the same thing. Do this ∂ upon $\partial\theta$. Put θ is equal to 0. You will generate the three rotation generators. You will generate the three rotation generators. These are given the name j. I have just pulled out the iota. These are given the name j. I have just pulled out the iota. So, I am just pulling out or absorbed in the definition of θ, let us say. Then the three rotation generators are given like this. Okay. Now, similarly, there could be three boosts. Boosts, remember, is cos hyperbolic β, sin hyperbolic β, then 0, 0, sin hyperbolic β, cos hyperbolic β, *0, 0,* and then *0, 0, 1, 0, 0, 0, 0, 1.* This will be first 2 cross 2 dimensional block is the boost block. Then do the same operation, take ∂ upon $\partial \beta$, put β is equal to 0. That will give you the generator of the boost along the X axis. And you will find out that you will get this operator, this matrix. So three boosts generator are this. Three rotation generators are this, obtained from $\partial \lambda$ upon $\partial\theta_i$, θ_i meaning along X axis what direction, y axis what direction, z axis what direction. And not is equal to zero then first you do this and put θ_i is equal to zero similarly for the boost matrix matrix along different components along x direction y direction g direction you can do the three boosts do this operation you will get these three matrices so these three rotation generators and these three boost generators are there previously i had these three rotation generators in position space. They are analog are this six generator in space time space. It is not only a position space but time position space. So, six generators are there. Previously rotations were satisfying the generators of rotations were satisfying this algebra which it should keep doing even in the four dimensional case. So, this is a rotation subgroup. But there is a boost as well so boost and boost commutator if i find out you will find out that they become the two of them will become third of the J meaning two boosts combined can be written in terms of rotation about one axis and so on one boost and one rotation equal is also commutes to one boost.
So we had previously three generators for rotation and the three rotations were satisfying this relation amongst each other for loading transformation we have three boosts and three rotations and their glee algebra is this the whole block the bigger block over here. And just like any representation or any matrix in whatever space was obtainable from their generators times the parameter of the transformation, here also we have six parameters, θ_1, θ_2, θ_3, β_1, β_2, β_3 . And their generators put together give you the representation of that. So what we will have to do? We will have to find out any n cross n dimensional matrices six of them which satisfy this relation. One simple realization we have at hand in space time this six matrices already do that. But we can explore more we can ask whether in four dimensional four cross four dimensional matrices space also are these the only choices? or there are more choices than that. Similarly, are there choices in not 4x4 dimensional space, let us say 2x2

dimensional space or 6x6 dimensional space which maintain this relation. So this is the soul of the Lorentz group SO group just like this was the sole of the rotation group. And then the various representation which we had written over here they all have the common feature that irrespective of their shape and size they were satisfying this relation. And the game here is also that irrespective of the dimension and the size we would have to find 6 matrices which satisfy this relation and that will generate me representations of Lorentz group. This is the simple simply given summary of this. One more thing what you can do is that from the generation of from the set of this three rotations and three boosts what we have *J* and Y which we have over here. We can construct some more things which are equivalent in description, but more easy to visualize things with. So, if we have three Js and three Ys, what we can do, we can combine them in a particular manner. That means I can define J_+ 3 J_{+}, J_+ 1, J_+ 2, J_+ 3, which is nothing but *1/2* of J_1 plus Y1.



In fact one can define $J^i_{\pm} = \frac{1}{2}(J^i \pm iY^i)$

s.t.

$[J^i_{\pm}, J^j_{\pm}] = i\varepsilon^{ijk} J^k_{\pm}$

{Two independent SO(3) groups}

$[J^i_{\pm}, J^j_{\pm}] = 0$

Lorentz *(2j+1)* ×*(2j'+1)* dimensional matrices.

- Scalar representation j= 0 = j'

0 × 0 = 1 × 1 = 1 × 1

- Spin representation *j = 0, j' =1/2*

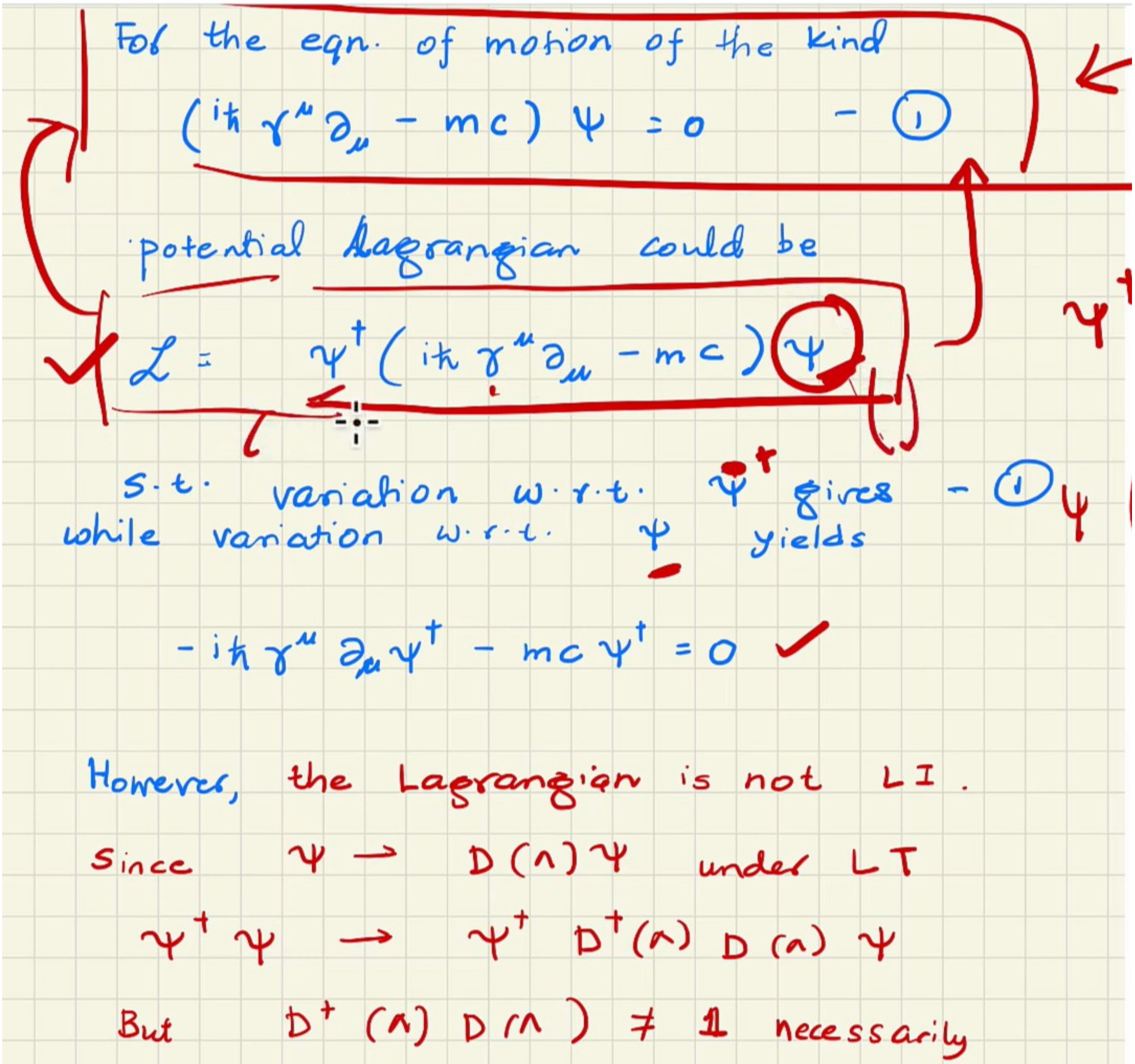
$$D(\Lambda)^{(0,1/2)} = 1 \times \exp\left(\frac{-i}{2}(\theta^i - i)\beta\sigma^i\right)$$

Actually, there should have been a *i*I believe, *i*here. So, I will define a J_+ which is summation of *J* and Y with *i*in between and J_- which is also *1/2J_i – iY_i*. So, I can define three components object J_+ and three component object J_- from this six component object *J* and Y put together. This is just a linear combination of *J* and Y. The crucial thing about this J_+ and J_- is that this J_+ talks to only J_+. It commutes to J_- . J_+ and J_- commutation is zero. Amongst Itself $J_{+\,\mathrm{I}}$ and *J* J_+ *J* Just commute to the same way as the rotations we are doing. $J_{+\,\mathrm{I}}$ J_+ *J* Is equivalent to $i\epsilon_{ijk}$ and J_+(k). Similarly J_- also does the same thing amongst each other that $J_{-\,\mathrm{I}}$ and J_- of *J* Is equivalent to *i* times $\epsilon_{ijk} J_-$ of k. So, and J_- d couple Into two independent SO3 groups, two Independent rotation groups. So, the statement is whatever was the low range group SO31 can also be identified as a two disjoint rotation groups. Their six generator operators, six generation matrices are equivalent to double copy of two rotation groups and individually each rotation group has its simple rotation algebra. So that means that I can obtain the representation of SO group from the two copies of representations of SO group, the rotation group which we have already done. The rotation group's representation we already know. For example, this is a rotation group in spin 1 object. This is a rotation group for a spin *1/2* object. So, I can take two copies of this and generate a SO3-1, the representation of a SO3-1 or representation of a low range group for the spiner objects. Okay. That means I have Now, in general so i would have a 2J_+ one dimensional representation meaning here I will have a 2j cross plus one cross 2J_+ one dimensional matrix and times another 2J_{+1} cross 2J_{+1} time times 2J_{+1} dimensional matrix since low range group is a double copy of two rotation groups so therefore its matrix structure will be just direct product of a rotation matrix times another rotation matrix put together. So, ultimately I would have this 2J_+ 1 cross 2J_+ 2*J'* plus 1 dimensional representation. I can take any *J* value, I can take any *J'* value, *J* was one representation, spin *J* representation and *J'* is another representation of the rotation group.

For instance, I can have most simple representation where *J* and *J'* both are 0. So, that means I should have written 1 cross 1 dimensional matrix. So, it is a 1 cross 1 dimensional representation, 2J_+ 1j is 0, both *J* and *J*'s are 0s, then it is 1 cross 1 dimensional representation which is just a simple 1 dimensional identity matrix cross 1 dimensional identity matrix. Similarly, one possibility of one representation of SO group is that The *J* here takes value spin *1/2* and the *J'* takes value spin 0. So, *J* is

1/2, *J′* is 0. Then this is called a spinor representation *½,* 0. First *J* is *1/2* and second *J′* is 0 and its representation is this is the, this is the matrix which generates Lorentz transformation on spin *½,* 0 object. This is called a Weyl Spinner effectively. And similarly there can be representation which is reversed way that first *J* is 0, the second *J′* is *1/2*. Here again it is called D_0, previously it was called a representation of *½,* 0. Remember this is *J* these representations are identified by two Lorentz group, one Lorentz group will representation can be *J* damaged identified with number *J*, another rotation groups representation can be identified with *J'*. There is no reason for *J* and *J′* to be the same, they can take their independent values. So, we are listing out possibilities.
First possibility was 0 0, the next possibility was *1/2* 0 or 0 *1/2*, the next possibility will be half half and so on so forth. So, we are just listing down the possibilities and just writing down the corresponding representation matrix. So, in *1/2* 0 representation this is a Lorentz transformation matrix, sigma is are the poly matrices, θ_i are the three rotations and β is our three boosts parameters. Similarly, for *0 1/2* representation, these are the three, these are the transformation matrix. This is how a spinor object of *0 1/2* rank will transform. Remember Dirac particle was a four-dimensional thing where the two blocks independently where the spin *1/2* is spin *1/2*. So, therefore, it is a spin *1/2* times spin *1/2* representation *J* and *J′* are both half. That means this transforms like this representation matrix take this object tensor it with this object that means d *1/2* 0 cross d 0 *1/2* that will define how the Dirac vector Dirac particle or Dirac field will transform. See this is the transformation matrix for a Dirac 4 cross 1 column vector. It is different from the usual $\lambda_{\mu\nu.}$ This is also 4 cross 4 dimensional matrix because this is half-half. So, half-half will mean here 2 cross 2 meaning 4 cross 4 dimensional matrix. But it is not $\lambda_{\mu\nu.}$ So this is how the Dirac particles are supposed to transform. Since these are spinors, spinors are just the half integer representation of the low range group. And the half integer low range group can be identified as a double copy of half integer rotation groups and therefore, this is how the low range transformation will act upon a Dirac spinor. It has a two block of spin half spinors, its low range transformation will be two blocks of spin half representation of the rotation group. So, this we have to keep in mind we can write down for example, what is this? For example, you know what is d *1/2* 0 which is this matrix $e^{i\theta i\beta i\sigma i}$. You can play around and write down for let us say rotation about X axis and no boost that means θ_1 will be 1 or parameter θ all other θ will be 0. And no boost means all the βs are 0. For that expand this exponential and find out what is the transformation matrix. Similarly for a boost you can find out what is the transformation matrix. And from this information you can also find out what is the representation matrix for the Dirac particle. So, here and here you have to put it the same copy of the d_λ *1/2 0* and d_λ *0 1/2*. There is a relative sign difference between *1/2* 0 and 0 *1/2* representation in their operators. So all these things go with a detailed discussion which is called chirality of the field and what not. We are not going to look at those in those details because we are not doing field theoretic. Our focus on this course is not about learning more subtle features out of fields. We will just want to develop interaction of quantum systems which we come across in labs to quantum fields. So therefore the useful information we are going to maintain only. The useful information for this discussion is that the Dirac particles, the wave function or the field which is going to be does not transform very trivially under low range transformation. It transforms with some matrix and that matrix has to be obtained from this and that put together in the diagonal block of a 4 cross 4 dimensional matrices. So, this is how the ψ transforms. ψ goes to the 4 cross 1 dimensional matrix goes to d_λ times 4 cross 1 where d_λ is this. And we have this equations of motion to obtain. So, one thing which we have realized in this discussion is how the Dirac field itself will transform under Lohit transformation. Then second thing which we wanted to know was to write down a consistent Lagrangian which will give me this equations of motion. Again if I have this equation of motion, you can find out that the Lagrangian which I am again proposing like I did for Klein-Gordon equation. The Lagrangian given over here correctly gives you this equation of motion if I do the variation with respect to ψ bar, $\psi^\dagger$ not ψ bar, $\psi^\dagger$. If I do the variation with respect to $\psi^\dagger$, I will get the Dirac equation. If I do the variation with respect to ψ, I will get the complex conjugate or the

Hermitian conjugate of this equation, the Dirac equation. So, this Lagrangian is a good candidate, just like there was a candidate Lagrangian for Klein-Cotton equation. This Lagrangian is a good candidate for starting our discussion on quantization of field. I can use this Lagrangian, find out the momenta corresponding to ψ and $\psi^\dagger$. Remember ψ is a complex field, it has real parts as well as imaginary part. Either you can work in terms of two fields, real components of ψ and imaginary components of ψ or you can take ψ and $\psi^\dagger$ pair which is equivalent. Ultimately any ψ is four dimensional object but I can write its real part times ψ real one ψ real two ψ real three ψ real four and imaginary part ψ real imaginary one ψ imaginary two ψ imaginary three ψ imaginary four so I will have a eight objects four components of the real part and four components of the imaginary part which I have to work with which is equivalent to having either real and imaginary pair of four cross one dimensional vectors or ψ and $\psi^\dagger$ which can be identified as linear independent four cross one dimensional vectors. So, I have a candidate Lagrangian which correctly generates for me the Dirac equation and I can use it to generate the phase space Hamiltonian and what not quantization whatever you have there.



For the eqn. of motion of the kind

$$(i\hbar\gamma^\mu\partial_\mu - mc)\psi = 0 \quad - (1)$$

potential Lagrangian could be

$$\mathcal{L} = \psi^\dagger(i\hbar\gamma^\mu\partial_\mu - mc)\psi$$

s.t. variation w.r.t. $\psi^\dagger$ gives $-(1)$ while variation w.r.t. ψ yields

$$-i\hbar\gamma^\mu\partial_\mu\psi^\dagger - mc\psi^\dagger = 0$$

However, the Lagrangian is not LI.

Since $\psi \to D(\Lambda)\psi$ under LT

$$\psi^\dagger\psi \to \psi^\dagger D^\dagger(\Lambda)D(\Lambda)\psi$$

But $D^\dagger(\Lambda)D(\Lambda) \neq \mathbb{1}$ necessarily

For the eqn. of motion of the kind

$(i\hbar\gamma^\mu\partial_\mu - mc)\psi = 0$ … (1)

Potential Langrangian could be

$L = \psi^\dagger(-i\hbar\gamma^\mu\partial_\mu - mc\,\psi = 0)\psi = 0$

s.t. Variation w.r.t. $\psi^\dagger$ gives – (1) while variation w.r.t ψ yields

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However, the Langrangian is not LI.

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$D^\dagger(\Lambda)D(\Lambda) \neq 1$ necessarily

However, one thing which should trouble us at this stage is this is fine that this is a C number as well because we wanted a real number kind of thing. Real number in the sense of this is ψ . This is a 4 cross 1 dimensional matrix. This is also some other 1 cross 4 dimensional matrix here. And in between, I have a 4 cross 4. So ultimately, they will combine together to give me a matrix of 1 cross 1 or a number which I will get. So our first demand is met that the Lagrangian should not be a matrix. It should give me a real number functional. This looks like happening, whatever we have written. However, the simple choice which we have written which does generate the equation of motion for the Dirac particle unfortunately turns out that it is not Lorentz invariant. This Lagrangian is not Lorentz invariant. How do I see that? This realization is that the transformation of ψ . This realization is that the transformation of ψ . This is the overall d_λ *1/2 1/2* which with the ψ transform. That means its dagger will transform with $D^\dagger$ of this. So, ψ transforms as $d\psi/d_\lambda\psi$, this is a matrix. $\psi^\dagger$ will transform to $\psi^\dagger$ times $D^\dagger(\lambda)$. That means the terms which are appearing like $\psi^\dagger\psi$ here or $\psi^\dagger$ of $\partial_\mu\psi$, those will transform with as this, $\psi^\dagger$ ψ will undergo, will become $\psi^\dagger$ $D^\dagger d\psi$. This ψ has transformed to $d\psi$, $\psi^\dagger$ has transformed to $\psi^\dagger D^\dagger$ We have in between a $D^\dagger$ multiplication. And unfortunately $D^\dagger$d in this case whatever we have obtained over here is not identity these are not unitary matrices which we can quickly see For example, even if the vector representation we take, the 4 plus 1 dimensional vector representation, there you can see that this is the rotation matrix, one about z axis, this is about y axis, this is about X axis. These are the three rotations which are possible. The three rotations are indeed generated by unitary matrices. You can take that the three rotation λ dagger λ is identity.

For the vector representation

$$(\Lambda)_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & +\sin\phi & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$(\Lambda)_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\phi & 0 & -\sin\phi \\ 0 & 0 & 1 & 0 \\ 0 & \sin\phi & 0 & \cos\phi \end{pmatrix}$$

$$(\Lambda)_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\psi & -\sin\psi \\ 0 & 0 & \sin\psi & \cos\psi \end{pmatrix}$$

Three rotations

$$\Lambda_i^{\dagger} \Lambda_i = \mathbb{1}$$

$i = 1, 2, 3$

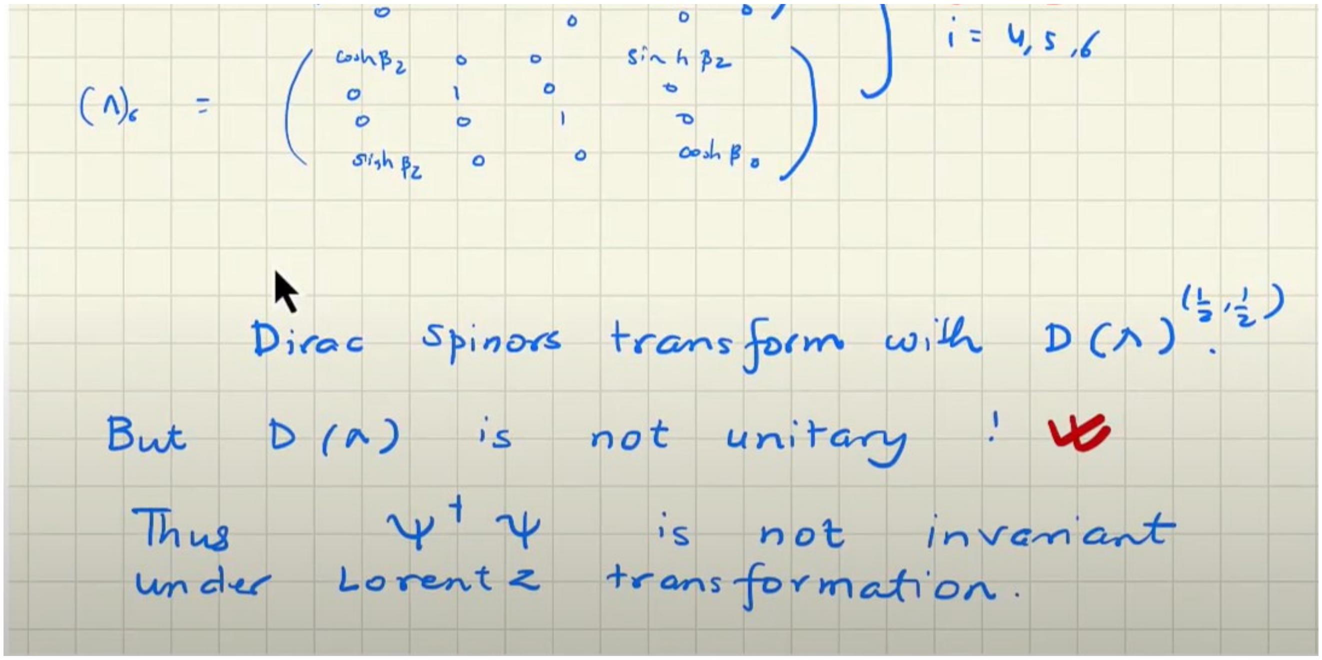
$$(\Lambda)_4 = \begin{pmatrix} \cosh\beta_x & \sinh\beta_x & 0 & 0 \\ \sinh\beta_x & \cosh\beta_x & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$(\Lambda)_5 = \begin{pmatrix} \cosh\beta_y & 0 & \sinh\beta_y & 0 \\ 0 & 1 & 0 & 0 \\ \sinh\beta_y & 0 & \cosh\beta_y & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Three boosts

$$\Lambda_i^{\dagger} \Lambda_i \neq \mathbb{1}$$

$i = 4, 5,$



For the vector representation

$$(\Lambda)_1=\begin{pmatrix}1 & 0 & 0 & 0\\ 0 & cos\theta & -sin\theta & 0\\ 0 & sin\theta & cos\theta & 0\\ 0 & 0 & 0 & 1\end{pmatrix}$$

$$(\Lambda)_2=\begin{pmatrix}1 & 0 & 0 & 0\\ 0 & cos\phi & 0 & -sin\phi\\ 0 & 0 & 1 & 0\\ 0 & sin\phi & 0 & cos\phi\end{pmatrix}$$

$$(\Lambda)_3=\begin{pmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & cos\psi & -sin\psi\\ 0 & 0 & sin\psi & cos\psi\end{pmatrix}$$

So three rotation $\Lambda_i^\dagger \Lambda_i = 1$ $\quad (\Lambda)_4=\begin{pmatrix}cosh\beta_x & sinh\beta_x & 0 & 0\\ sinh\beta_x & cosh\beta_x & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0\end{pmatrix}$

$$(\Lambda)_5=\begin{pmatrix}cosh\beta_y & 0 & sinh\beta_y & 0\\ 0 & 1 & 0 & 0\\ sinh\beta_y & 0 & cosh\beta_y & 0\\ 0 & 0 & 0 & 0\end{pmatrix}$$

So three boosts $\Lambda_i^\dagger \Lambda_i \neq 1$ for I = 4, 5, 6

$$(\Lambda)_6 = \begin{pmatrix} cosh\beta_z & 0 & 0 & sinh\beta_z \\ 0 & 0 & 1 & 0 \\ sinh\beta_z & 0 & 0 & cosh\beta_z \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Dirac spinors transform with $D(\Lambda)^{(1/2\,,\,1/2)}$

Thus $\psi^\dagger\psi$ is notinvariant under Lorentz transformation.

They are orthogonal and therefore it is just a special case of a unitary matrix. However, the three boosts which are possible, you can verify that they are not orthogonal or unitary. So, even for the vector representation which we have seen. So, even for the vector representation which we have seen. So, you can expect even for final representation it is not going to be unitary always. So, therefore it is not a unitary, d_λ is not a unitary and therefore ψ, the wave function ψ does not transform in a unitary way under no unit transformation. Schrodinger evolution which we have seen, unitary transformations, if something evolves with unitary transformation, then most of the physical quantities do not change. But if I write down the Lagrangian like this, which is just $\psi^\dagger$ and ψ multiplication or $\psi^\dagger$ and $\partial_\mu\psi$ multiplication, these things will not remain invariant under the transformation. So, with this realization, we stop over here. In the next class, we will see how to fix this once we have identified how the Dirac particles stop.

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