Foundation of Quantum Theory: Relativistic Approach

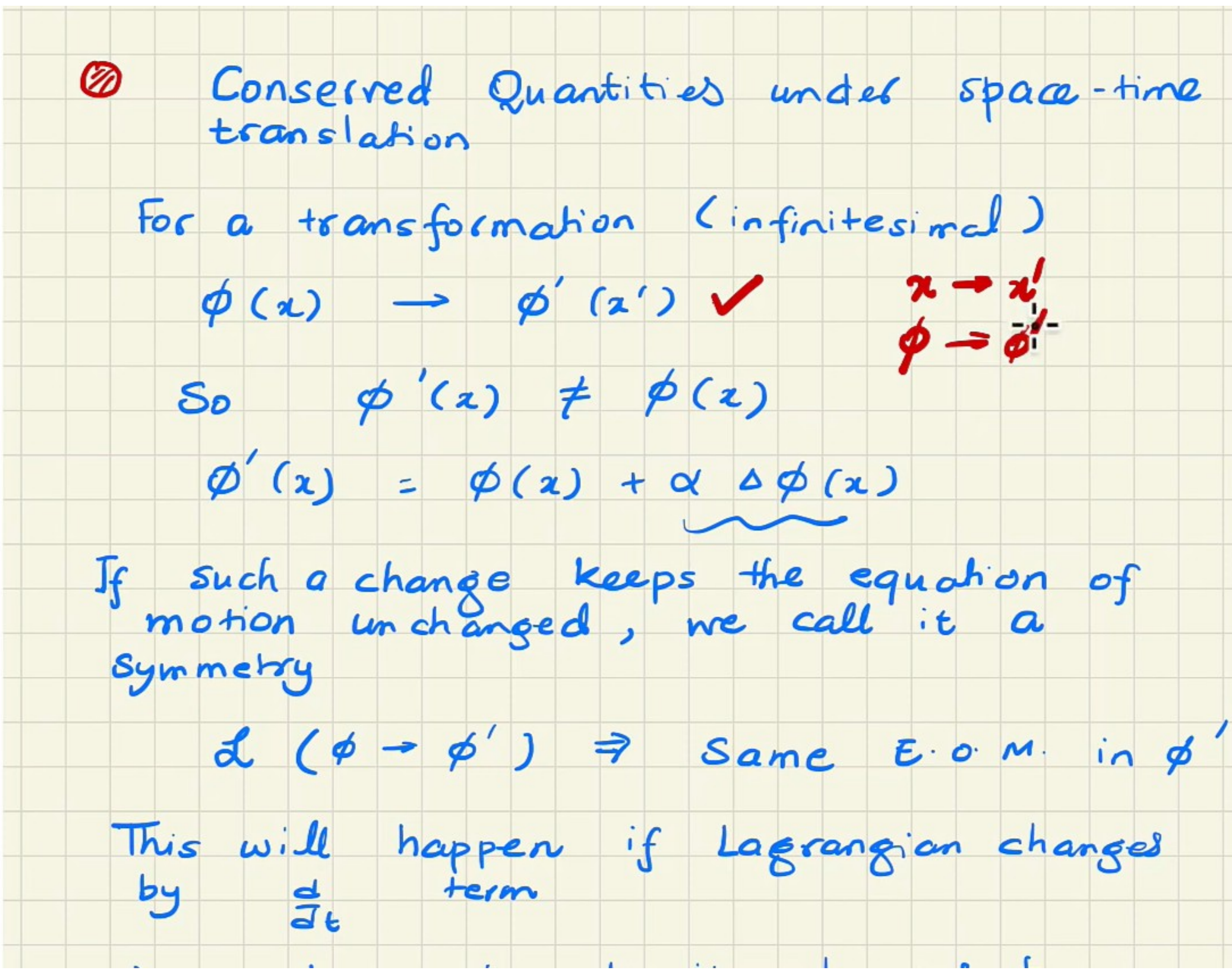
Quantum Field Theory 2.2

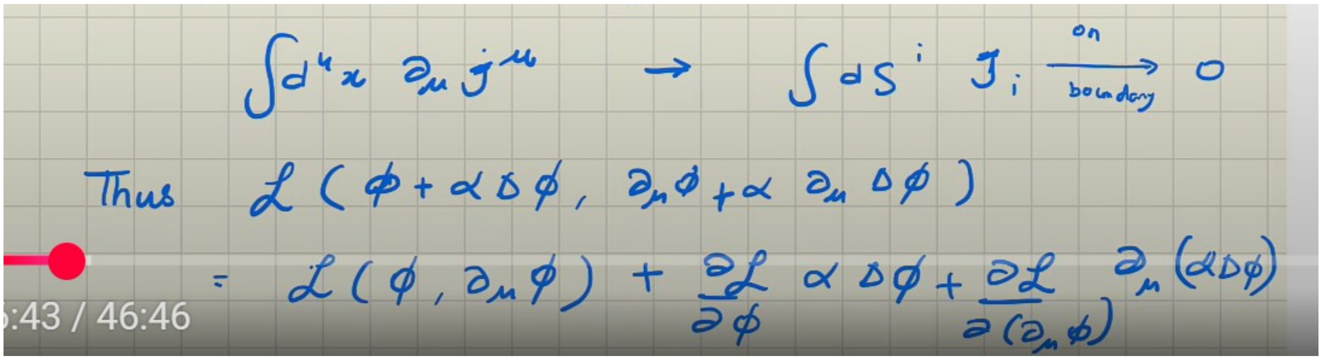
Prof . Kinjalk Lochan
Department of Physical Sciences
IISER Mohali

Conserved quantities in free fields

Lecture- 20

So in today's discussion session we are going to discuss about the conserved quantity associated with the action or the Lagrangian which we have written for fields and there are conserved operators corresponding to those if we demand it to be true at the quantum level as well.





◯ Conserved Quantities nder space-time translation

For a tarnsformation (infinitesimal)

$\phi(x) \to \phi'(x') \qquad x \to x'$
$\phi \to \phi'$
So $\phi'(x) \neq \phi(x)$
$\phi'(x) = \phi(x) + \alpha \Delta x$

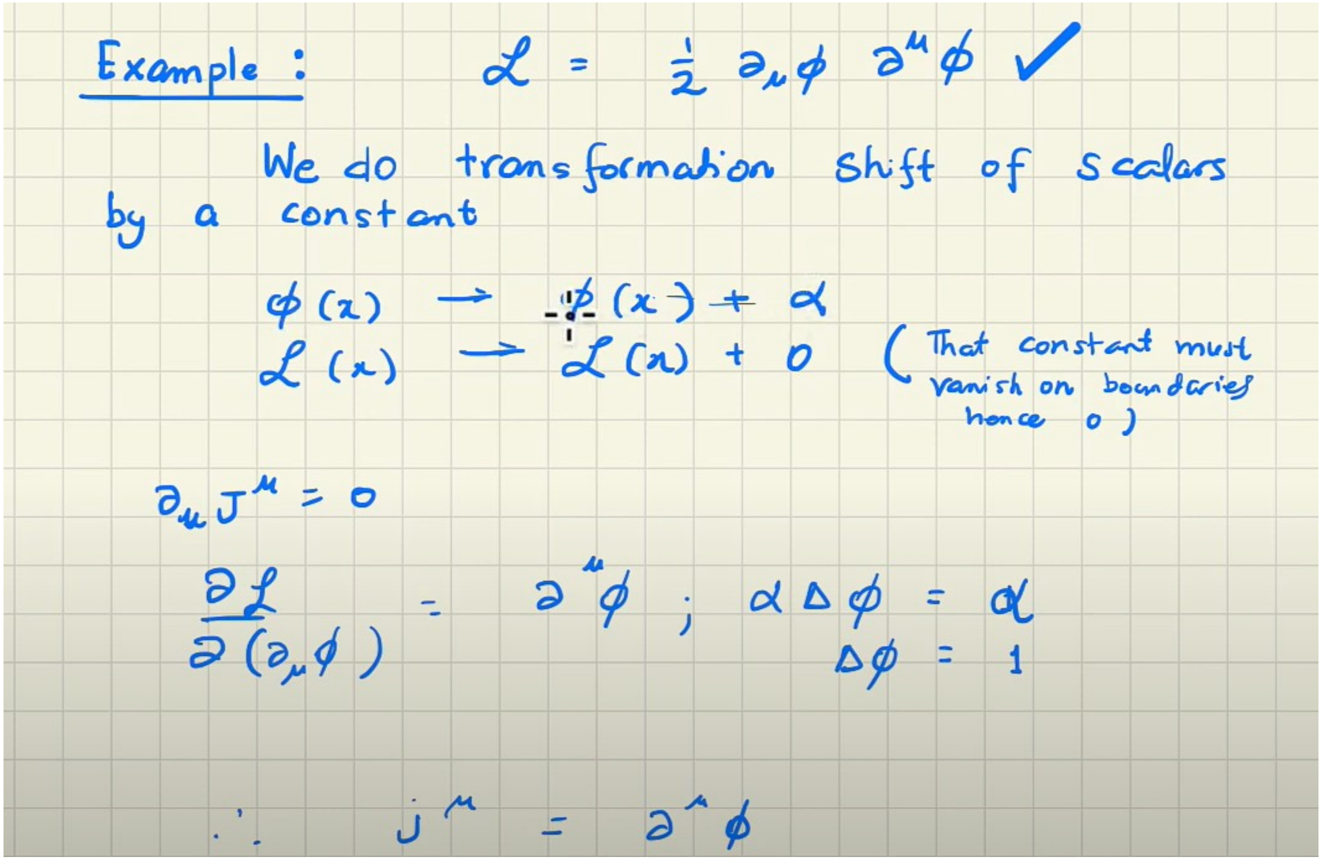
If such achange keeps the equation of the motion unchanged, we call it a symmetry.
L($\phi \to \phi'$) ⇒ same E or M in ϕ'
Thus will happen if Langrangian changes by d/ dt term

$$\int d^4 x \partial_\mu g^\mu \to \int dS^i J_i \to 0$$

$$Thus\ L(\phi + \alpha \Delta \phi, \partial_\mu \phi + \alpha \partial_\mu \Delta \phi) = L(\phi, \partial_\mu \phi) + \frac{\partial L}{\partial \phi} \alpha \Delta \phi + \frac{\partial L}{\partial \partial_\mu \phi} \partial_\mu (d \Delta \phi)$$

So we will learn about what are the conserved quantity at least at the classical levels first corresponding to the Lagrangian choice and then we will see what are the corresponding Hamiltonian corresponding quantum operators related to those symmetries. Okay, so let us recall in classical mechanics, the statement was if I had a system whose Lagrangian was given in terms of q, its time derivative and let us say maybe a dependency, explicit dependency on t. Then the following statement in the term of Noether's theorem was true, that if the q' is changed to some q'd such that the Lagrangian does not change, then there is a conserved quantity corresponding to this transformation. A typical example would be if I do a translation x going to x plus a and if the Lagrangian does not change at all such that it has either a constant potential throughout or no potential at least at all, in those cases Lagrangian does not change at all then there is a conserved quantity corresponding to this symmetry transformation which is called translation.
And we know from our analysis of classical mechanics momentum is the quantity which will be conserved if Lagrangian has this symmetry. Similarly if the Lagrangian has a symmetry under time translation. If the Lagrangian has a symmetry under triangle translation, there is again a conserved quantity corresponding to it, which is the Hamiltonian or the energy. So, this logic which we have learned previously, if under certain transformation, the Lagrangian does not change or it changes only by a total derivative term let us say. So, a total derivative of some function which is a function of q' let us say. Then we know under variational approach these kind of term do not change your equations of motion. So, the idea is I want either it f to be 0 or ∂f to be, ∂L to be time derivative of some function f.

If my change of q' generates change in Lagrangian of this kind, then I know there is a conserved quantity corresponding to this transformation. This is what we have learnt at the classical mechanics level. Now, we want to generalize that statement to field theoretic level as well. So, first we will do classical analysis and then we will see what kind of quantum operators get generated out of it. So in analogy to q' going to q' first we will change things from Lagrangian to Lagrangian density which depends on ϕ and it is all derivative not just on time. And right now we are not going to put any explicit dependency on either position or time. All the position and time dependency come through the dependency of position and time in ϕ. So, there is no direct dependency of Lagrangian density on x and t. All the dependency on x and t come via ϕ and its derivatives. So, the statement we are proposing that just like q' going to q'd, ϕ goes to ϕ'd and x goes to x'd. So, I am doing a transformation x going to x'd such that the functional form of ϕ goes to functional form of ϕ'd. A toy example would be, let us think of a function of ϕ in coordinate x which was x^2. It's plot would be something like this. Now, I do a translation. So, x' which is just x – a. In that case, in the new ϕ'd coordinate system, or x' coordinate system, the plot would look like slightly shifted. This will be 0 of x' coordinate and this was a 0 of x coordinate. This is a function ϕ'd and this is a function of ϕ. So, you see ϕ'd of x'd is just the same thing is just x'd plus a whole^{2}s. So you see not only coordinate has changed, x' and x are not the same thing. Similarly, x^2 and x' plus a whole^2 is not the same thing. So functional form also changes. So we have two fold changes. Once a change in position or position coordinate and then a functional structure of the function also changes. And in general at the same coordinate value, the two functions will not be the same. At same physical point, they will be the same. For example, this is a physical point.Its coordinate value is 0 in one frame and its coordinate value is a in another frame. So, this is the physical point. But it is not true that ϕ'd at $x'd$ is equal to 0 is same as ϕ at x is equal to 0. So therefore, in general, at the same coordinate value, the two functions will not agree to each other. There will be some difference between these two at the same coordinate value. And that difference we are going to write as α times $\partial\phi$. α is just a bookkeeping parameter like perturbation theory which we have done. $\partial\phi$ is essentially the change between the two functions at a given coordinate value x. If we want to keep this transformation as a symmetry transformation, we want that under this transformation of the field, the Lagrangian should change at most by such an element that the equations of motion do not change that in the ϕ' we get the same equations of motion as we were getting in the frame with x and ϕ. This is the similar thing which we did here that q going to q', the equations of motion in q' should be the same as the equations of motion when written in terms of q. So, there it was clear that if I had a Lagrangian which involved only a derivative with respect to time such that the total action was dt integral of L, then The Lagrangian was allowed to change at most by a total derivative of some function of q. In case of field theory, the same analogy if you want to track, the Lagrangian, the action this time is not just the time integral over Lagrangian density, it is the four-dimensional integration over the Lagrangian density. And in order to go to the boundary of this integral, Lagrangian should change by a divergence term. If the Lagrangian changes by a divergence term, that means the total change in the action will be just some α times $\partial_\mu J_\mu$. Now use the divergence theorem that will be equal to the surface integral times J. So if the Lagrangian density changes by a divergence term, I know the action will change at most by a boundary term. And if on the boundary the field values are fixed this boundary term will be under variation will vanish and therefore equations of motion would not change. So the analog of the classical mechanical system where the Lagrangian was allowed to change only up to total derivative now we have a case that the Lagrangian might Lagrangian density rather might change by a total divergence term.



Example: $L = \frac{1}{2} \partial_\mu \partial^\mu \phi$

We do transformation shift of scalars by a constant.

$\phi(x) \rightarrow \phi(x) + \alpha$

$L(x) \rightarrow L(x) + 0$ (That constant must vanish on boundaries hence 0)

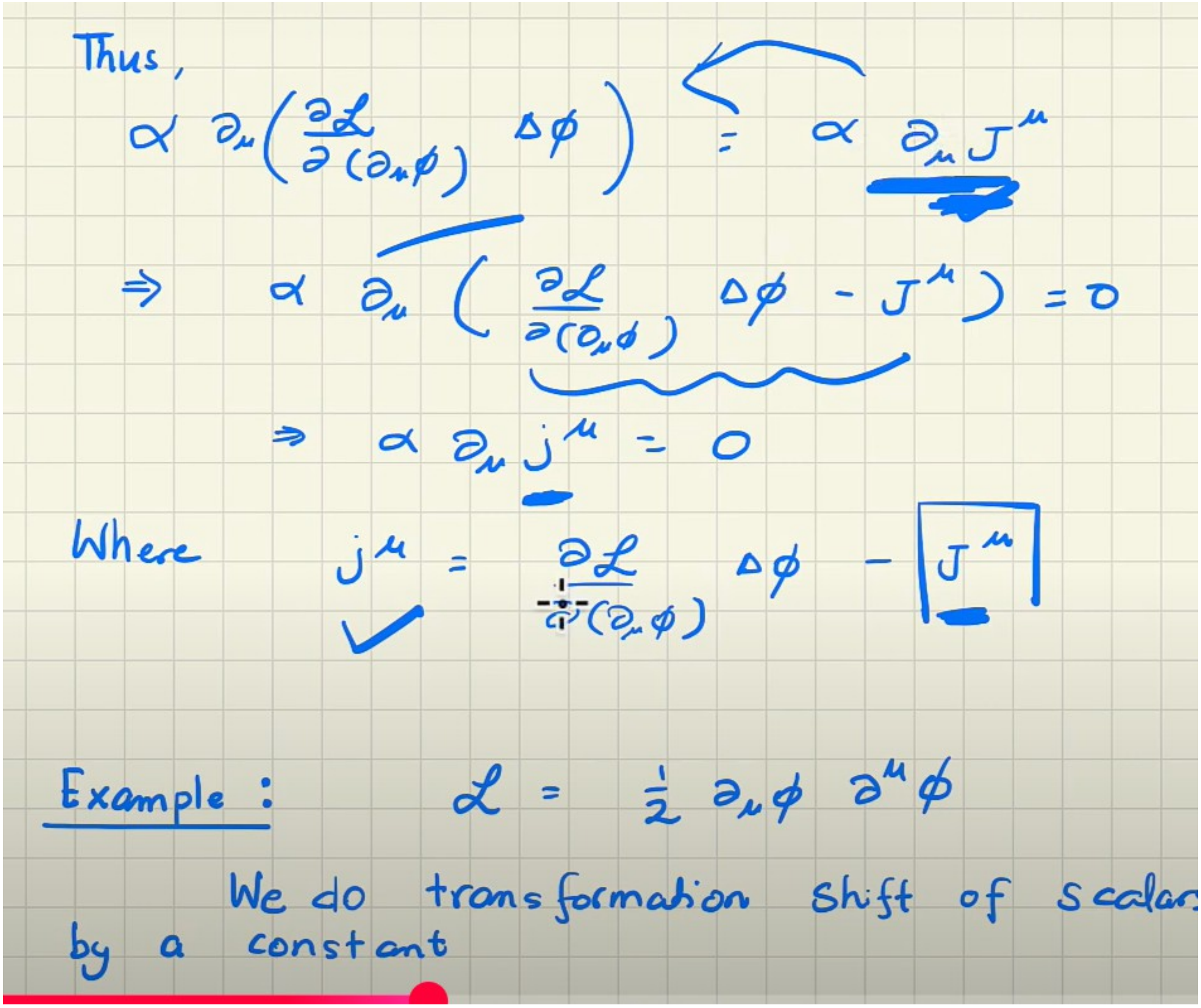
$\partial_\mu J^\mu = 0$

$\frac{\partial L}{\partial(\partial_\mu \phi)} = \partial^\mu$; $\alpha \Delta \phi = \alpha$

$\Delta \phi = 1$

$\therefore \; J^\mu = \partial^\mu \phi$

And this will give rise to the same equations of motion because using divergence theorem we can prove that the action the variation of action rather would not change because at boundary we have kept things fixed. So, in this case we can view the total change in the Lagrangian density to be a divergence term if it is a symmetry transformation then the change in ϕ by thismuch amount remember we had gone to ϕ from ϕ to ϕ' and ϕ' at the same location same coordinate value x was previous ϕ plus some difference term so that is what we have written in the new frame or new coordinate system x 'd. This would be the new field structure, previous field plus some other field and similarly the derivative will be previous derivative plus some other derivative. Now again as we are doing infinitesimal transformation as we discussed this α is going to be very small parameter, book keeping parameter. So we can do a Taylor expansion around the previous field configuration and its derivative. This is a function of two variable kind of thing and I have done some trans changes in both of them. So I can do the Taylor expansion around ϕ and $\partial_\mu\phi$. I will have a first order term in the change in ϕ like this and the first order term in change in the derivative of ϕ like that. So this would be the total change of the Lagrangian which we want to be a divergence of some vector. So first we are trying to ascertain what should be the form of the Lagrangian density change and that we should argue that better that should be a divergence of some term. So we have this change in the Lagrangian therefore Lagrangian density will be therefore this was the previous Lagrangian. So the difference will be this two terms over here. Some α is a bookkeeping parameter as we have discussed. Some α is a bookkeeping parameter as we have discussed. Now we can do the same trick which we had done previously that the second term over here which has a diverge the derivative the derivative of some α times $\partial\phi$, okay. That I can convert as, so we have a ∂L of ∂Times$\partial_\mu\phi$ times ∂_μ of, α I had pulled out, so I am just going to write ∂ϕ. This can be written as a ∂_μ of the whole term, this times $\partial\phi$, which is ∂L upon ∂ of $\partial_\mu\phi$, $\partial\phi$ – the derivative acting on this term ∂_μ of ∂L upon ∂ of $\partial_\mu\phi$ and then ∂ϕ just getting *multiplied*. So, this is what we had written the second term which is so the first term which is ∂_μ times this object is written over here and the second term with a – sign is written over here and this ∂L upon ∂ϕ was already present so i had just restructured the two terms using the total derivative conversion into this form okay so you see one we have doing that we have generated the total Lagrangian density the change in the total Lagrangian density to be made up of the total derivative of something and something times ∂ϕ. And again if I go to the action do this action is d4x of L ∂L that change in the action will automatically project this on the boundary and on the boundary the field configuration change in field configuration is 0. So, this will vanish on the boundary this will go away only this term will survive and that term we also want to be proportional to some J_μ. This already is proportional to some J_μ this can be thought of a J_μ vector.



Thus

$$\alpha\,\partial_{\mu}\left(\frac{\partial L}{\partial(\partial_{\mu}\phi)}\Delta\phi\right)=\alpha\,\partial_{\mu}J^{\mu}$$

$$\Rightarrow\alpha\left(\partial_{\mu}\left(\frac{\partial L}{\partial(\partial_{\mu}\phi)}\Delta\phi\right)-\alpha\,\partial_{\mu}J^{\mu}\right)=0$$

$\Rightarrow\quad \alpha\,\partial_{\mu}J^{\mu}=0$

where $\quad J^{\mu}=\frac{\partial L}{\partial(\partial_{\mu}\phi)}\Delta\phi - J^{\mu}$

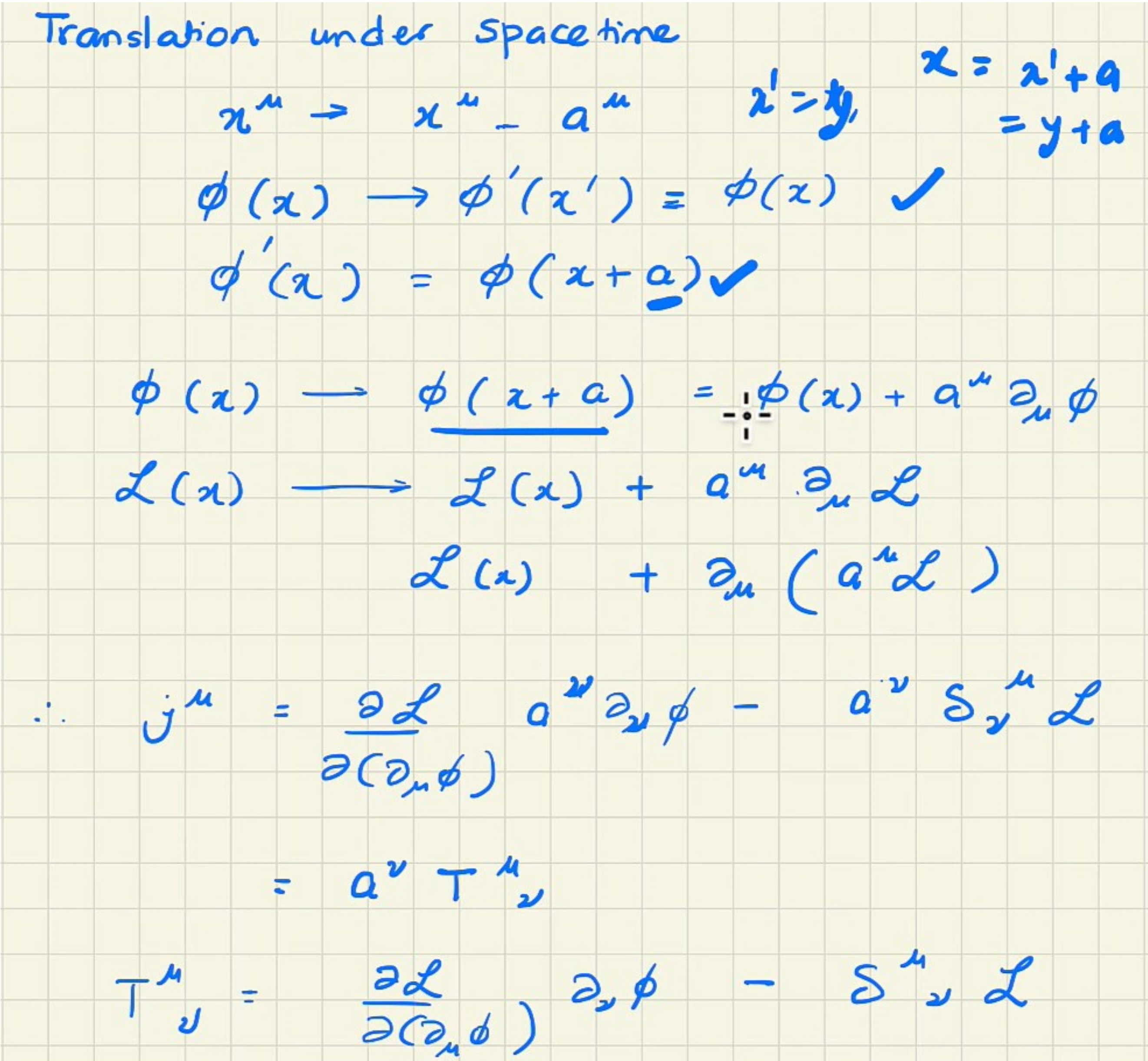
Example:

$$L=\frac{1}{2}\partial_{\mu}\partial^{\mu}\phi$$

We do transformation shift of scalar by a constant.

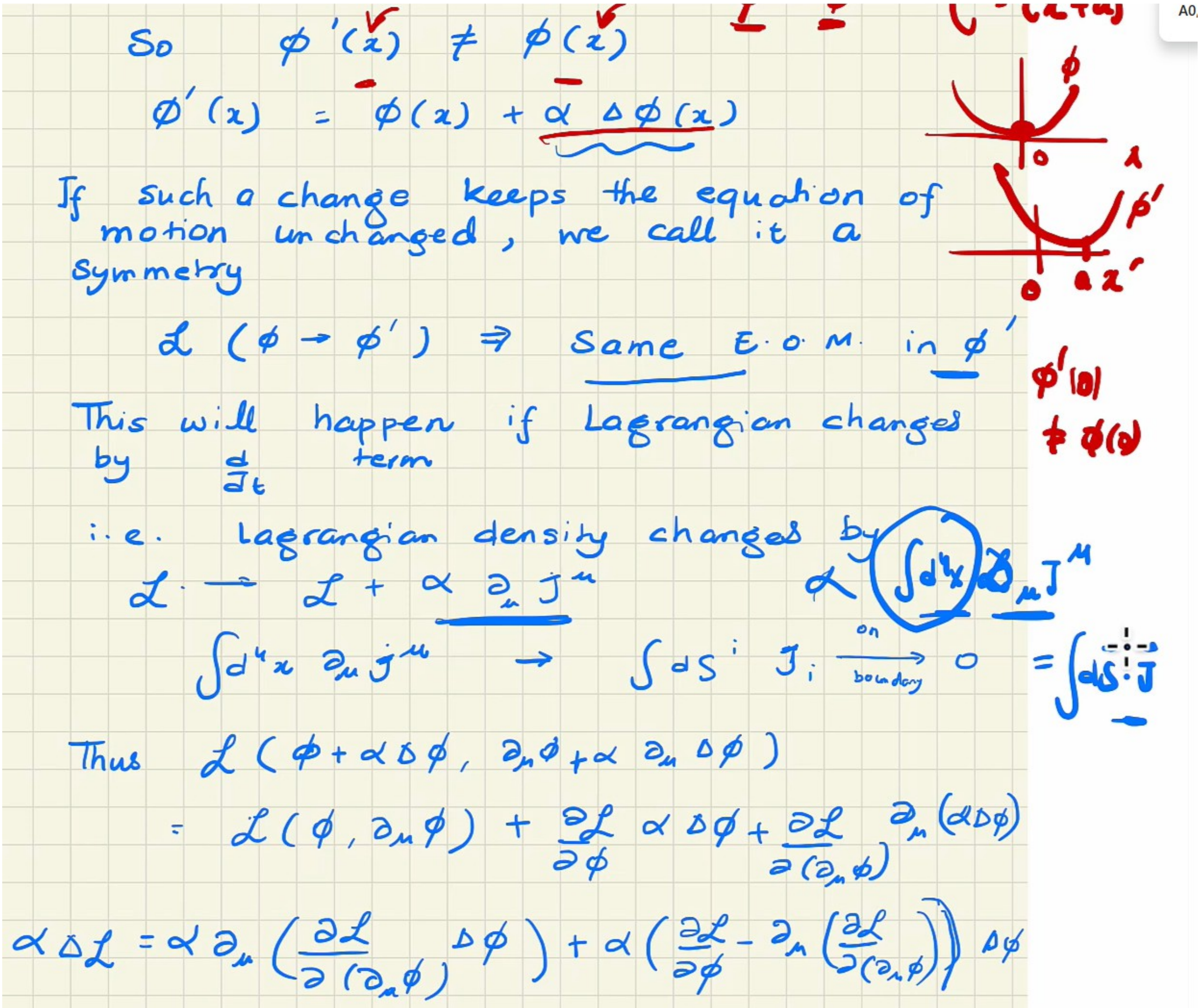
So, this will not bother you. This term is not clear that whether it is total divergence of something or not. But if we want it to be a symmetry transformation, we should note that it should be equal to some divergence of some J_μ. If we force this demand, then what we have, I can bring this term into this side and we will have equation like that and the whole thing can be defined as J_μ the whole thing can be defined as J_μ. So, now this should be 0 if you see that if I demand that to be total divergence that means the divergence of this – that is 0 which I am writing a small J_μ vector.

So the small J_μ is the term which was surviving and – some J_μ which we want it to be. So that is a conserved quantity. The small J_μ is a conserved quantity. So if I know what is capital J_μ, if I can cast it into a form of capital J_μ, I will know automatically. What is the protected quantity. So, you see divergence of J_μ is 0. Previously we wanted the Lagrangian to be divergence full itself, the change in the Lagrangian density to be a total divergence itself. The cost of having that is automatically that this combination is protected. That means its temporal derivative plus a spatial derivative added together is 0. This is like a flux conservation if you remember. That total the time derivative of something plus divergence of some spatial derivative is zero that means a quantity whose temporal structure is changing in time is equal to the flux going out of the boundary. So some conservation law emerges out automatically if we force it to be a total divergence kind of term. Let us see it via an example. That let us think of a most simple Lagrangian which we have discussed till now that Lagrangian is just kinetic term $\partial_\mu\phi\ \partial_\mu\phi$ no potential. And let us do a shift of a scalar the transformation which we are looking for is a shift of a scalar that ϕ all the scalar the ϕ the scalar field will change by this constant α. If I change ϕ by thismuch amount, this is my $\partial\phi$, meaning $\partial\phi$ is hiding here, $\partial\phi$ is 1.



So, if I change my scalar field by a constant, then you can see that the Lagrangian structure is such that the Lagrangian does not change at all. So, ∂L is 0. So, that is what it has automatically emerged out. And therefore, what we have over here, so what we have over here that small J_μ was supposed to be this, the derivative of this with respect to $\partial\phi$ which was $1 - J_\mu$. J_μ was the total change in the Lagrangian density itself. That total change in the Lagrangian density has turned out to be 0 in this case. That means capital J_μ is 0. So that means small J_μ which is going to survive was this first quantity itself. $\partial\phi$ is 1 and we have to compute this derivative. J_μ has been set to 0 because for this example Lagrangian has not changed or we can say that it has changed by a vector 0, divergence of a vector which is 0. So therefore, we have a divergence less vector which has changed the Lagrangian density. So therefore, we have a divergence less vector which has changed the Lagrangian density. So that means this small J_μ which is also this $\partial_\mu \partial_\mu\phi$ is 0. So, this is way we will do first we will formally find out what is the change in the Lagrangian that would be equal to this term α times ∂L and this we will try to equate it to some J_μ divergence of some J_μ and then we will compute this quantity explicitly Lagrangian density change with respect to $\partial_\mu\phi$ *multiplied* with $\partial\phi$ and the difference of this and that is a protected quantity. Okay so let us do one more example to make things clear. And that example will be for space-time translation. So, under space-time translation, the four space-time coordinate will change by some constant. I am writing it as x_μ goes to x'_μ let us say, which is equal to previous x_μ – some translation vector a_μ. This is four-dimensional translation vector in time, it will change by a 0 in spatial direction, it will change by a 1, a 2, a 3. Okay so under this case how would be the field transform field will be ϕ at x just like with the x^2 example it will go to ϕ'd of x′ and that should be equal to ϕ at x remember at the same physical location the field should agree so for example this is a physical location in another coordinate system it was written as this but physical location was this so its coordinate value was a in one frame and zero in another frame so ϕ at zero in the previous frame should be equal to ϕ'd at a in the new frame so therefore at the same physical location they should agree same coordinate locations they would not agree as we had seen previously So this is true under full translation that means ϕ'd at the location x is equal to $\phi(x)$ plus a from this identity you can put since x′d if i put x′d is equal to x that means x meaning some value let us say y then x would be x′d plus a which is y plus a. That means ϕ'd at y is equal to $\phi(y)$ plus a. I am just calling y as x, some variable. Under this ϕ will go to ϕ'd whose functional forms are related from its previous version like this. Again, since for infinitesimal transformation, I can do a Taylor expansion. I would write down ϕ x plus a as the previous function ϕ plus some difference which will just be a Taylor expansion term and a_μ plus $\partial_\mu\phi$. All right.

This is true for any scalar. Any scalar which is function of x should do that. And similarly, the Lagrangian. So, Lagrangian is also ultimately is a function on position space x. So, it should also change by the similar kind of thing where we had written the total derivative of the scalar times a_μ, total derivative of scalar times a_μ. Since a_μ is a constant, it does not depend from point to point, I can take this inside. So, this is the change in the Lagrangian. Remember, this was the ∂L which we talked about. It should better be a divergence of certain vector if it is a symmetry transformation. So that means whatever it has appeared over here, ∂L should be equal to some a_μ times ∂_μ or some parameter α times $\partial_\mu J_\mu$. That is what we have demanded. Now, since there are four symmetry transformation, one in time and three in space, you can think of only time translation, then only a_0 will survive, a_1, a_2, a_3 will be 0 or only a x-dimensional translation, then a_1 will be non-zero, all others will be 0. So, there are four symmetry transformation, a_0, a_1, a_2, a_3. And corresponding to all of them I want my Lagrangian to have some J_1, J_2,J_0, J_1, J_2, J_3, ∂_μ of J_{2_μ} and ∂_μ of J_{3_μ}. So, you see I can write the total change as a summation of all these total divergence terms. Now you can see that comparing this which has changed the total change in the Lagrangian is ∂_μ of a_μ L and I wanted it to be equal to some $\partial_\mu J_\mu$. So that means it is already giving you a hint over what is appearing over here that is your J_μ ultimately appearing for each$_\mu$ over here. So this is just a reflection will tell you that that quantity which is a chronicle $\partial_{\mu\nu}$ times the Lagrangian density should be your J_μ. What I am proposing if I call this quantity as J_μ then you can see that the Lagrangian has changed by $\partial_\mu J_\mu$. For this J_μ which is written over here the total change in the Lagrangian which is over in this circled quantity is nothing but divergence of this J_μ. So, I have identified the piece which can be written as a divergence of total divergence of a vector quantity which is the total change in the Lagrangian. And if it is a symmetry transformation it should better be equal to the indirect change in the Lagrangian which is supposed to come through change in the scalars. Remember for a symmetry transformation these two quantities the indirect change should be equal to the total change in the Lagrangian which is the total divergence of a vector. So therefore, the same thing I do, the total change in the Lagrangian corresponding to the four transformation a_0, a_1, a_2, a_3, you can do one by one or you can do it together that under total changes of all this, I would have a conserved quantity called small J_μ, which is this, which is this quantity, okay. What have we done?



So

$\phi'(x) \neq \phi(x)$

$\phi'(x) = \phi(x) + \alpha\Delta x$

If such a change keeps the equation of motion unchanged, we call it a symmetry.

L($\phi \to \phi'$) $\to$ same EOM in ϕ'

This will happen if Langrangian changes by $\frac{d}{dt}$ term

$$L \to L + \alpha\,\partial_\mu J^\mu \to \int dS \to 0 = \int d\,\vec{S}\cdot\vec{J}$$

Thus

$$L(\phi + \alpha\Delta\phi, \partial_\mu\phi + \partial_\mu\alpha\Delta\phi) = L(\phi, \partial_\mu\phi)\frac{dL}{d\phi}\alpha\Delta\phi + \frac{dL}{\partial(\partial_\mu\phi)}\partial_\mu(\alpha\Delta\phi)$$

$$\alpha\Delta L = \alpha(\partial_\mu(\frac{dL}{\partial(\partial_\mu\phi)}))\Delta\phi + \alpha(\frac{dL}{\partial\phi} - \partial_\mu(\frac{\partial L}{\partial(\partial_\mu\phi)}))\phi$$

$$j^1 = T^1{}_0 = \frac{\partial \mathcal{L}}{\partial(\partial_1 \phi)} \partial_0 \phi - \delta^1{}_0 \mathcal{L}$$

$$= \frac{\partial \mathcal{L}}{\partial(\partial_i \phi)} \partial_0 \phi \checkmark$$

For $\mathcal{L} = -\frac{1}{2} \partial_\mu \phi \, \partial^\mu \phi$ ✓

$$T^i{}_0 = \partial^i \phi \, \partial_0 \phi$$

$$\hat{T}^i{}_0 = \partial^i \hat{\phi} \, \partial_0 \phi = \partial_i \phi \, \partial_0 \phi \checkmark$$

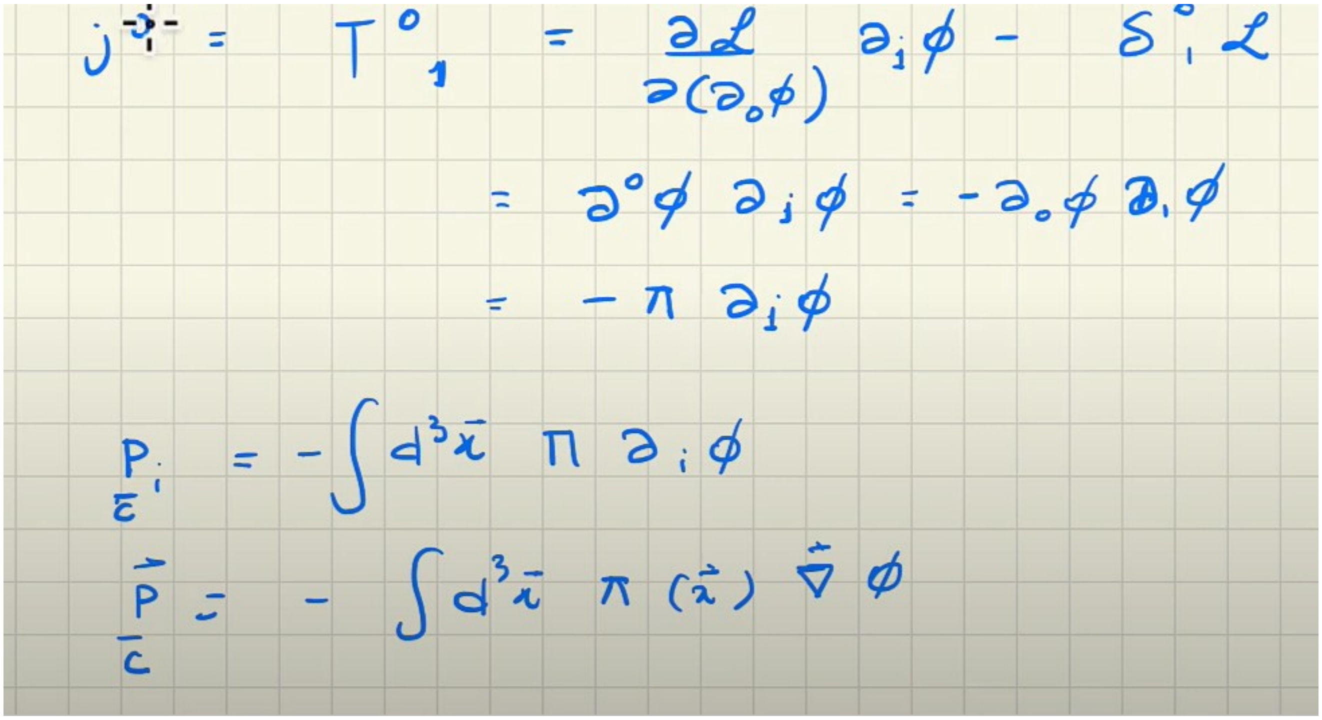
$\{ \eta_{ii} = 1 \}$

- For space translation

$$a^1 = 0 \quad , \quad a^0 = 0 = a^2 = a^3$$

$$j^\mu = T^\mu{}_1$$

$$j^0 = T^0{}_1 = \frac{\partial \mathcal{L}}{} \partial_1 \phi - \delta^0{}_1 \mathcal{L}$$



We did not do only one translation. It is four translations packet. On ealong the temporal directions and three along the spatial directions. And if I demand a symmetry transformation this to be a symmetry transformation for the Lagrangian density the implicit change in the Lagrangian through its dependency on ϕ should better be equal to divergence of some vector and that vector we have identified is this J_μ this$_\mu$ over here is getting summed over this means summation over μ similarly the $_\mu$ over here is also summed over so we have identified looking at the total change in the Lagrangian and looking at the total change in the field what should be the capital J_μ and then we have found out what should be the total J_μ small J_μ which is a conserved quantity. The form of the small J_μ which has emerged out of it has this structure that there are four quantities four symmetry transformations corresponding to each there should be a conserved quantity. So, therefore, there is a four parameter a_0, a_1, a_2, a_3 and corresponding to each v there is a unique J_μ just like $J_0\mu$, $J_1\mu$, $J_2\mu$ and $J_3\mu$. Collectively it is called $T_{\mu v}$. v, the lower$_v$ tells you which translation you are talking about, spatial translation or temporal translation, x translation or Y translation. And the upper T_μ tells you the four-dimensional vector corresponding to that$_v$. For only temporal transformation, only a_0 will survive and all other a_1, a_2, a_3 will be 0. So v will survive with value 0 over here and it has a vector now.$_\mu$ will take value 0, 1, 2, 3. Similarly, if only x directional translation you are talking about, only a_1 will survive. All other a_0, a_2, a_3 will be 0. So I will have a T_μ of 1 and μ can take 4 values. So there are 4 dimensional vectors for each lower index. So therefore, I have a two index quantity together which I can write which is $T_{\mu v}$ and look at this structure. This is the full $T_{\mu v}$ which we are talking about. This whole quantity is written like this. This is an elementary exercise for you to prove that the $T_{\mu v}$ which I am talking about is just this. Your exercise is to verify that given this definitions of $T_{\mu v}$, the total small J_μ which I have written upstairs is just with this $T_{\mu v}$, the total J_μ is this, that you can verify. So, there are four dimensional vectors J_μ, J_{0_μ}, $J1_\mu$, $J2_\mu$, J_{3_μ}, collectively I am calling it $T_{\mu v}$. WTherefore, it is called energy and momentum tensor.Actually you can see that four translations, temporal translation energy is conserved, three spatial translations, momentum are conserved.hy energy momentum tensor, it will be clear in a minute, but these are the four quantities. Actually you can see that four translations, temporal translation energy is conserved, three spatial translations, momentum are conserved. Therefore, it is called energy and momentum tensor. One index will talk about energy and three indices will talk about momentum. So, let us see it explicitly. For example, as we kept discussing, let us think of only temporal transformation, temporal translation. So, a_0 is going to be 1, all other quantities of translation parameters are 0. So, under this case, I know only one of the element a_0 is non-zero and similarly here also only one of the element a_0 will be surviving. All other quantities will become 0 because α 1, α 2, α 3, which is actually a_1, a_2, a_3, they are 0. So, only thing which will be surviving will be a_0 times $\partial_\mu J_{\mu 0}$. So, that is what it is surviving. J_μ 0, effectively the two index object we are going to call t, not a j. So, t_μ 0 will be surviving. That is the conserved quantity, the capital J_μ corresponding to that. Okay. So the capital J_μ is which the stress energy tensor which is of relevance for us is T_μ 0. So the capital J_μ is which the stress energy tensor which is of relevance for us is $T_{\mu 0}$. Now let us see what is the conserved quantity corresponding to meaning the, so the conserved quantity small J_μ is this T_{00}, this T_{00} is remember there is this term And there is this Lagrangian term which is coming about here. This was the capital J_μ. This was the capital J_μ. This has become the small J_μ completely which has got four different values corresponding to different differen T_μ. We are just selecting it for one particular a_0 is equal to one. Rest of them are zeros. So that means we have focused on $T_{\mu 0}$ kind of things what are its component its components are T_{00} and then J_1 small J_1 will be T_{10} small J_2 will be T_{30} and small J_3 will be T_{30}. So, just first focus on what is J_0 which is T_{00}. T_{00} from this definition will be $\partial L/\partial\partial_0\phi$, sorry here should be ∂ of $\partial_0\phi$ times $\partial_0\phi$ because μ and ν both have to be 0 and right hand side the second term will be $\partial_0 0$ talagrandian density. Remember this quantity over here we have seen it before that was the conjugate moment of Π. I have a pi times $\partial_0\phi$ – the Lagrangian density coming from here which we know by now is just the name of the Hamiltonian. So you see if I have a only time translation for the Lagrangian which we had written down the Lagrangian was just $\partial_\mu\phi\ \partial_\mu\phi$ let us say for that Lagrangian I would have a quantity which is energy total Hamiltonian or actually for this case we do not have to specify what is the Lagrangian only thing its kinetic term should be $\partial_\mu\phi\ \partial_\mu\phi$.

If the Lagrangian density has any structure – $V(\phi)$ as well then also this statement will go through that you have this object first derivative times $\partial_0\phi$ is π times $\partial_0\phi$ and this Lagrangian will involve ϕ as well. T_{00} will be the Hamiltonian. So, therefore under time translation a conserved quantity is present in the field theory picture as well which is Hamiltonian. Remember this is classical field theory yet we have not quantized anything as of now. You can try to find out what are the T_{i0} that means T_{10}, T_{20}, T_{30}. This is an exercise for you and I am just giving you the formula which you have to obtain. You will just get $\partial L/\partial_i\phi$ and then $\partial_{i0}L$. This will vanish because Kronecker δ survives only for $\partial_0 0$. Only the first term will remain. And for a given Lagrangian, you should be tempted to find out what is that a conserved quantity which we are talking about. Okay.

So, just do this exercise and you will find out that T_{i0} is $\partial_i\phi\ \partial_0\phi$ and T the quantum operator corresponding to that. Now, you can make all these things as quantum operators. But anyway, that is not very relevant as of now. Quantization we can do. We know how to do quantizations. All the ϕ's and pi's will be converted into operators. But first we are looking for their structure. So there are conserved quantity J_μ whose components are T_{i0} and T_{00} . T_{00} we have identified as the Hamiltonian density. T_{i0} is not quite something which we have seen before but it is something like a flux. Remember $T_{\mu 0}$ is supposed to satisfy this quantity. This equation that is to say $\partial_0 T_{00}$ should be equal to $\partial_i T_{i0}$. So, this is the flux of something being carried out. This has this structure remember ∂T of something plus divergence of something else is equal to 0. That means this quantity is rate of change is equal to – of the divergence of spatial divergence of something. That means this is a flux conservation equation. So this is some sort of a flux T_{i0} is some sort of a flux which we are talking about. So this is the structure which would emerge if I demanded only the temporal translation. Similarly, if I do the spatial translation along, let us say, one direction. So, this should be 1. So, that means a_1, let us say, along the x direction we are talking, the spatial translation. That means a_1 will be 1 and all other a_0, a_2 and a_3 will be 0. That means a_1 will be 1 and all other a_0, a2 and a3 will be 0. In this case, the conserved quantity small J_μ will be T_μ of 1. Remember, the$_\nu$ was talking about the$_\nu$ was talking about which direction translation we are talking about. If we are talking$_\nu$ is equal to 1 is surviving all other ν is Aν is 0 then only one quantity will survive which is T_μ 1. So, your conserved quantity is $T_{\mu 1}$. Again you can find out it has four component T_{01}, T_{21}, T_{11}, T_{21} and T_{31}. So, the zeroth component of that is T_{01}. Look at the definitions of $T_{\mu\nu}$ which we had given above. You have to put 0 and 1. This$_\nu$ should be 0 and ν should be 1 and you will get this quantity over here. This is $T_{\mu 1}$, sorry it should have been the other way, this should be$_\mu$, no that is fine, this should be μ was supposed to take value 0, so here it is 0 and μ was supposed to take value 1 which is over here and this will become 0 because of a *Kronecker delta* function. So, you will get $\partial_0\phi$, $\partial_i\phi$ which is this. Which $\partial_0\,\phi$ can be converted into lower $\partial_0\phi$ with a – sign. Remember ∂_μ of some function is equal to $\eta^{\mu\mu}$ where summation over μ is implied of $\partial^\mu f$. So, ∂_μ of ϕ is η_{00} of $\partial_0\phi$ upper ϕ. So, therefore I have a – sign. So, remember this $\partial_0\phi$ was again a Π. And this $\partial_1\phi$ I am just writing as $\partial_1\phi$ so if we are insistent on calling the conserved quantity along due to the spatial translation as a momentum then the momentum which we have obtained over here is this this is the definition of momentum there has to be a c parameter which is coming along because of the unit c is I am working in unit c is equal to 1 as we had discussed previously. But ultimately you would have this definition of momentum. This is the conserved current, this is the conserved current which is conserved quantity which is present due to symmetry along space translation. Previously we obtained The Hamiltonian was a conserved quantity, ∂_0 was a conserved quantity. So, just to write one thing, if you get something like $\partial_\mu J_\mu$, some J_μ is equal to 0, this is equal to $\partial_0 J_{0+}$ divergence of J, small J is equal to 0. If I do the d^3x integral of the whole equation. So then I will have a ∂_0 which is time derivative d^3x of J_{0+} or let us take it on the right hand side this is – of d^3x divergence of J again use gauss's divergence theorem it will project you at the boundary of this volume which is at infinity that is J dot ds at r tending to infinity. And if the J has the structure that it vanishes at infinity, right hand side becomes 0. That means the integral of J_0 does not change in time, its temporal derivative is 0. So, that is why t00 integral over d^3x will be a conserved quantity, that is the total Hamiltonian. Similarly, for spatial translation, the spatial translation the J_0 which is T_{01} its integration over d^3x will be a conserved quantity in time that will be the momentum and the momentum therefore will be the d^3x of this J_0. So, pi times $\partial_i\phi$ $\hbar\omega$ d^3x is the quantity which can be called as a conserved quantity corresponding to spatial translation. And therefore, it is the momentum, the momentum which we identify as a symmetry, the conserved quantity corresponding to space translation symmetry. Lastly, to look at how this operator looks in the quantum domain. For example, corresponding to time translation, the conserved quantity was the Hamiltonian, which was integration of T_{00} over the spatial volume, d^3s. The Hamiltonian operator we have already seen in the previous classes, that it would be just a dagger a times $\hbar\omega$ + *1/2* ∂_0. That was the Hamiltonian for scalar field of the Lagrangian, which we had written before. Now, we have landed up with a new operator, which is this new conserved quantity, which is this. So, I know how to write ϕ in terms of the a and a dagger and I know how to write Π in terms of a and a dagger. So, let us use that and write down those two quantities. Remember the ϕ and Π both will come with the integral quantityd^3k d^3k' 1 for ϕ and 1 for Π.

$$\frac{\hat{P}_i}{c} = -\hbar \int d^3\vec{x} \int \frac{d^3\vec{k}}{\sqrt{2\omega_k}} \int \frac{d^3\vec{k}'}{\sqrt{2\omega_{k'}}} \left(\frac{\omega_k}{ic}\right)\left(ik_i'\right) \left(\hat{a}_k e^{-i\omega_k t + i\vec{k}\cdot\vec{x}} - \hat{a}^\dagger_k e^{+i\omega_k t - i\vec{k}\cdot\vec{x}}\right)\left(\hat{a}_{k'} e^{-i\omega_{k'} t + i\vec{k}'\cdot\vec{x}} - \hat{a}^\dagger_{k'} e^{+i\omega_{k'} t - i\vec{k}'\cdot\vec{x}}\right)$$

$$= \frac{-\hbar}{ic} \int \frac{d^3\vec{k}}{\sqrt{2\omega_k}} \int \frac{d^3\vec{k}'}{\sqrt{2\omega_{k'}}}\, \omega_k\, ik_i' \int d^3x \Big(\hat{a}_k \hat{a}_{k'} e^{-i(\omega_k+\omega_{k'})t} e^{i(\vec{k}+\vec{k}')\cdot\vec{x}} - \hat{a}_k \hat{a}^\dagger_{k'} e^{-i(\omega_k-\omega_{k'})t + i(\vec{k}-\vec{k}')\cdot\vec{x}} - \hat{a}^\dagger_k \hat{a}_{k'} e^{i(\omega_k-\omega_{k'})t - i(\vec{k}-\vec{k}')\cdot\vec{x}} + \hat{a}^\dagger_k \hat{a}^\dagger_{k'} e^{-i(\omega_k+\omega_{k'})t} e^{-i(\vec{k}+\vec{k}')\cdot\vec{x}} \Big)$$

Using $\int d^3x \, e^{i(\vec{k}\pm\vec{k}')\cdot\vec{x}} = \delta(\vec{k}\pm\vec{k}')$

We have

$$\frac{\hat{P}_i}{c} = \frac{\hbar}{c} \int \frac{d^3k}{2} k_i \Big[\hat{a}_k \hat{a}_{-\vec{k}} e^{-2i\omega_k t} + \hat{a}_k \hat{a}^\dagger_k + \hat{a}^\dagger_k \hat{a}_k + \hat{a}^\dagger_k \hat{a}^\dagger_k e^{2i\omega_k t} \Big]$$

$$= \frac{\hbar}{c} \int d^3k \; k_i \Big[\hat{a}^\dagger_k \hat{a}_k + \frac{1}{2}\delta(0) \Big]$$

$$\frac{\vec{P}_i}{c} = -\hbar \int \frac{d^3\vec{k}}{\sqrt{2\omega_{\vec{k}}}} \int \frac{d^3\vec{k}'}{\sqrt{2\omega_{\vec{k}}'}} \frac{\omega_{\vec{k}}}{ic} iK'_i \left(\hat{a}_k e^{-\omega_k t + i\vec{k}\cdot\vec{x}} - \hat{a}^\dagger_k e^{+\omega_k t + i\vec{k}\cdot\vec{x}}\right)$$

=

$$\frac{-\hbar}{ic} \int \frac{d^3\vec{k}}{\sqrt{2\omega_{\vec{k}}}} \int \frac{d^3\vec{k}'}{\sqrt{2\omega_{\vec{k}}'}} \omega_{\vec{k}} iK'_i \int d^3x \left(\hat{a}_k \hat{a}_k' e^{-i(\omega_k+\omega_k')t} e^{i(\vec{k}+\vec{k}')\vec{x}} - \hat{a}_k \hat{a}^\dagger_k e^{-i(\omega_k-\omega_k')t + i(\vec{k}-\vec{k}')\vec{x}} - \hat{a}^\dagger_k \hat{a}_k' e^{-i(\omega_k-\omega_k')t - i(\vec{k}-\vec{k}')\vec{x}} + \hat{a}^\dagger_k \hat{a}^\dagger_k{}' e^{-i(\omega_k+}\right.$$

Using $\int d^3x \, e^{i(\vec{k}\pm\vec{k}')\cdot\vec{x}} = \delta(\vec{k}\pm\vec{k}')$

We have

$$\frac{\vec{P}}{c} = \frac{\hbar}{c} \int \frac{d^3k}{2k_i} k_i \left[\hat{a}_k \hat{a}_{\vec{k}} e^{-2i\omega_{\vec{k}} t} + \hat{a}_k \hat{a}^\dagger_k + \hat{a}^\dagger_k \hat{a}_k + \hat{a}^\dagger_k \hat{a}^\dagger_k e^{2i\omega_{\vec{k}} t}\right]$$

$$= \frac{\hbar}{c}\int d^3k\, k_i[\hat{a}_k^\dagger \hat{a}_k+\frac{1}{2}\delta(0)]$$

The special derivative of ϕ has to be taken that will throw up i times $k_{i.}$ So this will give me a k_i'. So I am writing ϕ as integration $\frac{d^{3k\,\prime}}{\sqrt{2\omega_{\vec{k}}}}$ and $\hat{a}_k\ \hat{a}_k^\dagger$. So if I do that, I would get $\frac{i}{k_i}$ after taking the spatial gradient operator. If I do the spatial gradient operator, I would get a iK. Okay and then the pi I have to write down the pi I have to write down the Π would be just ω_k/ic times this quantity $\hat{a}_k\,\hat{a}_k^\dagger$. So, remember when I take the gradient of ϕ This part will give me e to the power plus ik and this part will give me $-\ ik$. So, there will be a relative sign difference. So, both the terms ϕ and ∂ gradient of ϕ, ϕ and gradient of ϕ, both of them will have this structure, a relative sign between a and a dagger. This is simple algebra, you should try doing once, write down what is ϕ and write down what is gradient of, write down what is pi and write down what is the gradient of ϕ and you will get this kind of structure. Okay so you do the algebra slightly carefully you will find out that the d^3x of that has to be done because remember the momentum which is a conserved quantity which does not change in time is rather d^3x of the T_{01} okay so that spatial integral has also be has to be done that spatial integral so there is a x dependency here there is x dependency here here here. So you will have a product terms. This will multiply among, this will multiply to this, this will multiply to that, this will multiply to the first term, this will multiply to the second term. There will be four different multiplication. So I have written four differentmultiplication terms over here. All of them will have some exponential times $k + k'\dot{x}$ or $k - k'\dot{x}$ and so on. And there is a d^3x integral which is just over here. Again I will just urge you to do it slightly more carefully than what I am giving here. You can just verify whatever I am going to write over here whether you get it or not. So you will have Special integral I am going to do first and the special dependency is only in the exponential function here, here, here and here. And all of this integration are going to give me either of Dirac $\delta(k + k')$ or $\delta(k - k')$. You can see the structure. This will give you $k + k'$. This will give you $k - k'$. This will give you $k - k'$ and this will give you k plus k$'$. So it would be either equal to $k + k'$ *or* $k - k'$. And remember the identity which we had previously $\hat{a}_{-k} = \hat{a}_k^\dagger$ for real scalar field. Use those identity as well and you will get this kind of operator at your hand directly. And here you will see that you have a structure which is these integrals over here which are surviving over after doing the simplification they will just vanish because of odd integral structure these integrations this parts are symmetric under k going to – k while this is odd function this will vanish but for these two terms they will not vanish under integral and these will survive and you can figure out that the momentum integral the momentum operator which you are talking about is just this operator $\hat{a}_k^\dagger\ \hat{a}k$ times ki remember this was $\hbar\omega$ if it had been it would have become the hamiltonian operator but this is not $\hbar\omega$ so it is $\hbar k_i$ okay so. In this case again you can see that this is a vector component x component or y component or z component depending upon i. This is symmetric under k going to $-\ k$. So again the second part of this integral will vanish only the first part will survive.

This is not symmetric under k going to – k. So therefore ultimately this appears for a while but under integration vanishes. So at the end of the day you are left with pi upon c is equal to $\hbar/c$ times this. So, that tells you that momentum operator pi is equal to $\hbar$ times this integral d^3k and then ki, the ith component, forget about the vector sign, ith component of momenta is $\hat{a}_k^\dagger\hat{a}k$. It just counts how many excitations are there in which mode and associates a momenta $\hbar k$ to that mode and sums over all the modes. This is how you count, you calculate the momentum of a field. Total momentum of a field operator is this. Take its expectation in vacuum state, you will find out that is zero. Take its expectation in one k_0 state, you will find out that it is $\hbar_0$. k_0's ith component. This is an exercise for you. You can verify that it only counts about what is the mode present, which mode is present, how many of them and associates those many k i's to it. In this state only one excitation is present in k_0 mode and therefore it gives you $\hbar k_{0i}$. So, therefore this is the momentum operator which we will get after doing this computation in different states. Okay, so I wrap up the discussion on the scalar field and expectation. In the cases when we start dealing about interactions of atom with fields, we will see more examples of what kind of operators naturally appear. But now we are familiar with how to convert those operators in terms of $\hat{a}k$ and $\hat{a}_k^\dagger$, what are the conserved quantities. Using those we will compute things in later classes. So I wrap up the discussion on scalar field here. I will just briefly introduce the Dirac field structure which is very similar to this and light which is electromagnetic field structure which is also close by in couple of lectures coming from the next discussion session onwards to just give you complete glimpse of what kind of fields are present in nature. And their structure one has to be slightly careful about their spinorial structure or vector structure. But more or less central physics will revolve around this that you have to correctly identify what is the field operator and what are the other relevant operators which can be made from the field operators. So we will see those things coming up for Dirac fields in the next class onwards and later on for electromagnetic field.