Foundation of Quantum Theory: Relativistic Approach
Quantum Field Theory 2.1
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Field and Expectations
Lecture- 19

So, today we will move towards understanding the quantization of other fields.

Quantum field as oscillators

$$\hat{\phi}(x,t) = \int d^3\vec{k}\; \hat{\phi}(\vec{k},t)\, e^{i\vec{k}\cdot\vec{x}}$$

$$\ddot{\hat{\phi}}_{\vec{k}} + \omega_k^2\, \hat{\phi}_k = 0 \qquad \left\{ \; \omega_{\vec{k}} = \sqrt{k^2 c^2 + \frac{m_0^2 c^4}{\hbar^2}} \right.$$

$$\hat{\phi}_{\vec{k}} = \sqrt{\frac{\hbar}{2\omega_k}}\left(\hat{a}_{\vec{k}}\, e^{-i\omega t} + \hat{b}_k\, e^{+i\omega t} \right) \qquad \left\{ \begin{array}{l} \hat{\phi}(x,t) = \hat{\phi}^{\dagger}(x,t) \\ \hat{\phi}(\vec{k},t) = \hat{\phi}^{\dagger}(-\vec{k},t) \end{array} \right.$$

$$\hat{\phi}_{\vec{k}}^{\dagger} = \hat{\phi}_{-\vec{k}} \qquad \Rightarrow \qquad \hat{a}_k^{\dagger} = \hat{b}_{-k}$$

$$\therefore \quad \hat{\phi}_k = \sqrt{\frac{\hbar}{2\omega_k}}\left(\hat{a}_k\, e^{-i\omega_k t} + \hat{a}_{-k}^{\dagger}\, e^{i\omega_k t} \right)$$

In that case

$$\hat{\Pi} = \hbar^2\, \partial_0 \phi = \hbar^2 \int d^3\vec{k}\, (\partial_0 \phi)\, e^{i\vec{k}\cdot\vec{x}}$$

$$= (\hbar^2)\sqrt{\hbar} \int d^3 k \; \frac{e^{i\vec{k}\cdot\vec{x}}}{\sqrt{2\omega_{\vec{k}}}} \left(-i\frac{\omega_k}{c}\, \hat{a}_{\vec{k}}\, e^{-i\omega_{\vec{k}} t} + i\omega_k \frac{\hat{a}_{-\vec{k}}^{\dagger}}{c}\, e^{+i\omega_{\vec{k}} t} \right)$$

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$$\phi(\vec{x},t)=\sqrt{\hbar}\int\frac{d^3\vec{k}}{\sqrt{2\omega_{\vec{k}}}}\left(\hat{a}_{\vec{k}}\,e^{-i\omega_{\vec{k}}t}+\hat{a}_{\vec{k}}^\dagger\,e^{+i\omega_{\vec{k}}t}\right)e^{i\vec{k}\cdot\vec{x}}$$

$$\Pi(\vec{x},t)=\frac{\hbar^{5/2}}{ic}\int\frac{d^3k}{\sqrt{2\omega_{\vec{k}}}}\,\omega_{\vec{k}}\left(\hat{a}_{\vec{k}}\,e^{-i\omega_{\vec{k}}t}-\hat{a}_{\vec{k}}^\dagger\,e^{i\omega_{\vec{k}}t}\right)e^{i\vec{k}\cdot\vec{x}}$$

Real $\phi(\vec{x},t)$ and $\Pi(\vec{x},t)$ need

$$\hat{a}_k=\hat{a}_{-k}\;;\quad\hat{a}_k^\dagger=\hat{a}_{-k}^\dagger$$

Quantum field as oscillators

$$\hat{\phi}(x,t)=\int d^3\vec{k}\,\hat{\phi}(x,t)\,e^{i\vec{k}\cdot\vec{x}}$$

$$\ddot{\hat{\phi}}_{\vec{k}}+\omega_k^2\hat{\phi}_k=0\qquad\qquad\left(\omega_{\vec{k}}=\sqrt{k^2c^2+\frac{m_0^2c^4}{\hbar^2}}\right)$$

$$\hat{\phi}_{\vec{k}}=\sqrt{\frac{\hbar}{2\omega_{\vec{k}}}}\left(\hat{a}_{\vec{k}}\,e^{-i\omega_{\vec{k}}t}+\hat{b}_{\vec{k}}\,e^{i\omega_{\vec{k}}t}\right)\qquad\left(\begin{array}{l}\hat{\phi}(x,t)=\hat{\phi}^\dagger(x,t)\\[4pt]\hat{\phi}(\vec{k},t)=\hat{\phi}^\dagger(-\vec{k},t)\end{array}\right.$$

$$\therefore\ \hat{\phi}_{\vec{k}}^\dagger=\hat{\phi}_{\vec{k}}^\dagger\Rightarrow\hat{a}_{\vec{k}}=\hat{b}_{-k}$$

$$\therefore\hat{\phi}_k=\sqrt{\frac{\hbar}{2\omega_k}}\left(\hat{a}_k\,e^{-i\omega_kt}+\hat{a}_k\,e^{i\omega_kt}\right)$$

In that case
$$\hat{\Pi}=\hbar^2\partial_0\phi=\hbar^2\int d^2\vec{k}\left(\partial_0\phi\right)e^{i\vec{k}\cdot\vec{x}}$$

$$=\quad(\hbar^2)\sqrt{\hbar}\int\frac{d^3\vec{k}}{\sqrt{2\omega_{veck}}}\,e^{i\vec{k}\cdot\vec{x}}\left(\frac{-i\omega_k}{c}\hat{a}_{\vec{k}}\,e^{-i\omega_{\vec{k}}t}+\frac{i\omega_k}{c}\hat{a}_{\vec{k}}^\dagger\,e^{+i\omega_{\vec{k}}t}\right)$$

$$\hat{\phi}(x,t) = \sqrt{\hbar} \int \frac{d^3\vec{k}}{\sqrt{2\omega_{\vec{k}}}} \left(\hat{a}_{\vec{k}} e^{-i\omega t} + \hat{a}_{\vec{k}}^\dagger e^{i\omega t} \right) e^{i\vec{k}\cdot\vec{x}}$$

$$\hat{\Pi}(x,t) = \frac{\hbar^{3/2}}{ic} \int \frac{d^3\vec{k}}{\sqrt{2\omega_{\vec{k}}}} \omega_{\vec{k}} \left(\hat{a}_{\vec{k}} e^{-i\omega t} - \hat{a}_{\vec{k}}^\dagger e^{i\omega t} \right) e^{i\vec{k}\cdot\vec{x}}$$

Real $\phi(\vec{x},t)$ and $\Pi(\vec{x},t)$ need

$$\hat{a}_k = \hat{a}_{-k} \; ; \; \hat{a}_k^\dagger = \hat{a}_{-k}^\dagger$$

For example, we have discussed quantization of scalar fields, but in today's lectures onwards we will try to go to different kinds of fields. For example, Dirac fields which talks about fermions and later on in coming week we will be discussing about the gauge field quantization which is the electromagnetic waves or electromagnetic light quantization.

And in order to do that, we will just set the platform with exercises on scalar field itself. Since we have done the scalar field quantization of fresh, so things are much more clear in our head to go ahead with the computations. And in order to facilitate the computations with ease I will just point out certain things which i should have done in the previous classes but we missed out but anyway it is a good point here to clarify those ideas and move forward so remember how did we achieve the quantization of the scalar field first what we did we anticipated that the operator ϕ which gives me operators at each point of a space time and at all time so there is a operator field operator which gives you a scalar operator at all points of space time. That we anticipated as a time dependent operator in the Fourier space. So, this is how we anticipated how the operators will be talking, meaning how the operators in real space and Fourier space are related via a Fourier transform. Now, what it happens that this operator in the position space was supposed to satisfy the Klein-Gordon equation and in order to do that the Fourier counterpart of that which is $\phi(k)$ of t would be satisfying this differential equation which is the equation of motion of a harmonic oscillator if you recall. So, that is how we understood that the real time operator a free field operator in position space is a collection of harmonic oscillator in Fourier space put together. This Fourier transformation is just a limit of sum. So, these are infinitely many oscillators together constituting the position space operator. So far so good. That is what how we had proceeded forward to learn about the quantization of the field. Then we quantize this harmonic oscillator like equation just like we quantize harmonic oscillators. That means we wrote down $\phi(k)$ as set of $\hat{a}k$ and $\hat{a}_k^\dagger$ in order to make these $\phi(k)$'s just like oscillators. However, at this stage, we should have been careful, which I had not alerted you people in that point. But let me alert at this stage that when I am doing this and mâking an analogy with position space oscillator x double dot ωsquare x is equal to 0, on equation level they are looking exactly the same thing. However, remember when we wrote down x as $\sqrt{h}/2\omega$ times $\hat{a}k\, e^{-i\omega t}$ not k, there was no k for positional space. So, a $e^{i\omega t}a^\dagger e^{i\omega t}$. This way when we write, this is an inherent writing for a operator x which not only satisfies the differential equation for a harmonic oscillator, but is also a Hermitian operator. So, x is supposed to be $x^\dagger$. That is why this a and $a^\dagger$ with complex conjugation factors coming together with them has appeared. This ensures, this kind of decomposition

ensures that the related operator x is a Hermitian operator. Do we need to demand that for the field operator in Fourier space? That is not very clear that we should be applying to that for field operator in the Fourier space. I would know that the field operators in position of space should rather be Hermitian operators. So, what I want is ϕ in position space should be equal to its † in position space that does not mean that its Fourier counterpart will also be Hermitian. So, $\phi(k)^\dagger$ this is not implied if this is demanded this will not be true actually if you can see that if you demand that ϕ is equal to its † in position space, its Fourier counterpart satisfy a certain relation that $\phi(k)$t is $\phi^\dagger$ of – kt. So, it is not just plain simple conjugation it undergoes, k goes to – k under † operation. So, the $\phi(k)$ operator is not hermitian if we demand $\phi(x)$ operator to be hermitian and that is the cost we have to pay and we have to live with that's why now on i would not write $\hat{a}k + \hat{a}_k^\dagger$ because that was ensuring hermeticity of the operator since i do not want hermeticity in Fourier space i want this relation in Fourier space so i should find out what should satisfy what should be bk whether it should be $\hat{a}_k^\dagger$ or something else in order to make the position space operators Hermitian. So, that is what we are going to do that I am going to enforce this demand that $\phi(k)^\dagger$ is equal to ϕ of – k. That even work out easily will ensure that b of – k, b of – k over here is equal to $\hat{a}_k^\dagger$.

That means b k is a – k†. So, this is the choice we can move forward with. So, we have the statement that for real space, real scalar field, there could be different kinds of scalar field, complex scalar field, real scalar fields, which have different properties. For real scalar fields, this property is essential, that means $\phi(k)$ will become this. This $\phi(k)$ which was sitting over here is going to become this object where $\hat{a}k$ will be coming with $e^{-i\omega t}$ and $\hat{a}k$ a of – k† will be coming with $e^{+i\omega kt}$. And then the business as usual whatever we have done for operators moment, conjugate momenta and other things that should follow from there onwards. So, the cost of working with real scalar field over here is this. I will write down the Π like before which would be the derivative operator on ϕ and it will again be giving me some – i ω/c kind of things which this time only you have to be careful that $\hat{a}_k^\dagger$ has been replaced by a of – k†. Previously what we were writing for was a Fourier space Hermitian $\phi(k)$ that is not a real $\phi(x)$ which would be some complex scalar field but that is not what we want so I am specializing for real scalar field in position space so the cost would be wherever $\hat{a}_k^\dagger$ was appearing in the previous thing a of – k† will start appearing over there so this therefore will become my decomposition structure for real scalar field so whatever we did in the previous classes you have to just correct your nodes rather not correct your node but specialize your nodes for a real scalar field the decompositions becomes something like that. And one more thing what you can play a trick upon, let us look at this term and this term. The first term is $\hat{a}k\, e^{-i\omega_k t}\, e^{ik(\hat{x})}$ and there is a outside integration d^3k and $\sqrt{2\omega_k}$. The second term is similarly $d^3k\, \sqrt{2\omega_k}$, a – k$^\dagger\, e^{-i\omega_k}{}^{+i\omega kt}$ rather and $e^{ik(\hat{x})}$. Now, what you can do in this thing, in the second piece, what you can do is do a variable transformation k going to – k. Under k going to – k, you know d^3k goes to d^3k. it does not change at all, meaning all the limits go from – infinity to infinity kxky kz still. And $\sqrt{2\omega_k}$, ω_k is,ω_k depends on the vector k through its magnitude, through its magnitude.

So, it would not change at all. So, therefore, only thing it will happen under this k going to – k, the vector k going to – vector k, this integration can be written as $\hat{a}_k^\dagger$ now, previously it was a – k of † then i + i ωt $^{-ik(\hat{x})}$. So, I have just flipped the sign of k everywhere and therefore, this will therefore become the full thing will become $\hat{a}k\, e^{-i\omega ktik(\hat{x})} + \hat{a}_k^\dagger$ and the complex conjugate of this function which is appearing over here. Similarly for momentum operator as well. So here when I put a common $e^{ik(\hat{x})}$, then I would work with a – k†, a – k† in both sides. Alternatively, I can just bring these things in here and here and do a variable flip in the second integral so that I will have a $\hat{a}k$ times something $+ \hat{a}_k^\dagger$ times complete conjugate of that and so on. So, do not get surprised if you see such things appearing in my notes as well that I would write ϕ, real scalar field ϕ as either this decomposition in the box I have written over here or $\hat{a}k\, e^{-i\omega t}{}^{+ik(\hat{x})} + \hat{a}_k^\dagger\, e^{+i\omega t}{}^{-ik(\hat{x})}$. Both are same thing for real scalar field. So, let us move forward with this understanding. I will just summarize in terms of how to write Hamiltonian for real scalar field and other things and then we will move on to do some exercises.

Then, $\mathcal{H} = \int d^3\vec{k}\ \hbar\omega_{\vec{k}} \left(\hat{\tilde{a}}_{\vec{k}}^{+} \hat{\tilde{a}}_{\vec{k}} + \frac{1}{2}\delta^3(0) \right)$ ✓

Thus, a useful set of variables leading to harmonic oscillator uantization is

$$\mathcal{L} = -\frac{1}{2}\left(-\frac{1}{c^2}(\partial_t\phi)^2 + (\nabla\phi)^2 - \frac{m^2c^2}{\hbar^2}\phi^2 \right)$$

$$\phi(\vec{x}) = \sqrt{\hbar} \int \frac{d^3\vec{k}}{\sqrt{2\omega_k}} \left(\hat{a}_k\, e^{-i\omega_{\vec{k}}t + i\vec{k}\cdot\vec{x}} + \hat{a}_k^{+}\, e^{i\omega_{\vec{k}}t - i\vec{k}\cdot\vec{x}} \right)$$

$$\Pi(\vec{x}) = \frac{\sqrt{\hbar}}{i} \int d^3\vec{k}\ \sqrt{\frac{\omega_{\vec{k}}}{2}} \left(\hat{a}_k\, e^{-i\omega_{\vec{k}}t + i\vec{k}\cdot\vec{x}} - \hat{a}_k^{+}\, e^{i\omega_{\vec{k}}t - i\vec{k}\cdot\vec{x}} \right)$$

$$\omega_{\vec{k}} = \sqrt{k^2c^2 + \frac{m^2c^4}{\hbar^2}}$$

$$[\hat{a}_{\vec{k}}, \hat{a}_{\vec{k}'}] = 0 = [\hat{a}_{\vec{k}}^{+}, \hat{a}_{\vec{k}'}^{+}]$$

$$[\hat{a}_{\vec{k}}, \hat{a}_{\vec{k}'}^{+}] = \delta(\vec{k} - \vec{k}')$$

$$\mathcal{H} = \int d^3\vec{k}\ \hbar\omega_{\vec{k}} \left(\hat{a}_{\vec{k}}^{+} \hat{a}_{\vec{k}} + \frac{1}{2}\delta^3(0) \right)$$

Then,

$$H = \int d^3\vec{k}\,\hbar\,\omega_{\vec{k}}\left(\hat{\tilde{a}}^{\dagger}_{\vec{k}}\hat{\tilde{a}}_{\vec{k}} + \frac{1}{2}\delta^3(0)\right)$$

Thus, a useful set of variables leading to harmonic oscillating quantization is

$$L = -c\frac{\hbar^2}{2}\left(\frac{-(\partial_t\phi)^2}{c^2} + (\partial_x\phi)^2 + (\partial_y\phi)^2 + (\partial_z\phi)^2 - \frac{m^2c^2}{\hbar^2}\phi^2\right)$$

$$\Pi(\vec{x},t) = \frac{\sqrt{\hbar}}{i}\int d^3\vec{k}\sqrt{\frac{\omega_{\vec{k}}}{2}}\left(\hat{a}_{\vec{k}}e^{-i\omega_{\vec{k}}t + i\vec{k}\cdot\vec{x}} - \hat{a}^{\dagger}_{\vec{k}}e^{i\omega_{\vec{k}}t - i\vec{k}\cdot\vec{x}}\right)$$

where $\quad \omega_{\vec{k}} = \sqrt{k^2c^2 + \frac{m^2c^4}{\hbar^2}}$

$$[\hat{a}_k, \hat{a}_{k'}] = 0 = [\hat{a}^{\dagger}_k, \hat{a}^{\dagger}_{k'}]$$

$$H = \int d^3\vec{k}\,\hbar\,\omega_{\vec{k}}\left(\hat{\tilde{a}}^{\dagger}_{\vec{k}}\hat{\tilde{a}}_{\vec{k}} + \frac{1}{2}\delta^3(0)\right)$$

 So, as we saw for this setting I would get the harmonic oscillator Hamiltonian which will be $\hbar\omega_k$ and the number operator for each k mode and then there is a vacuum energy for each k mode that would be coming about. And the box here tells us the usual way of variables again, but we have discussed previously. So, here you see I have done this decomposition which I was talking about. $\hat{a}k$ times $e^{-i}\omega_k{}^t$ + $ik(\hat{x})$ and just Hermitian conjugate of that term over here such that the ϕ in the Position space becomes Hermitian and similarly you can see that this and those things are conjugate of each other. This time they are coming with a – sign to compensate for that. I have i over here such that the Π here as well is a Hermitian operator. So now with this choice of variables and the selection of the operators relation with each other I have a real scalar field. Previously we were discussing about a real Fourier space field which would have been a complex scalar field of a special kind. That would not be most general complex scalar field. But in this course we are not going to discuss about complex scalar field. Those become more interesting for gauge field theories interacting with particles. If a scalar field has a charge or something. However, we are not going to deal about that in more details. We will touch upon that. So therefore, most of the discussions I am going to base upon in this course is for real scalar field. So this box, therefore, is a good set of information for real scalar field. You can come back again and again to just cross check your structure that I have a Lagrangian for a real scalar field, which is given by this. The field operator is written in terms of creation and annihilation operator like this. Its conjugate momentum is this. In Fourier space, there is a harmonic oscillator structure for each different k and their frequency are related to the mass and the k like this. They satisfy the standard creation annihilation structure. commutation relation and with this choices we land up on a collection of harmonic oscillator Hamiltonian which is solved like that. And with this choices the Lagrangian is obtained through the integration of the Lagrangian density first over the volume and then over the time. So, this is the whole package for a real scalar field. Similarly, one can construct a package for complex scalar field as well, complex scalar field of the kind which we discussed as well.So, you can do this as at your leisure as an exercises, but for now we are going to go ahead with exercises of computations for real scalar field. So, now we what we have learnt previously as well that for now the each oscillator for example oscillator with identified with wave number k, it can have a vacuum state 0 of k Or it can have a first one particle excited state or two particle excited state or three particle excited state so on.

Calculations on Fock space

1. $\langle 0 | \hat{\phi}(\vec{x}) | 0 \rangle$

$\langle 0 | \hat{a}_k | 0 \rangle \, e^{-i\omega t + i \vec{k}_0 \cdot \vec{x}}$

$\Rightarrow \langle 0 | \sqrt{\hbar} \int \frac{d^3\vec{k}}{\sqrt{2\omega_{\vec{k}}}} \left(\hat{a}_k e^{-i\omega_{\vec{k}} t + i \vec{k} \cdot \vec{x}} + \hat{a}_k^\dagger e^{+i\omega_{\vec{k}} t - i\vec{k}\cdot\vec{x}} \right) | 0 \rangle$

$= \sqrt{\hbar} \int \frac{d^3\vec{k}}{\sqrt{2\omega_{\vec{k}}}} \boxed{\langle 0 | \hat{a}_{\vec{k}}^\dagger | 0 \rangle} \, e^{i\omega_{\vec{k}} t - i\vec{k}\cdot\vec{x}}$

$= 0$

2. $\langle 1_{\vec{k}_0} | \hat{\phi}(\vec{x}) | 1_{\vec{k}_0} \rangle$

$= \sqrt{\hbar} \langle 0 | \hat{a}_{\vec{k}_0} \int \frac{d^3 k}{\sqrt{2\omega_{\vec{k}}}} \left(\hat{a}_k e^{-i\omega_{\vec{k}} t + i \vec{k} \cdot \vec{x}} + \hat{a}_k^\dagger e^{i\omega_{\vec{k}} t - i\vec{k}\cdot\vec{x}} \right) \hat{a}_{\vec{k}_0}^\dagger | 0 \rangle$

$= \sqrt{\hbar} \int \frac{d^3\vec{k}}{\sqrt{2\omega_{\vec{k}}}} \left[\langle 0 | \left(\hat{a}_{\vec{k}_0} \hat{a}_{\vec{k}} \hat{a}_{\vec{k}_0}^\dagger \right) | 0 \rangle \, e^{-i\omega_{\vec{k}} t + i\vec{k}\cdot\vec{x}} \right.$

$\left. + \langle 0 | \left(\hat{a}_{\vec{k}_0} \hat{a}_{\vec{k}}^\dagger \hat{a}_{k_0}^\dagger \right) | 0 \rangle \, e^{i\omega_{\vec{k}} t - i\vec{k}\cdot\vec{x}} \right]$

$\langle 0 | \hat{a}_{\vec{k}_0} \hat{a}_k \hat{a}_{k_0}^\dagger | 0 \rangle = \langle 0 | a_{\vec{k}_0} \left[\delta(\vec{k} - \vec{k}_0) + \hat{a}_{\vec{k}_0}^\dagger \hat{a}_{\vec{k}} \right] | 0 \rangle$

$= \delta(\vec{k} - \vec{k}_0) \langle 0 | a_{\vec{k}_0} | 0 \rangle = 0$

Similarly $\langle 0 | \hat{a}_{\vec{k}_0} \hat{a}_{\vec{k}}^\dagger \hat{a}_{k_0}^\dagger | 0 \rangle = 0$

$\langle 1_{\vec{k}_0} | \hat{\phi}(\vec{x}) | 1_{\vec{k}_0} \rangle = 0$

Calculation in Fock Space

1. $\langle 0_{\vec{k}} | \hat{\phi}(\vec{x}) | 0_{\vec{k}} \rangle$

$\Rightarrow$

$\langle 0_{\vec{k}} | \sqrt{\hbar} \int \dfrac{d^3 \vec{k}}{\sqrt{2\omega_{\vec{k}}}} \left(\hat{a}_{\vec{k}} e^{-i\omega t + i\vec{k}\cdot\vec{x}} + \hat{a}_{\vec{k}}^{\dagger} e^{i\omega t - i\vec{k}\cdot\vec{x}} \right) | 0_{\vec{k}} \rangle$

$=$

$\sqrt{\hbar} \langle 0_{\vec{k}} | \hat{a}_{\vec{k}}^{\dagger} | 0_{\vec{k}} \rangle e^{i\omega_p t - i\vec{k}\cdot\vec{x}}$

2. $\langle 0 | (\hat{a}_{\vec{k0}} \int \dfrac{d^3 \vec{k}}{\sqrt{2\omega_{\vec{k}}}} \left(\hat{a}_{\vec{k}} e^{-i\omega t + i\vec{k}\cdot\vec{x}} + \hat{a}_{\vec{k}}^{\dagger} e^{i\omega t - i\vec{k}\cdot\vec{x}} \right) \hat{a}_{\vec{k}}^{\dagger} \hat{a}_{\vec{k}_0}) | 0 \rangle$

$=$

$\sqrt{\hbar} \int \dfrac{d^3 \vec{k}}{\sqrt{2\omega_{\vec{k}}}} \left[\langle 0 | \hat{a}_{\vec{k}_0} \hat{a}_{\vec{k}} \hat{a}_{\vec{k}_0}^{\dagger} | 0 \rangle e^{-i\omega t + i\vec{k}\cdot\vec{x}} \right] \langle 0 | \hat{a}_{\vec{k}_0} \hat{a}_{\vec{k}_0}^{\dagger} \hat{a}_{\vec{k}} | 0 \rangle$

$\left[\langle 0 | \hat{a}_{\vec{k}_0} \hat{a}_{\vec{k}} \hat{a}_{\vec{k}_0}^{\dagger} | 0 \rangle \right] = \langle 0 | \hat{a}_{\vec{k}_0} \left[\delta(\vec{k} - \vec{k}_0) + \hat{a}_{\vec{k}_0}^{\dagger} \hat{a}_{\vec{k}} \right] | 0 \rangle = \delta(\vec{k} - \vec{k}_0) \langle 0 | 0_{\vec{k}_0} | 0 \rangle = 0$

Similarly $\left[\langle 0 | \hat{a}_{\vec{k}_0} \hat{a}_{\vec{k}_0}^{\dagger} \hat{a}_{\vec{k}_0}^{\dagger} | 0 \rangle \right] = 0$

$\left[\langle 1_{\vec{k}_0} | \hat{\phi}(\vec{x}) | 1_{\vec{k}_0} \rangle \right] = 0$

And similarly for k' as well, there can be vacuum at k' or one particle can be excited at k' or two particles could have been excited at k' and so on. So, these are the structure for different k. A full state of the field has to tell me exactly about in which k modes how many particles are there. For example, a typical ψ could be let us say 0 particle in wave number k0, 1 particle in wave number k_1, then 0 particle in wave number k_2 and then similarly 3 particles in wave number kn and so on. So, these are infinitely many different harmonic oscillators with identified with number $k_0, k_1, k_2,.... k_n$, infinitely many of them. You can think of it as a collection of three harmonic oscillators would have a state, first particles, first oscillator state, second oscillator states and third oscillator state. Similarly, we are doing for infinitely many of them. This could be one possible structure or I can put some coefficient over here c1 + c2 could be some three particles in k_o, one particle in k_1 again and two particle in k_2, no particle in k_n. So all different combinations you could try are possible or are plausible set of wave functions which a field can have. This is a decomposition for oscillators in Fourier space, but you know there is a one-to-one map to the position space as well. If I want to know what does the field operator does on these states, I would have to act ϕ operator on this. So, that is what we are going to do. And a good choice of a state to learn many things about is a vacuum state. Vacuum state is all the oscillators are there in their ground state. k0 is also having no particle, k_1 is also not having any particle, k_2 is also not having any particles and so on.

So, this is a unique state. Vacuum is a unique state of the theory and most of the things which we want to know about the structure which we will be interested about in this course as well will revolve around vacuum. So, in this course we want to know what the quantum fields intrinsic structure does on to the

system. When we apply electric field more or less we know what the atom does, the Stark effect or Zeeman effect where we apply electric field or magnetic field. The structure which will be most interesting for us is when we do not apply anything, then also there should be quantum fluctuations in that. And those quantum fluctuations will be coming from a state of a vacuum kind where nothing is excited everything is in vacuum. So this is a good choice of a state to work things with and we are going to do most of the business with this and we will see even if you take states like ψ like this non-vacuum state many of the properties of the non-vacuum state stem from their structure in the vacuum state itself as well this will become clear as we go forward in the course but as of now I am going to do business starting with the vacuum state For instance, I would want to know what is the field doing, what is the structure of the field operator in the vacuum state. So, one way to answer that is that obtain the expectation value of the field in the vacuum state. So, what I would do, I would write first decompose ϕ in its harmonic oscillator avatar and then squeeze it between the vacuum state. So, therefore, I have written ϕ as a collection of infinitely many harmonic oscillators, something which we have learned to do and then squish that operator between the vacuum state. Remember this vacuum over here means this. So, let me just first make a clean identification mark on this that vacuum means this state. So, each of the operator $\hat{a}_k$ will search for its own counterpart state. That means $\hat{a}_{k1}$ will try to find what is the state structure in the k_1 thing. So, it will find no particle here. Similarly, $\hat{a}_k$ n will try to find what is happening over here and it will find no particle in k_n as well. So, all of the ks are put to vacuum and different-different $\hat{a}_ks$ running with the changing of k will search for the corresponding structure of state in their their individual Hilbert subspace let us say or Fock basis element. So, therefore the structure is like this, this vacuum state is just an acronym of this expanded version and ultimately what will happen is that I will have this operator squeezed between 0 and 0 from left and right and this operator squeezed between 0 and 0 from left and right okay and $e^{i\omega t + ik x \hat{x}}$ in field theory position is not an operator time is not an operator k is not an operator and ω is not an operator so they will go out this integral will also go out so I will have two terms one with 0 $\hat{a}_k$ 0 and then $e^{-i\omega t}$ + k x and the other is 0 $\hat{a}_k^\dagger$ 0 and $e^{+i\omega kt \; -ik(\hat{x})}$. Out of these two terms, the first term will vanish because $\hat{a}_k$ annihilates the vacuum. So, remember this $\hat{a}_k$ is being integrated over d³k that means it is $\hat{a}_{k1} + \hat{a}_{k2} + \hat{a}_{k3} + \hat{a}_{k4}$ under this integration it means different $\hat{a}_ks$ are coming and hitting this 0 and they are trying to find out in their corresponding wave number what is the configuration of the state. It will find for wave number kn let us say it will find there is a vacuum it will annihilate that. So, for all operators coming under this integration. Each time this annihilation operator will find the vacuum acting upon and kill the state. So therefore, this first squeezing will not survive. Only the second squeezing is going to survive and which is this. Only the second squeezing is going to survive and which is this. Ok, $\hat{a}_k^\dagger$ is squeezed between 0 and 0. But you see $\hat{a}_k^\dagger$ hitting the 0 from the left is just the Fourier counter part or not Fourier complex, Hermitian counterpart of this statement. If you write this, this will become 0 $\hat{a}_k^\dagger$. But we already know this thing is 0 and its † should be 0 as well. Therefore, this operator is also annihilating the $<|0$. The $\hat{a}_k^\dagger$ annihilates the $<|0$. $\hat{a}_k$ annihilate the ket 0, $\hat{a}_k^\dagger$ will annihilate the $<|0$. So, therefore, even this squeezing will not survive.

That means in the vacuum state, the field does not have any expectation value, it has a 0 expectation value. So, there is no expectation value in the vacuum state. This looks like a good definition of a vacuum that there is no expectation value which is surviving. See, this is not a very unique statement about vacuum. For example, if I do one particle excited state, which is non-vacuum, that means one of the k, let us say k first, k_0, I try to excite now. So, let us say this state I am going to talk about, which is ψ_{1k0}. This state will be called collectively $1k_0$ because all other things are in vacuum. If I compute $_{1k}$ expectation value of the field in the $1k_0$ state, just let me correct the, just rewrite what was vacuum such that this detour was just to tell you what is $1k_0$ state. So, once we have identified what is $1k_0$ state, I will squeeze operator 5k between these two things. So, previously you saw that $\hat{a}_k$ and $\hat{a}_k^\dagger$ operators were getting squeezed between vacuum from both sides. This time they will get squeezed $1k_0$, $1k_0$ from left

and right. And what more? $1k_0$, the state $1k_0$ can be written as $\hat{a}_k^\dagger$ at wavelength k_0 acting on vacuum. So, vacuum used by creation operator excites one particle into wave number k_0. I am going to write the state $1k_0$ as $\hat{a}_{k0}^\dagger$ and similarly its bra which would be hermitian counterpart of that would be vacuum $\hat{a}_{k0}$ coming from the right. This is not 0. Had it been an $\hat{a}_k^\dagger \hat{a}_k^\dagger$ then it would have been 0, but if it is $\hat{a}k$ this is not 0. Similarly here this is non-zero $\hat{a}_k^\dagger$ hitting the kit zero is not zero had it been only $\hat{a}k$ if this was not there then it would have been a zero as well but right now they survive okay so fine so let us go ahead and see once they survive what do they tell us about its structure So you see now I have three operators, one $\hat{a}k$ coming from here, then $\hat{a}k$ coming from here and then $\hat{a}_k^\dagger$ coming from here. They get squeezed and similarly for this term, $\hat{a}k$ not coming from here, $\hat{a}_k^\dagger$ coming from here and $\hat{a}_k^\dagger$ coming from here, that gets squeezed between vacuum. So now we have three objects squeezed for the first term and three objects squeezed for the second term. First term is $\hat{a}k$, $\hat{a}k$, $\hat{a}_{k0}$, $\hat{a}k$, $\hat{a}_{k0}^\dagger$, $\hat{a}_{k0}$, $\hat{a}k$, $\hat{a}_{k0}^\dagger$. And this exponential factor and the integration are tagging along with this. And the second term is $\hat{a}_{k0}$, $\hat{a}_k^\dagger$, $\hat{a}_{k0}^\dagger$. So, $\hat{a}_{k0}$, $\hat{a}_{k0}$, $\hat{a}_k^\dagger$, $\hat{a}_{k0}^\dagger$ and this exponential getting tagged along with this. Now, the crucial and important thing is this. Previously, I had a single $\hat{a}k$ which was getting squeezed between vacuum and single $\hat{a}_k^\dagger$ which was getting squeezed between the vacuum. This time we have three of them getting squeezed between the vacuum. Now, what you can do?

You can try to find out what happens to these terms. I have an $\hat{a}k$ and $\hat{a}_k^\dagger$ here. $\hat{a}_k$ and $\hat{a}_k^\dagger$ here can be flipped using a commutator because we know $\hat{a}_k$ and $\hat{a}_k^\dagger$ Let us say $\hat{a}_{k0}^\dagger$, commutator would be $\delta(k-k_0)$. That means $\hat{a}_k\hat{a}_k^\dagger$, $\hat{a}_k \, \hat{a}_{k0}^\dagger$ rather would be equal to $\delta(k-k_0) + \hat{a}_{k0}^\dagger\hat{a}_k$, which is this. The same thing over here using the commutator can be written like this. What do we earn out of it? First of all, $\hat{a}_k$ has come on the right-hand side. That will annihilate the vacuum. Because I know whenever $\hat{a}_k$ hits the vacuum, it kills it. Previously, $\hat{a}_{k0}^\dagger$ was coming on the right-hand side. It was not annihilating the vacuum. Using the commutator, I flip things. And therefore, it annihilates that. So, I am not worried about this term anymore. What about $\delta(k-k_0)$? $\delta(k-k_0)$ is not an operator. It is identity. So this will also not do anything. So I will have just $\hat{a}_k$ not surviving with $\delta(k-k_0)$, $\delta(k-k_0)$ which will come out and 0 $\hat{a}_k$ not 0 will be there. Again $\hat{a}_k$ not is hitting the 0. So it will annihilate that. So therefore the whole term over here, the three operators squeezing is 0. Similarly, you can verify even for the second three operators squeezing over here, that is also identically 0. So, that means even in the first excited state the field expectation is 0. You can see the vacuum the field expectation was 0 which was good for us, but there are other states as well where field expectations are 0. This is not surprising because in harmonic oscillators also you know that in vacuum state which was something like a Gaussian like this the field expectation is 0. But for any symmetric wave function as well, if you can think of a symmetric wave function, there also if you find out the field expectation remains 0, the harmonic oscillator expectation remains 0. So, harmonic oscillator ground state x operator in the ground state of harmonic oscillator is 0. But it is not unique to ground state only, there could be other excited state, remember x expectation in position space means $\partial_x\psi$ of x mod square from – infinity to infinity. If this $\psi(x)$ is a symmetric function then overall function becomes odd and odd function under integration from a – infinity to + infinity is 0. So, any symmetric function will also make the expectation value of the field of the oscillator 0. That is what exactly happening here. This $1k_0$ is not vacuum, but its structure is something like that of a symmetric state in the oscillator structure. So therefore, field expectation being 0 is necessary for vacuum, but it is not sufficient for vacuum. You cannot claim that if field operator's expectation is vacuum, that means we are talking about vacuum state. Oh sorry, if the expectation value of the field operator is x_0, then it necessarily does not mean we are talking about vacuum state. There can be other states as well whose expectation value remain 0 in those states.

So, the field operators expectation value is not going to distinguish between vacuum and a similarly poised or similarly expectation profile non-vacuum state like $1k_0$ and even $2k_0$, $3k_o$ if you can try to find out for all those states you would find out that the field expectation is vanishing. So, there are infinitely

many states where field expectation vanishes. In fact, if you do the exercise for the conjugate momenta operator Π as well, for example, this Π operator whose decomposition which we have learned previously how to write again in terms of $\hat{a}_k$ and $\hat{a}_k^\dagger$ which is over here then also the same story will unfold.

If I take Π and it squeeze between 0, then either single $\hat{a}_k$ will get squeezed between 0 or $\hat{a}_k^\dagger$ will get squeezed between 0 and they will annihilate the vacuum state like the Π operator does. So, even Π operator, even the Π operator would be having a 0 expectation value in vacuum. And this is again the same computation, you will see only the exponential factors, there is a relative sign between $\hat{a}_k$ and $\hat{a}_k^\dagger$ for pi, nothing more, there is i and there is a complex conjugate –. But individually both these operators which were being added for ϕ and getting subtracted for pi, these are the operators which are getting added for, so this operator and this operator get added for ϕ. and the same operators get subtracted for pi. So, the operators are the same, the operators which are constituting ϕ and Π are the same just there is a relative sign difference. However, what we saw that the operators themselves get squeezed between the states 0 or $1k_0$ to get annihilated. So, that becomes 0. So, the same thing will be true for Π as well. And therefore,So, the same thing will be true for Π as well. you will realize that the Π also has a 0 expectation value. either in the ground state or in the $1k_0$ state or $2k_0$ state, $3k_0$ state what we talked about. So, field expectation value or its conjugate is not going to differentiate between the vacuum and its excited state of the similar profile. So, we need more operators to distinguish between a vacuum and non-vacuum states. ϕ and Π are not able to distinguish between vacuum and non-vacuum state. So, one such operator could be the Hamiltonian. The Hamiltonian operator between the vacuum, the squeeze between the vacuum state or the non-vacuum state of the kind which we have discussed $_{1k}0$, both these squeezings are non-zero.

Why is that?

You can see that the Hamiltonian operator itself is made up of two pieces. One is the number operator $\hat{a}_k^\dagger \hat{a}_k$, this is the operator and then there is identity operator times $\delta(k-k_0)$. So, this object over here is some surviving operator, meaning it will not be going to 0 for any state because it is identity. So, this $\delta(0)$ times half is going to survive always unless this $\hat{a}_k^\dagger \hat{a}_k$ expectation cancels it. But I know this is a number operator and number operator is bounded from below to be 0 or above, meaning it cannot be taking negative value. So, there is no case where Hamiltonian can become 0. So, Hamiltonian will come at least with this much of energy value and that is the case for the vacuum state. If I take the Hamiltonian squeeze it between the vacuum state, I would have a two operators which will be surviving. One is $\hat{a}_k^\dagger \hat{a}_k$ which is the number operator squeezed and then there is a half times $\delta(0)$ times identity operator which is squeezed. This will be going to 0 because of this again $\hat{a}_k$ annihilating the vacuum state. However, the second part will be surviving and that will ascribe some energy to the vacuum state. So, therefore, the energy of a vacuum state is given by the expectation value of this will be obtained as half times $d^3k\ \hbar\omega_k$ and $\delta(0)$. This $\delta(0)$ is the delta function in the Fourier space, so that is proportional to the volume of the position space. So, you see that this is the Hamiltonian energy density and total volume multiplied together give you the total Hamiltonian. And this is the energy the total Hamiltonian comes up with or this is the energy density vacuum state comes up with. However, once we do for the $1k_0$ state, previously the two operators were getting squeezed between 0, this time they will get squeezed between $_{1k}0$. So, I will have a term which is $1k_0\ \hat{a}_k^\dagger \hat{a}_k,\ _{1k}0$. So, remember there is an integration d^3k upon times $\hbar\omega_k$ and then there is a half term which is also there. So, this is $1k_0$ and $\delta(0)$ times identity on $_{1k}0$. So, this is still going to be half times $\delta(0)$ while this time this operator $\hat{a}_k^\dagger \hat{a}_k$ which was getting squeezed between 0 previously to become 0, this time it will not be 0. And it will exactly give you, if you do this exercise slightly carefully and pay attention to what we are doing, you will get a $\delta(0)$ times 1. So, overall it will become $3/2\ d^3k\hbar\omega_k\ \delta(0)$. So, the Hamiltonian operators separates the two state. The field operator does not, the conjugate momentum operator does not. They give you the

same expectation value which is 0, but the Hamiltonian tells them apart. That the vacuum comes with the minimum energy possible which is this. There is a one particle excitation which gives you additional energy cost per unit volume of 1 or 1 meaning $\hbar\omega_k$. for each k and in the whole space the $\delta(0)$ volume multiplied so overall it becomes three half so therefore we can have different different operators which will try to describe different different expectation value and also presumably the fluctuations or variances in them.

So, we can try to find out what is the variance in the vacuum state, what is the fluctuation, what is the correlator. One interesting object which we will need and we will see later on is two point correlation that $\phi(x)$ and $\phi(y)$ in the vacuum state or any other state. Most of the time we will discussing about vacuum as we discussed but this can be computed in any other state. This is analog of x of t and x of t′ in harmonic oscillator. So, these kind of two-point correlator is like two different position correlators in harmonic oscillator which will be interesting and handy. Those kind of things which will be important we will learn in coming weeks. But as of now, this is the standard way we can compute things using the property of annihilation operator and creation operators and their commutation relations. In the next class, we will just wrap up the discussions on the calculational suspects, trivial calculation suspect by discussing conserved charges, conserved quantities in these kind of actions, the trivial field theory.