Foundation of Quantum Theory: Relativistic Approach
Quantum Field Theory 1.4
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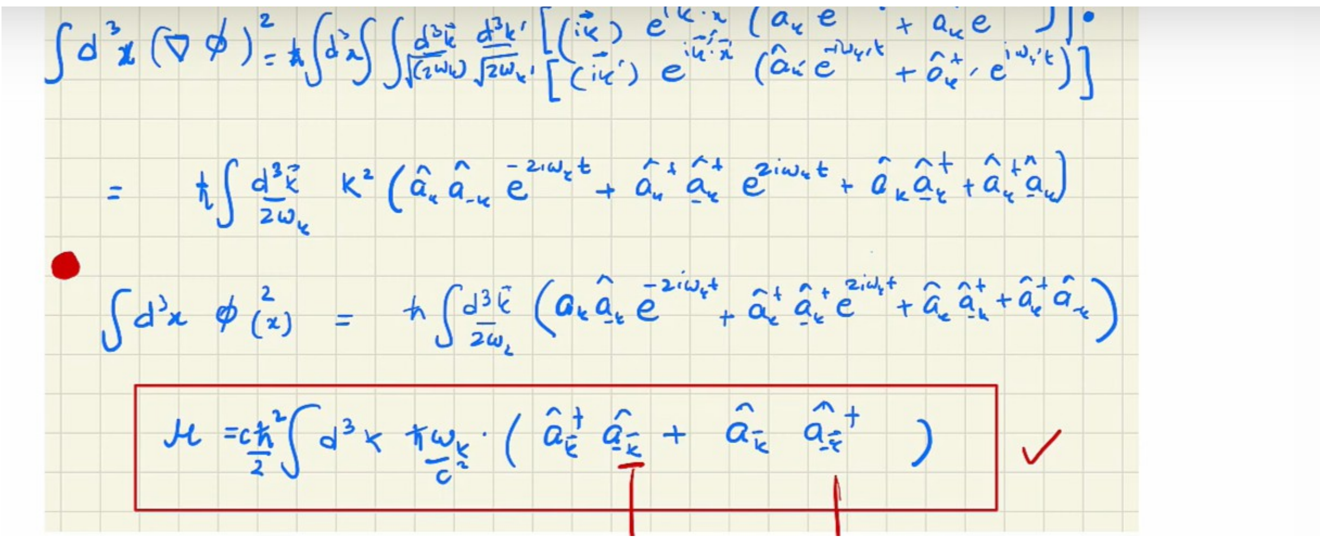
Quantum Scalar Field 2
Lecture- 18



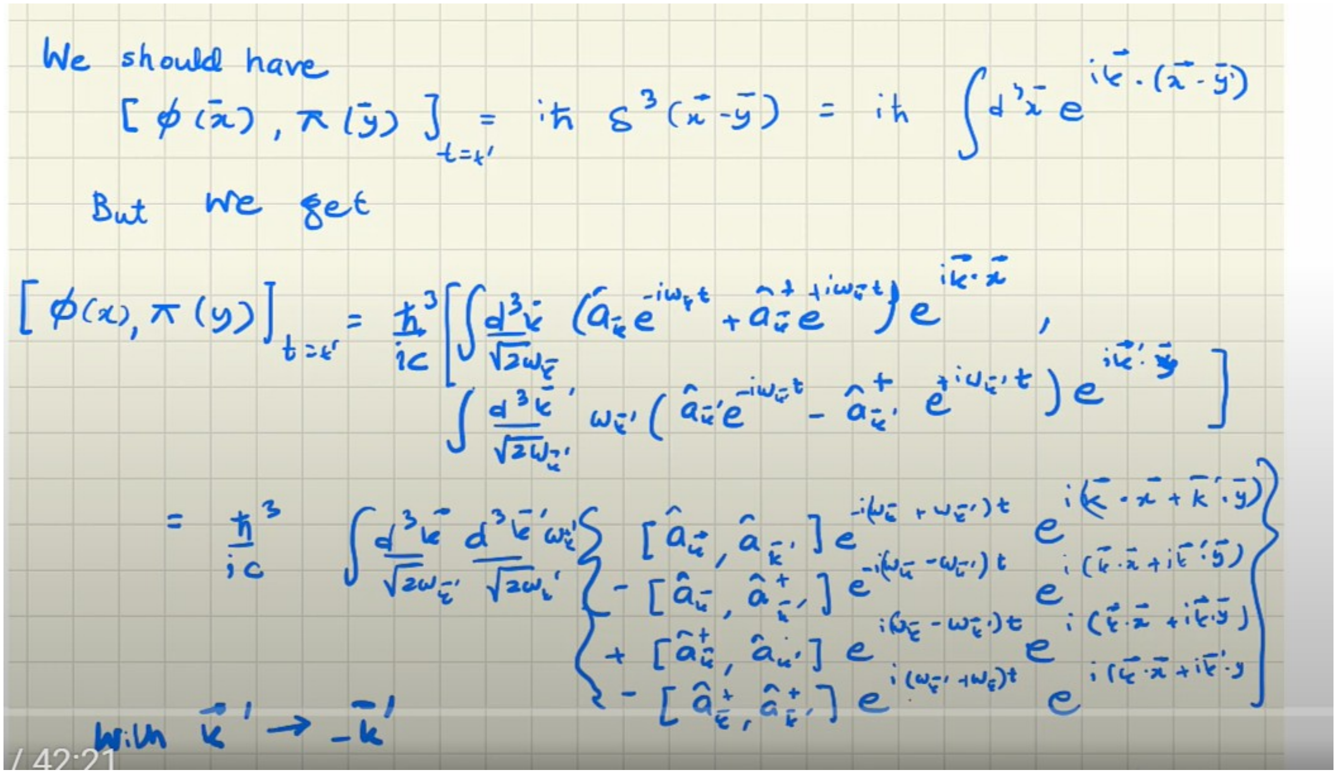
$$\int d^{3x}(\nabla_x\phi)^2=\hbar\int d^3x\int\int\frac{d^3\vec{k}}{\sqrt{2\omega_k}}\frac{d^3\vec{k}'}{\sqrt{2\omega_k}}[i\vec{k}e^{i\vec{k}\vec{x}}(\hat{a}_k e^{-i\omega_k t}+\hat{a}_k^{\dagger}e^{i\omega_k t})][i\vec{k}'e^{i\vec{k}'\vec{x}}(\hat{a}_{k'}e^{-i\omega_k' t}+\hat{a}_{k'}^{\dagger}e^{i\omega_k' t})]$$

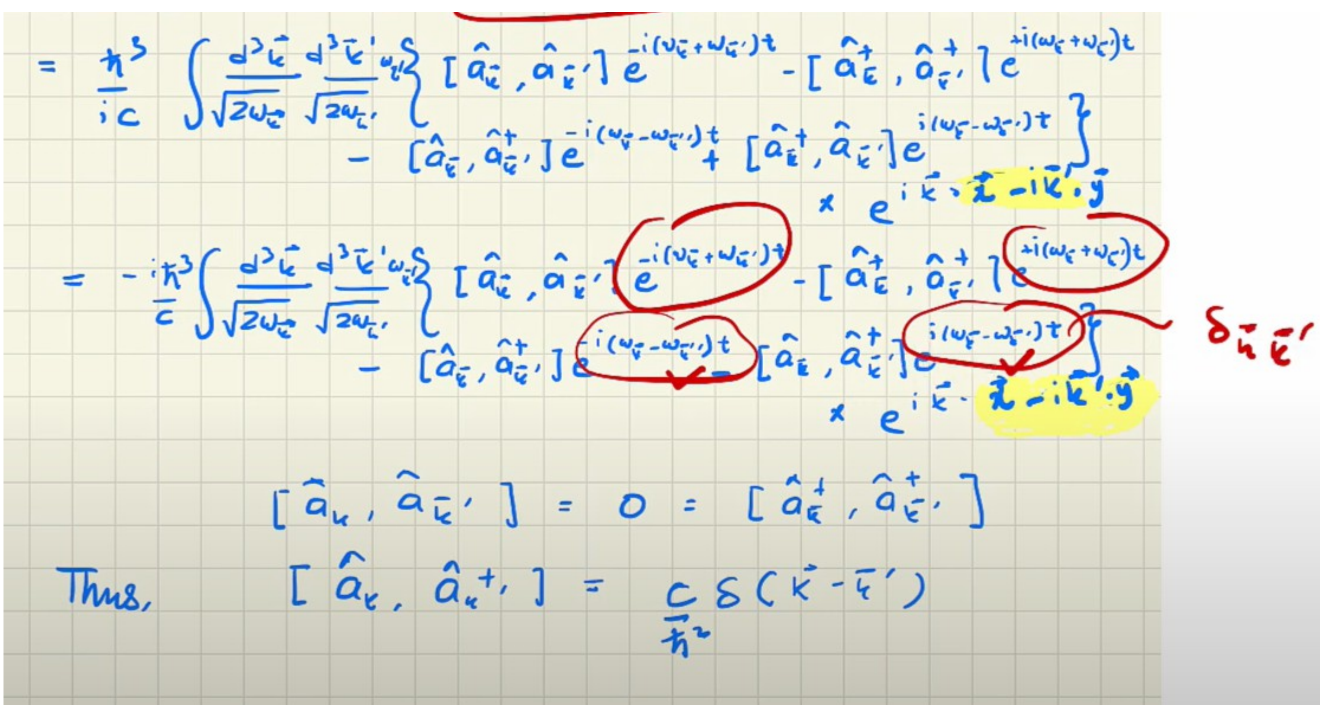
=

$$\hbar\int\frac{d^3\vec{k}}{2\omega_k}k^2(\hat{a}_k\hat{a}_{-k}e^{-2i\omega_k t}+\hat{a}_k^{\dagger}\hat{a}_{-k}^{\dagger}e^{2i\omega_k t}+\hat{a}_k\hat{a}_{-k}^{\dagger}+\hat{a}_k^{\dagger}\hat{a}_{-k})$$

$$H=\frac{c\hbar^2}{2}\int d^3k\frac{\hbar\omega_k}{c^2}(\hat{a}_{\vec{k}}^{\dagger}\hat{a}_{-\vec{k}}+\hat{a}_{\vec{k}}\hat{a}_{-\vec{k}}^{\dagger})$$

Today we will move forward towards full quantization of the scalar field and try to see what kind of states and expectations value for various things we do get from the quantization of the harmonic oscillators for each k-mode and then summing it over to obtain the full field to be harmonic oscillator like up to absorption of certain coefficient which we wanted to know and the commutator relations between $\hat{a}_k$ and $\hat{a}_k^{\dagger}$. So remember a of – k was also equal to $\hat{a}_k$ and a of – $k^{\dagger}$ was also equal to $\hat{a}_k^{\dagger}$. This was true because of the reality conditions of the field to be harmonic oscillator like up to absorption of certain coefficient which we wanted to know and the commutator relations between $\hat{a}_k$ and $\hat{a}_k^{\dagger}$. So remember a of – k was also equal to $\hat{a}_k$ and a of – $k^{\dagger}$ was also equal to $\hat{a}_k^{\dagger}$. This was true because of the reality conditions of the field. So ultimately we do have up to certain constant $c\hbar^2/2$. We have a structure which is almost like harmonic oscillator.





With $k' \rightarrow k$

We should have

$$[\phi(\bar{x}), \Pi(\bar{y})]_{t=t'} = i\hbar\,\delta^3(\vec{x}-\vec{y}) = i\hbar \int d^3k\, e^{\vec{k}\cdot(\vec{x}-\vec{y})}$$

But we get

$$[\phi(x), \Pi(y)]_{t=t'} = \frac{\hbar^3}{ic}\left[\int \frac{d^3\vec{k}}{\sqrt{2\omega_{\vec{k}}}}(\hat{a}_{\vec{k}}e^{-i\omega_{\vec{k}t}} + \hat{a}^{\dagger}_{\vec{k}}e^{+i\omega_{\vec{k}t}}), \int \frac{d^3\vec{k}\,'}{\sqrt{2\omega_{\vec{k}\,'}}}(\hat{a}_{\vec{k}}\,' e^{-i\omega_{\vec{k}'t}} - \hat{a}^{\dagger}_{\vec{k}}\,' e^{+i\omega_{\vec{k}'t}})\right]$$

$$= \frac{\hbar^3}{ic}\int \frac{d^3\vec{k}}{\sqrt{2\omega_{\vec{k}}}}\int \frac{d^3\vec{k}\,'}{\sqrt{2\omega_{\vec{k}'}}}\omega_k{}' \begin{pmatrix} [\hat{a}_{\vec{k}}, \hat{a}_{\vec{k}'}]e^{-i(\omega_k+\omega_k{}')t}e^{i(\vec{k}\cdot\vec{x}+\vec{k}\,'\cdot\vec{y})} \\ -[\hat{a}_{\vec{k}}, \hat{a}^{\dagger}_{\vec{k}'}]e^{-i(\omega_k-\omega_k{}')t}e^{i(\vec{k}\cdot\vec{x}+\vec{k}\,'\cdot\vec{y})} \\ +[\hat{a}^{\dagger}_{\vec{k}}, \hat{a}_{\vec{k}'}]e^{-i(\omega_k-\omega_k{}')t}e^{i(\vec{k}\cdot\vec{x}+\vec{k}\,'\cdot\vec{y})} \\ -[\hat{a}^{\dagger}_{\vec{k}}, \hat{a}^{\dagger}_{\vec{k}'}]e^{-i(\omega_k+\omega_k{}')t}e^{i(\vec{k}\cdot\vec{x}+\vec{k}\,'\cdot\vec{y})} \end{pmatrix}$$

$$\frac{\hbar^3}{ic}\int \frac{d^3\vec{k}}{\sqrt{2\omega_{\vec{k}}}}\frac{d^3\vec{k}\,'}{\sqrt{2\omega_{\vec{k}'}}}\omega_{\vec{k}}\left([\hat{a}_{\vec{k}}, \hat{a}_{\vec{k}}\,']e^{-i(\omega_{\vec{k}}+\omega_{\vec{k}'})t} - [\hat{a}^{\dagger}_{\vec{k}}, \hat{a}^{\dagger}_{\vec{k}'}]e^{-i(\omega_{\vec{k}}+\omega_{\vec{k}'})t} - [\hat{a}_{\vec{k}'}, \hat{a}^{\dagger}_{\vec{k}}]e^{-i(\omega_{\vec{k}}-\omega_{\vec{k}'})t} + [\hat{a}^{\dagger}_{\vec{k}}, \hat{a}_{\vec{k}'}]e^{-i(\omega_{\vec{k}}-\omega_{\vec{k}'})t}\right) \times e^{ik(\dot{x}-\dot{y})}$$

Using $e^{-i(\omega_{\vec{k}}-\omega_{\vec{k}'})t} = \delta(\vec{k}-\vec{k}\,')$

$$[\hat{a}_{\vec{k}}, \hat{a}_{\vec{k}}\,'] = 0 = [\hat{a}^{\dagger}_{\vec{k}}, \hat{a}^{\dagger}_{\vec{k}}\,']$$

Thus,

$$[\hat{a}_{\vec{k}}, \hat{a}^{\dagger}_{\vec{k}}\,'] = \frac{c}{\hbar^2}\delta(\vec{k}-\vec{k}\,')$$

We have $\hbar\omega_k$ an extra c square is there which I can pull out here and then $\hat{a}_k{}^{\dagger}\hat{a}_k + \hat{a}_k\hat{a}_k{}^{\dagger}$ this kind of structure is there if you recall that was also present in the harmonic oscillator business as well so if we quantize ordinary harmonic oscillator at the end of the day we do get the hamiltonian almost like this but there the commutators between $\hat{a}_k$ and a and $a^{\dagger}$ in their in quantum mechanics there is no k dependency in a so a and $a^{\dagger}$ was identity using that we obtained converted this into $\hat{a}_k{}^{\dagger}\hat{a}_k$ + Actually it becomes twice of this + identity 1. And the outside half makes everything into single times of $\hat{a}_k{}^{\dagger}\hat{a}_k$ and 1 */2*. So similar kind of thing we can expect if we do have the commutator between $\hat{a}_k$ and $\hat{a}_k{}^{\dagger}$ of a similar kind. So that is what we will try to see. How do we get it from? So first of all, recall what we had done. We had identified the dynamic field ϕ. We obtained its conjugate momenta Π by just going to the Legendre transformation, creating $\phi\dot{\phi}$ 4 Π. And then the usual structure of a Poisson bracket structure should suggest me that there should be a ϕ and Π Poisson bracket, which should be identity. This is particularly you have seen that in quantum mechanics or even classical physics of particles. If canonical variable is *q, p* is identified as $\partial L / \partial \dot{q}$ and then you have a structure like qp is equal to identity not identity but one in classical mechanics it is one and this poisson bracket later on is converted into commutators when you convert the q and p into operators in phase space the poisson bracket becomes commutator and right hand side becomes $i\hbar$ okay now suppose there are two particles classically let us say q_1 and q_2 So I would have a q_1 p1 and $q_2 p_2$ Then you would have structure like qi pj should be equal to δ_{ij}. That means q_1's Poisson bracket with p_2 should be 0, it should be non-zero only with p1 and vice versa. q2's Poisson bracket should be non-zero only with p_2 and with p_1 its Poisson bracket should be 0. So that is captured by δ_{ij}. . Okay if two particles are there jointly their poisson bracket is written like that and therefore jointly their commutator would be written like this qi pj is equal to $i\hbar\ \delta_{ij}$. so you see each particle knows about whom to commute and whom not to commute with so this is relevant because in field theory we do not have one oscillator we have hundreds and infinitely many of oscillators because of the k dependence. For each k we have an oscillator and therefore each k should know about whom to commute and whom to not commute with. Even in the position space, we have different operators at different positions. So if this is a position slice, then here there are some operators, ϕ here and Π here. And similarly some other location there, ϕ there and Π there. So, this will be function of x2 and this will be function of x1. So, therefore this ϕ should know that its Poisson bracket with respect to this position should be non-zero and with respect to some other position it should be zero and vice versa. So, ultimately in the same spirit of multi-particle structure of Poisson bracket or commutator we should anticipate the ϕ and Π commutator should have this delta function behavior that it only becomes non-zero for the same point and since it is a continuous variable Kronecker delta, δ_{ij}. , which was true for Dirac, for a continuous discrete variables is getting replaced by Dirac delta function, which does not become 1 at its argument 0, but shoots up to infinity at the cost of shrinking its width. So, Dirac delta function is the continual generalization of the Kronecker delta. So, we expect ϕ and Π to have this kind of Poisson bracket from its structure. So, let me erase this and make it clean space so that we can clearly see where we are headed. So, this is what a natural space structure converted into commutator should give me. And I know this delta function can be written as a Fourier integral of x – y. So, delta function is known to be d^3k, it should be k here. $d^3k\ e^{ik(\dot{x}-\dot{y})}$. So, this is what we should be able to obtain and since we know ϕ and Π individually can be written in terms of $\hat{a}_k$ and $\hat{a}_k^\dagger$ which we previously did this boxed equation ϕ and Π can be written like this. So, I will feed the information of the whole ϕ here and the feed the information of whole Π there into the commutator and try to recover what is the commutation relation between $\hat{a}_k$ and $\hat{a}_k^\dagger$ and so on. So that's what I do. This $\hbar$ cub*e* divided by *i*c will come from √h factor which was coming from ϕ. And remember Π was coming with √h times $\hbar^2$ by *i*c. So together they will give you this $\hbar$ cube *i*c kind of thing which is outside. Then inside I will write down all the operators with their coefficients. So this is the operator description for ϕ. And this is a operator description for pi. Each one of them is an integral over all k's. I have written two different k integral and k' integrals. Operators part are only $\hat{a}_k$'s and $\hat{a}_k^\dagger$s.

$$\mathcal{H} = \frac{\hbar^2}{2c}\int d^3\vec{k}\ \hbar\omega_{\vec{k}}\left(\hat{a}_{\vec{k}}\hat{a}^{\dagger}_{\vec{k}} + \hat{a}^{\dagger}_{\vec{k}}\hat{a}_{\vec{k}}\right)$$

$$= \frac{\hbar^2}{2c}\int d^3\vec{k}\ \hbar\omega_{\vec{k}}\left[2\hat{a}^{\dagger}_{\vec{k}}\hat{a}_{\vec{k}} + \frac{c}{\hbar^2}\delta(0)\right]$$

$$= \int d^3\vec{k}\ \hbar\omega_{\vec{k}}\left(\frac{\hbar^2}{c}\hat{a}^{\dagger}_{\vec{k}}\hat{a}_{\vec{k}} + \delta(0)\right)$$

Many of the times it is convenient to abosrb $c\hbar^2$ in the $\mathcal{L}$ into redefinition of $\phi \rightarrow \tilde{\phi} = \sqrt{c}\,\hbar\,\phi$

$$\mathcal{L} = -\frac{\hbar^2}{2}c\left[-\frac{(\partial_t\phi)^2}{c^2} + (\partial_x\phi)^2 + (\partial_y\phi)^2 + (\partial_z\phi)^2 - \frac{m^2c^2}{\hbar^2}\phi^2\right]$$

$$\rightarrow -\frac{1}{2}\left[-\frac{1}{c^2}(\partial_t\tilde{\phi})^2 + (\nabla\tilde{\phi})^2 - \frac{m^2c^2}{\hbar^2}\tilde{\phi}^2\right]$$

So, $\tilde{\pi} = \dfrac{\partial\mathcal{L}}{\partial(\partial_t\tilde{\phi})} = \partial_t\tilde{\phi} = \dfrac{1}{c}\dot{\tilde{\phi}}$

$$\tilde{\phi}(\vec{x},t) = \hbar\sqrt{\hbar c}\int\frac{d^3\vec{k}}{\sqrt{2\omega_{\vec{k}}}}\left(\hat{a}_{\vec{k}}e^{-i\omega_{\vec{k}}t} + \hat{a}^{\dagger}_{\vec{k}}e^{+i\omega_{\vec{k}}t}\right)$$

$$\tilde{\Pi}(\vec{x},t) = \frac{\hbar\sqrt{\hbar c}}{i}\int\frac{d^3k}{\sqrt{2\omega_{\vec{k}}}}\omega_{\vec{k}}\left(\hat{a}_{\vec{k}}e^{-i\omega_k t} - \hat{a}^{\dagger}_{\vec{k}}e^{i\omega_{\vec{k}}t}\right)e^{i\vec{k}\cdot\vec{x}}$$

$\Rightarrow$ If we define $\sqrt{c}\,\hbar\,\hat{a}_k = \hat{\hat{a}}_k$

$$\tilde{\phi}(\vec{x},t) = \sqrt{\hbar}\int\frac{d^3\vec{k}}{\sqrt{2\omega_{\vec{k}}}}\left(\hat{\hat{a}}_{\vec{k}}e^{-i\omega_{\vec{k}}t} + \hat{\hat{a}}^{\dagger}_{\vec{k}}e^{+i\omega_{\vec{k}}t}\right)$$

$$\tilde{\Pi}(\vec{x},t) = \frac{\sqrt{\hbar}}{i}\int\frac{d^3k}{\sqrt{2\omega_{\vec{k}}}}\omega_{\vec{k}}\left(\hat{\hat{a}}_{\vec{k}}e^{-i\omega_k t} - \hat{\hat{a}}^{\dagger}_{\vec{k}}e^{i\omega_{\vec{k}}t}\right)e^{i\vec{k}\cdot\vec{x}}$$

$$H=\frac{c\hbar^2}{2}\int d^3\vec{k}\,\frac{\hbar\omega_k}{c^2}(\hat{a}^\dagger_{\vec{k}}\hat{a}_{-\vec{k}}+\hat{a}_{\vec{k}}\hat{a}^\dagger_{-\vec{k}})$$

$$L=c\frac{\hbar^2}{2}\left((\frac{-1}{c^2})(\partial_t\phi)^2+(\partial_x\phi)^2+(\partial_y\phi)^2+(\partial_z\phi)^2\right)$$

$$(\frac{-1}{c^2}\frac{\partial^2\phi}{\partial t^2}-\frac{\partial^2\phi}{\partial x^2}-\frac{\partial^2\phi}{\partial y^2}-\frac{\partial^2\phi}{\partial z^2})=\frac{m^2c^2}{\hbar^2}\phi$$

$$\Pi=\partial\frac{L}{\partial(\partial_0\phi)}=c\hbar^2\partial_0\phi=\hbar^2\dot{\phi}$$

$$H=\Pi(\partial_0)-L=c\frac{\hbar^2}{c^2}(\dot{\phi})^2-L$$

$$=\quad c\hbar^2\left(\frac{1}{2c^2}(\frac{\partial\phi}{\partial t})^2+\frac{1}{2}(\nabla\phi)^2+\frac{m^2c^2\phi^2}{2\hbar^2}\right)=c\hbar^2\left(\frac{\Pi^2}{\hbar^4}+\frac{1}{2}(\nabla\phi)^2+\frac{m^2c^2\phi^2}{2\hbar^2}\right)$$

$\Rightarrow$

$$\tilde{\phi}(\vec{x},t)=\hbar\sqrt{\hbar c}\int\frac{d^3\vec{k}}{\sqrt{2\omega_{\vec{k}}}}\left(\hat{a}_{\vec{k}}e^{-i\omega_{\vec{k}}t}+\hat{a}^\dagger_{\vec{k}}e^{i\omega_{\vec{k}}t}\right)$$

$$\tilde{\Pi}(\vec{x},t)=\hbar\frac{\sqrt{\hbar c}}{i}\int\frac{d^3\vec{k}}{\sqrt{2\omega_{\vec{k}}}}\omega_{\vec{k}}\left(\hat{a}_{\vec{k}}e^{-i\omega_{\vec{k}}t}-\hat{a}^\dagger_{\vec{k}}e^{i\omega_{\vec{k}}t}\right)e^{i\vec{k}\vec{x}}$$

$\Rightarrow$ If we define $\sqrt{c}\,\hbar\,\hat{a}_k=\hat{\tilde{a}}_k$

$$\tilde{\phi}(\vec{x},t)=\sqrt{\hbar}\int\frac{d^3\vec{k}}{\sqrt{2\omega_{\vec{k}}}}\left(\hat{\tilde{a}}_{\vec{k}}e^{-i\omega_{\vec{k}}t}+\hat{\tilde{a}}^\dagger_{\vec{k}}e^{i\omega_{\vec{k}}t}\right)$$

$$\tilde{\Pi}(\vec{x},t)=\frac{\sqrt{\hbar}}{i}\int\frac{d^3\vec{k}}{\sqrt{2\omega_{\vec{k}}}}\omega_{\vec{k}}\left(\hat{\tilde{a}}_{\vec{k}}e^{-i\omega_{\vec{k}}t}-\hat{\tilde{a}}^\dagger_{\vec{k}}e^{i\omega_{\vec{k}}t}\right)e^{i\vec{k}\vec{x}}$$

So, these are the operators. Okay. Remember each of $\hat{a}_k$ comes with $e^{-i\omega t}$ or – iω_k' t here. But each of $\hat{a}_k^\dagger$s come with + e + i ωt. Okay. So now this is the structure operator parts here are only $\hat{a}_k$. So commutator will have four kind of term when this goes and tries to do a commutation with this.This one goes and tries to do a commutation with this with a negative sign. This one goes and tries to do a commutation with the first and this one goes and tries to do a commutation with the last term over here. So ultimately you will get four terms. Their exponentials in the time will combine in that particular way and in their exponentials in the position will also combine in that particular way okay so this should be fine enough you should just write it down collect all the commutators and you will get the next equation the big curly bracket equation here so this is what ϕ Π commutator at equal time so this is at equal time. Remember when we did commutator of classical particles as well, the q and p are supposed to be at the same time, then only it becomes $i\hbar$. If q is at time t1 and p is at time t2, then you do not get $i\hbar$. So, this is called equal time commutation relation. Equal time commutation relation which also gets a short name ETCR. So, times have to be the same. So, you see the coefficients of both ω_k and $\omega_{k'}$ are the same because we are talking about same time, but we are not talking about the same position. So, therefore, coefficients of k and k's are different. Now, one more thing I would do before going ahead, I would just do the integral d^3k'd is also there here. What I am going to do is a variable transformation. I am going to relocate each k'd into – k'd. So, you know the integral measures would not change. So, if d^3k was running from – infinity in all its component k_x, k_y, k_z. it will go into d^3 again remain the same with a – sign, let us call it k″. So then it also becomes the same thing. So the measure does not change under this redefinition.ω_k does not change because it is not sensitivesignature of K. It becomes –, it becomes K square. And we know $\hat{a}_k$ and $\hat{a}_k^\dagger$s are also symmetric under k going to – k. So, nothing will change up to the last term here, only here k' will become negative, all other terms will remain the same. Nothing changes under k going to – k. So, ultimately you will have exponential $e^{ik(\dot{x}-\dot{y})}$. So, overall factor of all the four commutators, the spatial part. Previously they were all $+ k\dot{x} + k'\dot{y}$. Now under redefinitions of k going to – k, it will become $e^{ik(\dot{x}-\dot{y})}$. So this is what you would get. Now let us try to compare things. I want answer should be $i\hbar$. times integral in d^3k times $e^{ik(\dot{x}-y)}$. First of all, there is no time dependency. It should be true for all times. So, the time argument of this ϕ and Π should be the same, but that time argument could be anything. It could be t equal to 0, 25, 30. The answer on the right hand side should not change. So, it should be time independent. So first of all, I will try to locate and try to see how the time elements could be taken out. So you see time appears here, time appears here, time appears here and time appear here. So you see in the last two terms, in this and this, time dependency can be taken out ifω_k is equal toω_k'. Then these portions will drop off. So from here I want to have $\delta(k')'$. If they become the same thing, then the time dependency in the last two terms goes away. But look at the first two terms, $(\omega_k + \omega_k')d$, that can never be set to 0. Time dependency cannot be killed becauseω_k cannot be equal to – of $\omega_k'd$ because ωs are positive quantities. They are square $\sqrt{(k^2c^2 + m^2c^4/\hbar^4)}$. So therefore, there is no way by selecting ω_k and ω_k', I can kill this term. So only way this term can go away if the commutators they are getting multiplied with, they go out, they become 0. So ultimately the demand which we should have is that this ω_k and ω_k' should be same which demands k and k' should be the same and then these two terms will go will lose their time dependency for first two terms to lose their time dependency only their commutators need to vanish there is no other way so ultimately The demand of time-independency will force me to have this a commutator to be 0. This should be 0. This should also be 0 and these two terms are roughly the same thing with a flip of a – sign. So, remember here I had a + over here, I should have a + here, I had just flip $\hat{a}_k^\dagger\hat{a}_k'$ was $\hat{a}_k^\dagger\hat{a}_k^\dagger$ commutator $\hat{a}_k$ was there, so I just flipped their order to obtain in a similar kind of structure. If both of them become the $\delta(k-k')$, the time dependency goes away and even the exponential, I had $e^{ik(\dot{x}-\dot{y})}$ with a $\delta(k-k')$, this will also become $e^{ik(\dot{x}-\dot{y})}$, something which we were looking for. So, in order to obtain the proper commutator, The commutation relation set which I have got should be this boxed equation. This equation over here. I should get. So now once this commutator relations are with us, we can go back to the Hamiltonian and try to find out how does the Hamiltonian change. So remember there is an extra c $/\hbar^2$ factor we want. In the commutator because that has to cancel this $\hbar^3/c$ factor to leave us with $\hbar$ factor outside so we wanted just a $i\hbar$ factor outside we are getting $(i\hbar)^3/c$ so therefore This c and $\hbar^2$ which are coming extra should be adjusted by the commutator. They should envelop that to get us the correct thing. So, we have this relation therefore at our hand. If that is the relation that means $\hat{a}_k^\dagger$ should be equal to this commutator value which is $c\hbar^2\ \delta(0)$. Because when I put k is equal to k' I will get a $\delta(0)$ + $\hat{a}_k^\dagger$. Sorry, so I should have written, so this is fine, + $\hat{a}_k^\dagger\hat{a}_k$. Okay, so this is what I would get. So, therefore, we can in the Hamiltonian, which we had obtained previously, $\hat{a}_k\hat{a}_k^\dagger$ was appearing over here.

To this, I would just write down $c/\hbar^2$ $\delta(0)$ and $\hat{a}_k^\dagger\hat{a}_k$. To this, I would just write down $c/\hbar^2$ $\delta(0)$ and $\hat{a}_k^\dagger\hat{a}_k$. So, that is what I will get twice of $\hat{a}_k^\dagger\hat{a}_k$ and c /$\hbar$ c /$\hbar$ and $\delta(0)$ okay so therefore from outside as we discussed this hamiltonian had the structure if you remember hamiltonian had a structure which was Outside we had a, so we have to go to a couple of steps back to just look at where we had, here it is. The Hamiltonian had outside $c\hbar^2/2$. So, that c $\hbar^2/2$ would be remaining, just a second. Right and this c^2 we had pull up pulled out so we had $c\hbar^2/2c^2$ ultimately outside I have a $\hbar^2$ /2c so that's what would remain outside and inside I will get twice of $\hat{a}_k^\dagger\hat{a}_k + c/\hbar^2$ $\delta(0)$. now the time is there to bring this whole factor inside If I bring the whole factor inside, here the two will cancel each other. $\hbar^2/c$ will be remaining with $\hat{a}_k^\dagger\hat{a}_k$ and + $\delta(0)$ will be obtained. So, you see this will become our Hamiltonian. For it is almost like harmonic oscillator. Each harmonic oscillator has this much energy. This is the number operator for each k. And instead of 1, I am getting a $\delta(0)$, which is the artifact of a continuous theory. Continuous theory would have $\delta(k-k')$. Chronicle delta will be replaced by a Dirac delta. $\delta(0)$ will come about. And then I have a d3k, summation over all such oscillators in momentum space.

So only thing is just that this $\hbar^2/c$ might be troubling you. So what we can do we can absorb it in the redefinition of various things. For example I can redefine the field as ϕ tilde as $\sqrt{c\hbar\phi}$. So if remember if that we did our Lagrangian which we started with was also containing a – $\hbar^2$ /2 times c and this was the Lagrangian. If I do this redefinition of thing, I will get this c times $\hbar^2$ absorbed in ϕ and I will get a ϕ tilde everywhere. So, extra factor outside, not *1 /2,* this c $\hbar^2$ factor can be absorbed into redefinition of ϕ. Then you can get a redefinition of pi, which is a conjugate momenta. So, ϕ will go to ϕ tilde, Π will go to Π tilde. Then things will start coming into order. You would get uh ϕ tilde as this because previously you had this much for you had this much for the usual ϕ with a c will come so x extra √h /c uh extra h /√c will come about so this h and √c will be extra compared to the previous definition and similarly for Π as well this $\hbar$ and √c will be extra after absorbing the outside factor. So you will get this kind of new ϕ tilde and Π tilde and in the same spirit as we rescaled the ϕ if I rescale the $\hat{a}_k$ and $\hat{a}_k^\dagger$ absorbing these coefficients I will have a a tilde k which is just the same factor multiplied to previous $\hat{a}_k$ as the new ϕ gets defined after absorbing these factors into ϕ. If I do in this all things that means I absorb this √c /√c times $\hbar$ everywhere the correct ϕ and Π tilde in terms of their $\hat{a}_k$ tilde and $\hat{a}_k^\dagger$ tilde would be exactly like harmonic oscillator without any extra $\hbar$ or c factors. So all those things I have absorbed and the Lagrangian becomes simple enough and Hamiltonian in that case would be exactly like harmonic oscillator thing without any troublesome factor of $\hbar^2/c$.

Then, $\mathcal{H} = \int d^3\vec{k}\ \hbar\omega_{\vec{k}} \left(\tilde{a}^{\dagger}_{\vec{k}} \tilde{a}_{\vec{k}} + \frac{1}{2}\delta^3(0) \right)$

Thus, a useful set of variables leading to harmonic oscillator quantization is

$$\mathcal{L} = -\frac{1}{2}\left(-\frac{1}{c^2}(\partial_t \phi)^2 + (\nabla\phi)^2 - \frac{m^2c^2}{\hbar^2}\phi^2 \right)$$

$$\phi(\vec{x}) = \sqrt{\hbar}\int \frac{d^3\vec{k}}{\sqrt{2\omega_k}} \left(\hat{a}_k e^{-i\omega_{\vec{k}} t + i\vec{k}\cdot\vec{x}} + \hat{a}^{\dagger}_k e^{i\omega_{\vec{k}} t - i\vec{k}\cdot\vec{x}} \right)$$

$$\Pi(\vec{x}) = \frac{\sqrt{\hbar}}{i}\int d^3\vec{k}\ \sqrt{\frac{\omega_{\vec{k}}}{2}} \left(\hat{a}_k e^{-i\omega_{\vec{k}} t + i\vec{k}\cdot\vec{x}} - \hat{a}^{\dagger}_k e^{i\omega_{\vec{k}} t - i\vec{k}\cdot\vec{x}} \right)$$

$$\omega_{\vec{k}} = \sqrt{k^2c^2 + \frac{m^2c^4}{\hbar^2}}$$

$$[\hat{a}_{\vec{k}}, \hat{a}_{\vec{k}'}] = 0 = [\hat{a}^{\dagger}_{\vec{k}}, \hat{a}^{\dagger}_{\vec{k}'}]$$

$$[\hat{a}_{\vec{k}}, \hat{a}^{\dagger}_{\vec{k}'}] = \delta(\vec{k} - \vec{k}')$$

$$\mathcal{H} = \int d^3\vec{k}\ \hbar\omega_{\vec{k}} \left(\hat{a}^{\dagger}_{\vec{k}} \hat{a}_{\vec{k}} + \frac{1}{2}\delta^3(0) \right)$$

Then,

$$H=\int d^3\vec{k}\,\hbar\,\omega_{\vec{k}}\left(\ddot{\hat{a}}^{\dagger}_{\vec{k}}\ddot{\hat{a}}_{\vec{k}}+\frac{1}{2}\delta^3(0)\right)$$

Thus, a useful set of variables leading to harmonic oscillating quantization is

$$L=-c\frac{\hbar^2}{2}\left(\frac{-(\partial_t\phi)^2}{c^2}+(\partial_x\phi)^2+(\partial_y\phi)^2+(\partial_z\phi)^2-\frac{m^2c^2}{\hbar^2}\phi^2\right)$$

$$\Pi(\vec{x},t)=\frac{\sqrt{\hbar}}{i}\int d^3\vec{k}\sqrt{\frac{\omega_{\vec{k}}}{2}}\left(\hat{a}_{\vec{k}}e^{-i\omega_{\vec{k}}t+i\vec{k}\cdot\vec{x}}-\hat{a}^{\dagger}_{\vec{k}}e^{i\omega_{\vec{k}}t-i\vec{k}\cdot\vec{x}}\right)$$

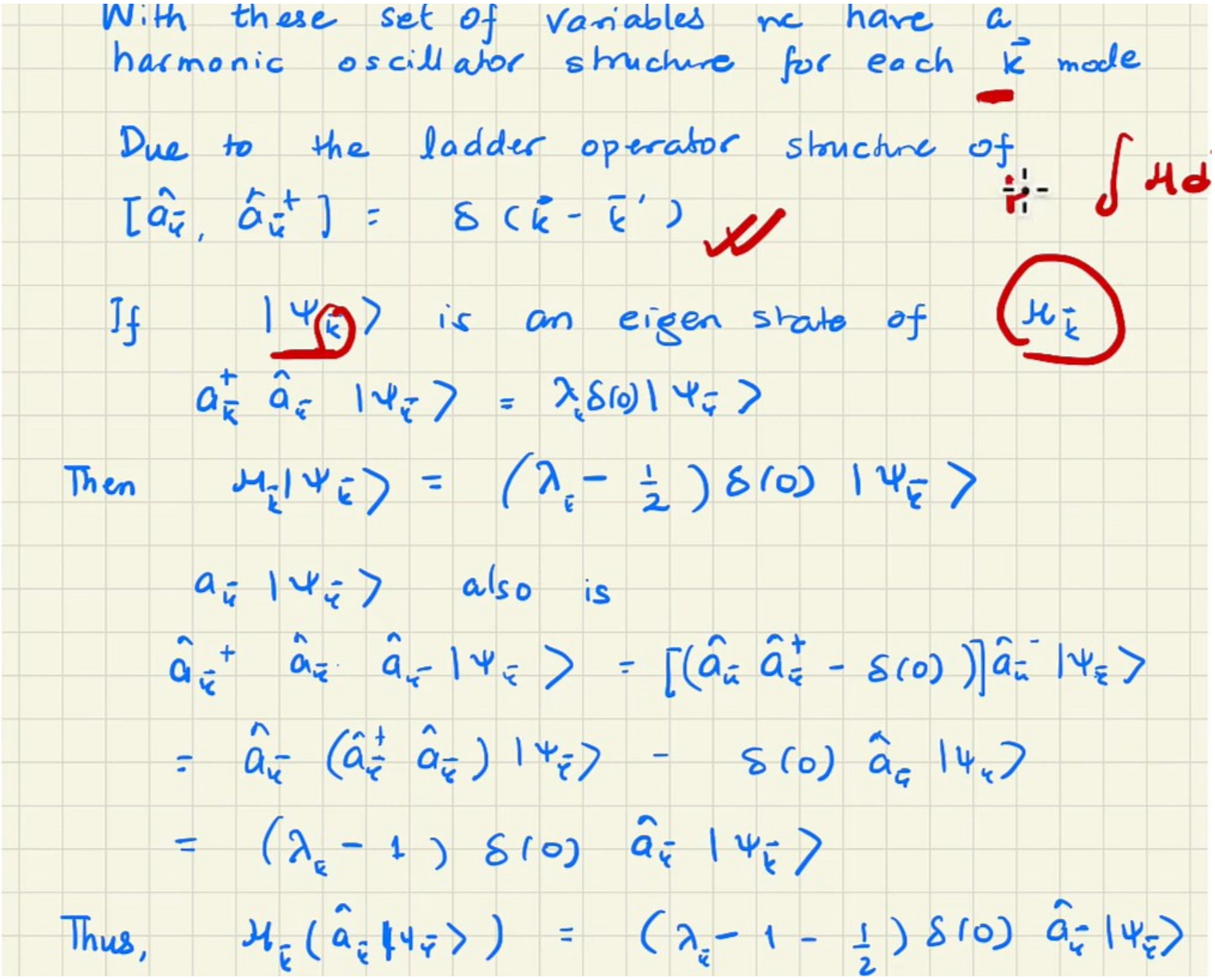
where $\omega_{\vec{k}}=\sqrt{k^2c^2+\frac{m^2c^4}{\hbar^2}}$

$$[\hat{a}_k,\hat{a}_{k'}]=0=[\hat{a}^{\dagger}_k,\hat{a}^{\dagger}_{k'}]$$

$$H=\int d^3\vec{k}\,\hbar\,\omega_{\vec{k}}\left(\ddot{\hat{a}}^{\dagger}_{\vec{k}}\ddot{\hat{a}}_{\vec{k}}+\frac{1}{2}\delta^3(0)\right)$$

So I could have started the story with reabsorbing the ϕ. So I have just showed you a way where you start with a general ϕ and then you land up to harmonic oscillator up to a coefficients constant coefficient that you can absorb and overall in many textbook actually you will find this thing being told from the beginning with a mysticism called something called a natural units so we are very close to that that we have absorbed $\hbar$ and c as much as possible. Not completely, we could have absorbed even c's appearing here and there into this.

And most of the textbooks tell you that they are going to write down something called $\hbar$ is equal to 1 and c is equal to 1 units. That means they will not care of presence of $\hbar$ here and presence of c there. But still if you want to focus that you want to keep all $\hbar$ and c together a good set to do business would be this Lagrangian where we have ate up the extra $\hbar^2 c$ factor coming from outside. And this Π this Π ϕ pair which is just like harmonic oscillators and lastly you will get exact the exactly the same commutator between $\hat{a}_k$ and $\hat{a}_k^{\dagger}$ just like a ordinary harmonic oscillator which we had done previously and therefore the hamiltonian will become just a collection of all oscillators put together only signature of this being a continuous theory is appearance of this $\delta^3(0)$ rather than a one so in harmonic oscillator case remember this function which was appearing apart from the number operator counts the state of the energy of the vacuum state suppose there was a vacuum state for oscillator there you knew that even for vacuum state there is even for vacuum state there is an energy which is obtainable from half factor so half $\hbar\omega$ *is* the energy of a vacuum state if you excite further this $\hat{a}_k^{\dagger}\hat{a}_k$ adds particle to this which each come with n $\hbar$ ωhow many particles you have excited the same game is appearing over here this infinite $\delta(0)$ is actually the energy of the this is the energy of the vacuum state and then you are summing over all vacuum of all of these things in a continuous theory will give you infinite amount of energy and over that the finite part is the excitation energy coming from the excitations of particle this counts this number operator times $\hbar\omega$ counts the excitation or energy excitation above the vacuum energy just like it did for harmonic oscillator. half factor so half $\hbar\omega$ *is* the energy of a vacuum state if you excite further this $\hat{a}_k^{\dagger}\hat{a}_k$ adds particle to this which each come with n $\hbar\omega$ how many particles you have excited the same game is appearing over here this infinite $\delta(0)$ is actually the energy of the this is the energy of the vacuum state and then you are summing over all vacuum of all of these things in a continuous theory will give you infinite amount of energy and over that the finite part is the excitation energy coming from the excitations of particle this counts this number operator times $\hbar\,\omega$counts the excitation or energy excitation above the vacuum energy just like it did for harmonic oscillator.



With these set of variables se have a harmonic oscillator structure for each $\vec{k}$ mate.
Due to the ladder operator structure of

$$[\hat{a}_k, \hat{a}^{\dagger}_{k'}] = \delta(\vec{k} - \vec{k}')$$

$$\hat{a}_k, \hat{a}^{\dagger}_{k'} |\psi_{\vec{k}}\rangle = \lambda_f \delta(0) |\psi_{\vec{k}}\rangle$$

Then

$$H_{\vec{k}} |\psi_{\vec{k}}\rangle = (\lambda_f - \frac{1}{2}) \delta(0) |\psi_{\vec{k}}\rangle$$

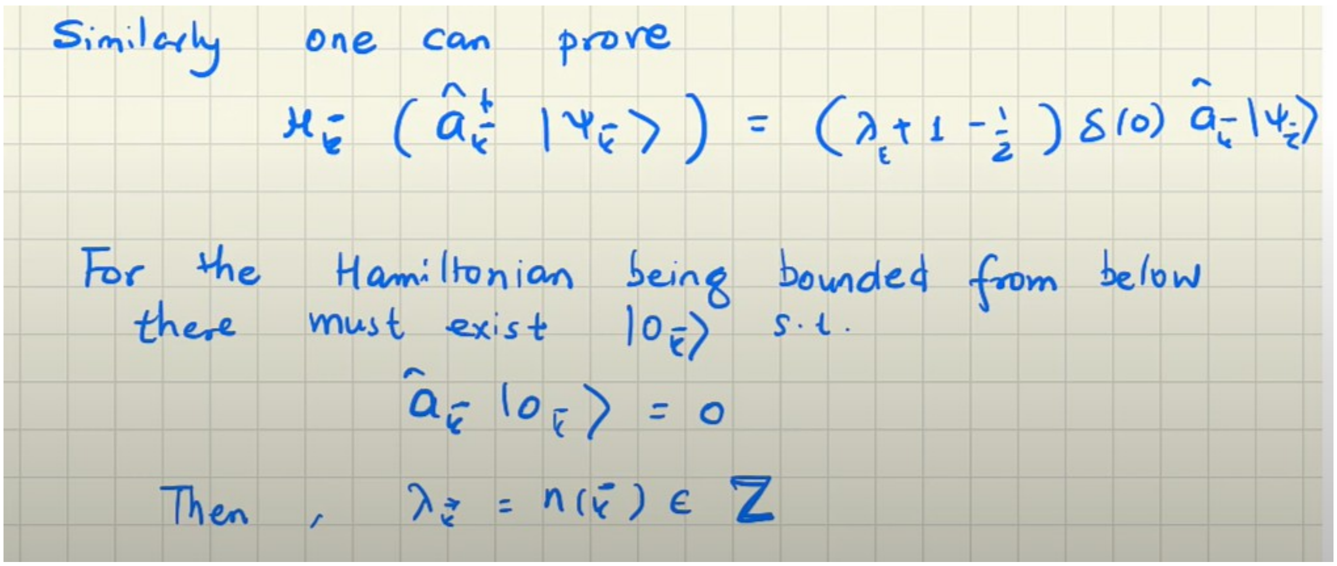
$a_{\vec{k}}|\psi_{\vec{k}}\rangle$ *also is* $\quad \hat{a}^{\dagger}_{\vec{k}}\hat{a}_{\vec{k}}\hat{a}_{\vec{k}}|\psi_{\vec{k}}\rangle=[\hat{a}_k\hat{a}^{\dagger}_k-\delta(0)]\hat{a}_{\vec{k}}|\psi_{\vec{k}}\rangle$

$= \quad \hat{a}_{\vec{k}}(\hat{a}^{\dagger}_{\vec{k}}\hat{a}_{\vec{k}})|\psi_{\vec{k}}\rangle-\delta(0)\hat{a}_{\vec{k}}|\psi_{\vec{k}}\rangle$

$= \quad (\lambda_f-1)\delta(0)\hat{a}_{\vec{k}}|\psi_{\vec{k}}\rangle$

Thus,

$$H_{\vec{k}}(\hat{a}_{\vec{k}}|\psi_{\vec{k}}\rangle)=(\lambda_{\vec{k}}-1-\frac{1}{2})\delta(0)\hat{a}_{\vec{k}}|\psi_{\vec{k}}\rangle$$



Simalarly one can prove

$$H_{\vec{k}}(\hat{a}_{\vec{k}}|\psi_{\vec{k}}\rangle)=(\lambda_{\varepsilon}+1-\frac{1}{2})\delta(0)\hat{a}_{\vec{k}}|\psi_{\vec{k}}\rangle$$

For the Hamitonian being bounded from eblow there must exist $|0_{\vec{k}}\rangle$ s.t.

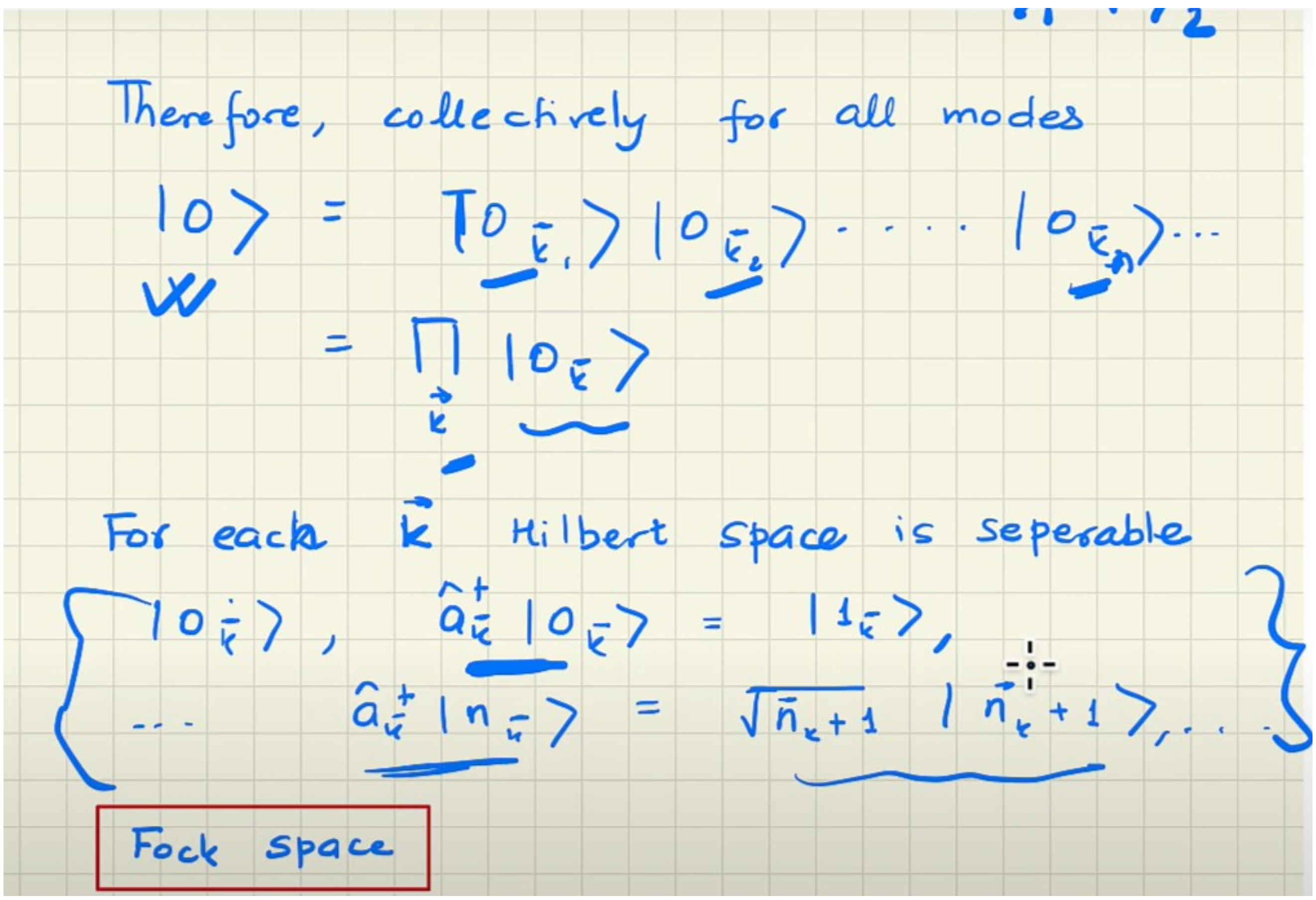
$$\hat{a}_{\vec{k}}|0_{\vec{k}}\rangle=0$$

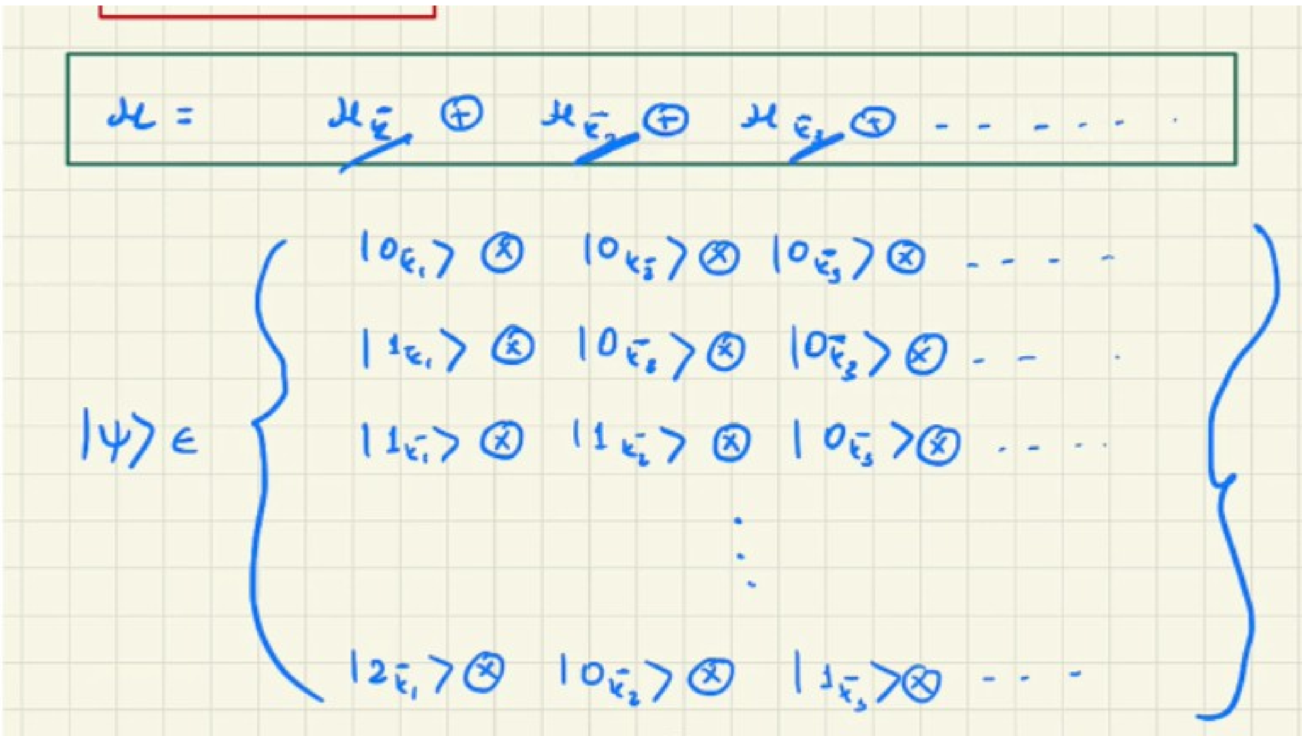
Then, $\lambda_{\vec{k}}=n_{(\vec{k})} \quad \epsilon Z$

Only thing of the interest or change here for a continuous theory you are getting an infinite as the energy of the vacuum compared to a finite half which was previously present. So now we will go further and try to identify the existence of such a vacuum and how to identify different particle modes which can exist over here. So, again we will start the story similar to the harmonic oscillator case, we will assume that suppose there is a Eigen state of a Hamiltonian and then we will try to see what does $\hat{a}_k$ and $\hat{a}_k^{\dagger}$ do, where do we that takes us and what kind of states are generated. We have a harmonic oscillator structure for each k mode and we have a commutator structure just like a analog of a ordinary oscillator with a delta function appearing this time rather than a Kronecker delta. Now, let us assume that $\psi_{\vec{k}}$ is an eigenstate of Hamiltonian at k. Remember, we had identified Hamiltonian as d^3x of Hamiltonian density. Total Hamiltonian was integration of Hamiltonian density d^3x. The same thing can be written as a d^3k *H(k)* as well. So, total Hamiltonian can be written like this as well. This is an exercise for you to find out what is the expressions of *H(k)* and actually you can see *H(k)* is nothing but the harmonic oscillator, this is your *H(k)*. So, now we have a Hamiltonian for each k. This is like a harmonic oscillator for each k. Therefore, there should be eigenstate for each oscillator as well. This is like a harmonic oscillator for each k. Therefore, there should be eigenstate for each oscillator as well. In the definition of *H(k)*, Already there is a trivial part. Trivial part in the sense of this $\delta^3(0)$ times half does not come with any operator. That means it is coming with identity operator. So that part acting on $|\psi_{\vec{k}}>$ will give you $|\psi_{\vec{k}}>$ back. So that is eigenstate part is respected by that term. The only non-trivial part is that the first number operator should act on the $|\psi_{\vec{k}}>$ and give back you state $|\psi_{\vec{k}}>$ with some eigenvalue times $\delta(0)$ let us say. Let us say this is the energy eigenvalue for this operator, $\lambda_{\vec{k}}$ times $\delta(0)$. I am just redefining my eigenvalue to be $\lambda_{\vec{k}}$ times $\delta(0)$. This is my demand. So if that is the case, then total energy of the whole system should be $\lambda_{\vec{k}}$ + half. So, sorry, I have written a – here and there. I do not know why, but let us see this. It should be, okay. So, total energy in this definition, if I put that eigenstate of $\hat{a}_k^\dagger\hat{a}_k$ gives you eigenvalue $\lambda_{\vec{k}}$ times $\delta(0)$, then the total Hamiltonian acting on the $|\psi_{\vec{k}}>$ will give me first term will give me $\lambda_{\vec{k}}$ times $\delta(0)$. The second term will give me half times $\delta(0)$. So, together combined I will get $\lambda_{\vec{k}}$ + half $\delta(0)$. So, this would be the total eigenvalue of the Hamiltonian. So, I am saying if I take eigenstate of $\hat{a}_k^\dagger\hat{a}_k$ operator with eigenvalue $\lambda_{\vec{k}}$ times $\delta(0)$. Then that state is also eigenstate of total Hamiltonian with eigenvalue $\lambda_{\vec{k}}$ + half $\delta(0)$, all right. So, now let us try to see what do we get What do we get in terms of if I act $\hat{a}_k$ operator on $|\psi_{\vec{k}}>$. So, let us see whether it acts as a ladder operator, raising operator or lowering operator or not. So, first I take $|\psi_{\vec{k}}>$ 'd and define it like this. And I want to know what happens when h of k acts on $\psi_{\vec{k}}$'d. What do I get? So there are again in h of k there are two pieces one is half $\delta(0)$ which is trivial that will give me the $\lambda_{\vec{k}}'$ back with a half $\delta(0)$. Only non-trivial operation I have to be aware about is $\hat{a}_k^\dagger\hat{a}_k$ acting on $|\psi_{\vec{k}}>'$. $|\psi_{\vec{k}}>'$ is $\hat{a}k$ times $|\psi_{\vec{k}}>$. So there I can play the game that I would now have this $\hat{a}_k^\dagger\hat{a}_k$ here appearing. I can flip it using the commutator into this so $\hat{a}^\dagger_k\, \hat{a}_k$ was appearing over here I can flip it using the commutator as we had done in past as $\delta(0)\ \hat{a}_k\hat{a}_k^\dagger$ okay so that is what I will get so here so I have in hurry i had changed the signature here as well so here it should not change only here last step it should have changed okay all right so using the commutator properly I will get a $\hat{a}_k\hat{a}_k^\dagger - \delta(0)$ would be $\hat{a}_k^\dagger\hat{a}_k$ which is coming from the commutator we just wrote above. From this commutator when you put k is equal to k*′* you will get this relation over here. So, I have flipped the left two operators into this and $\hat{a}k$ which was appearing over here still survives like this. Now we go forward there will be two terms once this combining with this and then $\delta(0)$ combining with this.

So first I will get three $\hat{a}_k\ \hat{a}_k\ \hat{a}_k^\dagger\ \hat{a}_k$ coming from this and this put together. Just let me clean it up so that we can see. So we will have three $\hat{a}_k\ \hat{a}_k^\dagger$ and $\hat{a}_k$ which is over here and then – of $\delta(0)$ times this which is fine. And we already know what does $\hat{a}_k^\dagger$ appearing here times $\hat{a}_k$ acting on $|\psi_{\vec{k}}>$ does. $\hat{a}_k^\dagger\ \hat{a}_k$ acting on $|\psi_{\vec{k}}>$ used to give me $\lambda_{\vec{k}}$ times $\delta(0)$. So that will still be maintained. $\lambda_{\vec{k}}$ from here I will get $\lambda_{\vec{k}}$ times $\delta(0)$ and one $\hat{a}_k$ will be left outside. Together that will become $\lambda_{\vec{k}}$ times $\delta(0)$. and $\hat{a}_k$ acting on $|\psi_{\vec{k}}>$ and from this side $\hat{a}_k$ acting on $|\psi_{\vec{k}}>$ with a $-\ \delta(0)$. So, put together everything I would have a $\lambda_{\vec{k}} - 1\ \delta(0)$ with $\hat{a}_k\ |\psi_{\vec{k}}>$. Remember $\hat{a}_k\ |\psi_{\vec{k}}>$ was defined to be $|\psi_{\vec{k}}>$ 'd. So, we had written somewhere which I might have erased right now. So, what happened when I operated $\hat{a}_k^\dagger\ \hat{a}_k$ on $\hat{a}_k\ |\psi_{\vec{k}}>$. I got back $\hat{a}k\ |\psi_{\vec{k}}>$ with eigenvalue $\lambda - 1\ \delta(0)$. That means $\hat{a}k$ has reduced the eigenvalue by 1. Only $|\psi_{\vec{k}}>$ was eigenvalue with λ. $\hat{a}k$ times $|\psi_{\vec{k}}>$ is reduced eigenvalue by 1. So therefore the Hamiltonian will also go down one step. I will get a $\lambda - 1$ + half this time. Okay and similarly you can prove the just like harmonic oscillator if i did for $\hat{a}_k^\dagger$ the same business you will realize that this time eigenvalues go up by one so what $\hat{a}k$ does $\hat{a}k$ reduces the eigenvalue of $|\psi_{\vec{k}}>$ by unit one times $\delta(0)$ and $\hat{a}_k^\dagger$ enhances the eigenvalue with unit one times $\delta(0)$. Only thing which might have been worrying you why I had taken $\delta(0)$ in this definition of Feigen value. This is because you see I have two $\hat{a}k$'s multiplying. Two $\hat{a}k$ multiplying should give you something of the dimension of $\delta(0)$. It cannot be that left hand side is dimensionless quantity and right hand side is a dimensional. This is a Dirac delta in volume in $\hat{a}_k$ space. Dirac delta in k space is like a dimension of volume. Total volume so this is what we should also get after two $\hat{a}k$s act the state should not remain the state is whatever dimension of the state is there it would give you the same thing but extra factor here and extra factor here should have the same dimension so if i call this as a real number then something like a volume should multiply that is why I have written a $\delta(0)$.

Okay so just to be consistent I have redefined λs in as λ times $\delta(0)$ so you see it doesn't matter ultimately in this way if I write what happens $\hat{a}k$'s take the states down by one units $\hat{a}_k^\dagger$'s take the states up by one unit so for each k oscillator $\hat{a}k$ and $\hat{a}_k^\dagger$ indeed behave as raising and lowering operator and everything of harmonic oscillator comes true. And in the same spirit, just like for ordinary harmonic oscillator, if there exists a lower most state, if the Hamiltonian is bounded from below, that is to say, if there is a minima of the energy, then at some stage, $\hat{a}k$'s operation should act on that minimum energy state and give you zero. Because if it is not 0, some other state, I will know that it will be eigenstate of Hamiltonian with one reduced unit of $\delta(0)$ energy less. And then again acting on that, I will get one step down further and further and further. This process will only terminate once you hit a 0, not 0 ket. So 0, number 0. That is the state which is completely annihilated by this. And I know this $\hat{a}k$'s go down by unit 1. In any case, it will be annihilated only if this λ case are integer numbers. Then only going down, I will hit 0 after few steps. If there were not integers, let us say 1.5 or something, I will never hit this 0 after even operating 10 times or 20 times. So, two things are realized. There is a eigenstate with 0 eigenvalue, let us say, and that fixes that all other eigenstates of the Hamiltonian or number operator would have integer ns.





Eqn

Therefore, collectively for all modes

$$|0\rangle = |0_{\vec{k}_1}\rangle, |0_{\vec{k}_2}\rangle \cdots |0_{\vec{k}_n}\rangle \cdots$$
$$= \prod_{\vec{k}} |0_{\vec{k}}\rangle$$

For each $\vec{k}$ Hilbert space is separable

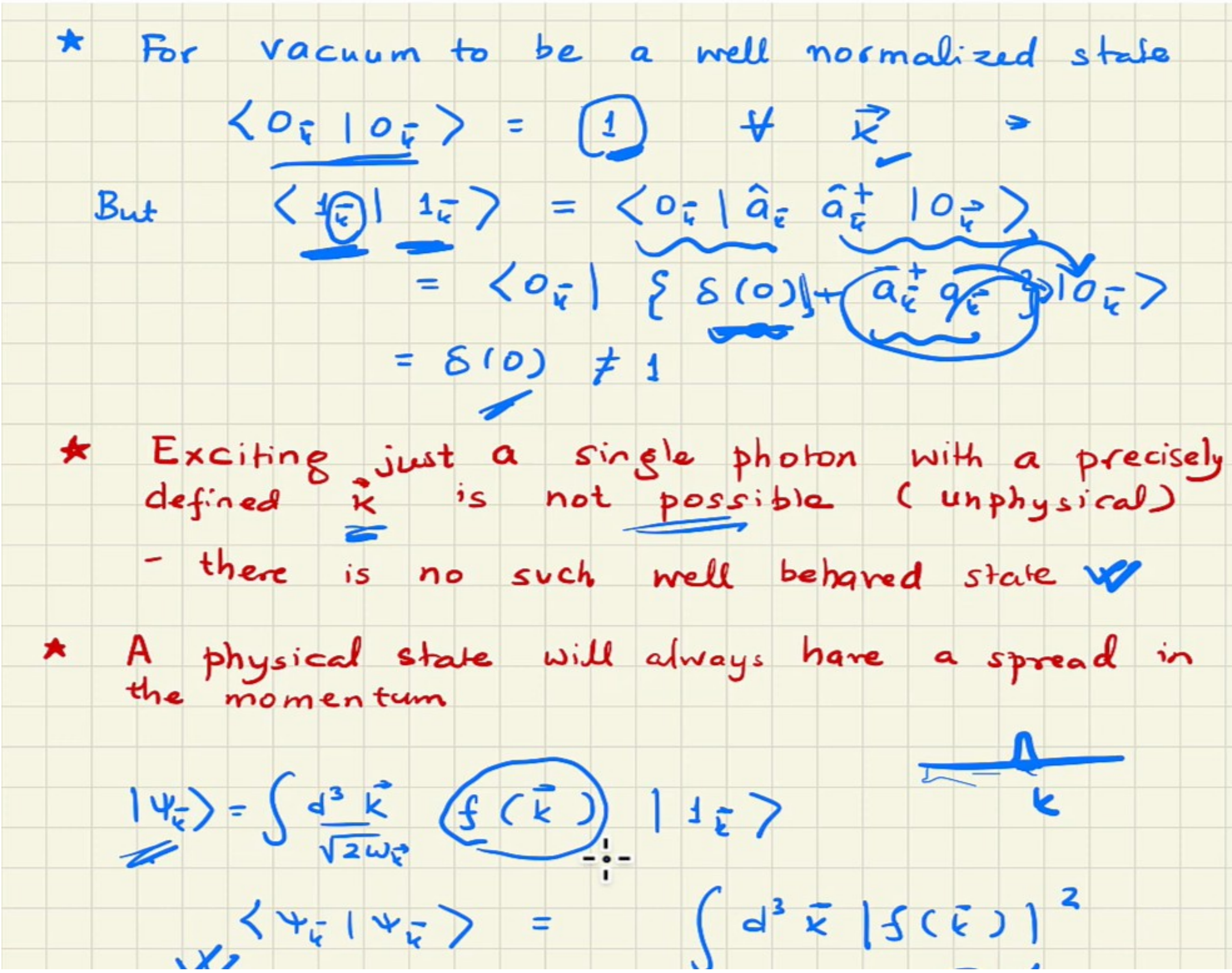
$$|0_{\vec{k}}\rangle, \hat{a}_{\vec{k}}|0_{\vec{k}}\rangle = |1_{\vec{k}}\rangle$$
$$\cdots \quad \hat{a}_{\vec{k}} = \sqrt{\vec{n}_k + 1}|\vec{n}_k + 1\rangle$$

Fock space

$$H = H_{\vec{k}_1} H_{\vec{k}_2} H_{\vec{k}_3} \ldots ..$$

$$|\psi_{\vec{k}}\rangle \epsilon \begin{pmatrix} |0_{\vec{k}_1}\rangle\, |0_{\vec{k}_2}\rangle\, |0_{\vec{k}_3}\rangle \ldots \\ |1_{\vec{k}_1}\rangle |1_{\vec{k}_2}\rangle |1_{\vec{k}_3}\rangle \ldots \\ \cdot \\ \cdot \\ \cdot \\ |2_{\vec{k}_1}\rangle\, |2_{\vec{k}_2}\rangle\, |2_{\vec{k}_3}\rangle \ldots \end{pmatrix}$$

There is a eigenstate with 0 eigenvalue, let us say, and that fixes that all other eigenstates of the Hamiltonian or number operator would have integer ns. And therefore, we have just talked about one oscillator. If there are two oscillators, I know the total vacuum state is the oscillator 1's vacuum and oscillator 2's vacuum. And now we are talking about infinitely many of them for each k. So the total vacuum of the full Hilbert space will be vacuum of first oscillator, vacuum of second oscillator, vacuum of nth oscillator and so on. That is collectively written as a product k and vacuum state for each k. And the operators can be raised and lowered in each k segment by their respective $\hat{a}_k^\dagger$ operators or $\hat{a}_k$ operators. So you see for each k I can either have a 0 k which is a vacuum of that oscillator set or I can have first excited state which is $\hat{a}_k^\dagger$ acting upon $\hat{a}_k$ or I can have second excited state while two $\hat{a}_k^\dagger$s act on the vacuum and similarly nth eigen state or n + one of eigen state by obtaining $\hat{a}_k^\dagger$ acting on n k and you will get just like an ordinary oscillator you get you will get this kind of structure so this whole set of oscillator for each k exists, all set of states for each k exists and you can have a superposition of that to write down a valid oscillator for that k mode. And the Fock space is defined as collection of all such Hilbert spaces of all ks together. Total Hamiltonian is oscillator number 1, oscillator number 2, oscillator number 3 and also oscillators put together. Their Hamiltonians are in direct sum. While their states are in direct product. So you see you can have first oscillator in any of these states or their superposition. Second oscillator could be any of these states this or this or that or their superposition and so on. So there can be possible states which are like vacuum of all oscillators or first oscillator is in first excited state all other oscillators are in vacuum. Or first two oscillators are in excited state all others are in vacuum and so on similarly first oscillator is in second excited state all others are in vacuum or first two oscillators are in excited state double excited state and all others are in vacuum or one oscillator is in double eigen state one is in vacuum third one is again in first excited state and so on you can think of those infinite many proper infinitely many combinations each oscillator can have any of the eigenstate vacuum or their support position and similarly for all others. So, this description is called the Fock space. There is a vacuum for each k, there is a Hilbert space of each k and denser product of all such states is called the Fock space. Lastly, we will just see how to do business with such states.



★For vacuum to be a well normalized state

$\langle 0_{\vec{k}} | 0_k \rangle = 1 \, \forall \, \vec{k}$

But $\langle 1_{\vec{k}} | 1_k \rangle = \langle 0_{\vec{k}} | \hat{a}_{\vec{k}} \hat{a}^{\dagger}_{\vec{k}} | 0_{\vec{k}} \rangle$

=

$\langle 0_{\vec{k}} | \delta(0) \hat{a}_{\vec{k}} \hat{a}^{\dagger}_{\vec{k}} | 0_{\vec{k}} \rangle = \delta(0) \neq 1$

★Exciting justa single photon with a pecisely defiend k⃗ is not possible (unphysical)

- There is no such well behaved state.

★A physical state will alwya shave a spread in the momentum.

$$|\psi_{\vec{k}}\rangle = \int d^3 \frac{\vec{k}}{\sqrt{2\omega_{\vec{k}}}} f(\vec{k}) | f_{\vec{k}} \rangle$$

So

$\langle a | b \rangle = \int d^3 \vec{k} \, |f(\vec{k})|^2$

First, we will want at least the vacuum state of the theory to be well defined and well That means normalizations of vacuum for each oscillator should be equal to 1 that we demand that for all case ground state should be well defined let us say. If I do so, then I realize the first excited state do not remain very well defined.
Why is that? Because first excited state is obtainable from operation of $\hat{a}_k^\dagger$ on to vacuum. So this is first excited state$_{1k}$. This is its conjugate. When I take inner product between$_{1k}$ and$_{1k}$, I will get this $\hat{a}_k\hat{a}_k^\dagger$ in between. And this $\hat{a}_k\hat{a}_k^\dagger$ using the commutator can be written as a $\delta(0) + \hat{a}_k^\dagger\hat{a}_k$. This is the usual commutation usage. And this $\hat{a}k$ acting on vacuum will kill it. So therefore this term will touch this vacuum and make it 0. Only this term will survive. $\delta(0)$ is not an operator it means there is identity here $\delta(0)$ will come out and 0 k 0 k will get inner product with itself and I will get a one so ultimately $\delta(0)$ will survive which is not one it is not well normalized state it is infinity rather so that tells you exciting just a single photon with a single k is not a physical process it cannot have a well-defined state so just exciting a single k is just not possible it is unphysical it is not a well-behaved state so therefore what one should really look for we do not excite at a particular k but we have a width some spread i do not exactly know where I have excited I can just say that I have excited around k value in some width that width is given by *f(k)* so i would have state that i do not know which k exactly it is it is some k centered maybe around some k_0 value but some distribution should be there and you can find out that such a state's norm is obtainable from the L2 integration of *f(k)*.
So, if FKs are L2 function, Lebesgue functions of order 2, then the states can be well normalized. So, in order to have a well normalized state, the outcome or the cost one has to pay is that you do not claim exact knowledge of which photon of which momentum has been excited. You just say that most likely this photon is excited and there can be some width. This is something like something like uncertainty principle in some sense that you do not have precise knowledge of momenta that is barred from you if you demand precise knowledge of momenta then the wave will become plane wave in position space and therefore you would not have normalizability.
So in order to have a normalized state you need to have some superpositions of k and therefore your states in position space cannot be plane waves so this is what is exactly replicated here as well so lessons scalar field have exact harmonic oscillator like structure for each k h is harmonic oscillator for each k product over all case so not product sum over all k energy is sum over all k states are like you could have product over well-defined states in all k or summation of them. So let us say ck kind of this. okay so this kind of superpositions in different case and product over all things just like we saw it could be zero on wave number one first excited state in wave number two and then everything else in vacuum + first two things are in vacuum k_1 k_2 and nth ninth state is in third eigenstate and so on so all these things can be constituted and superposed with only thing is that different hilbert space elements will be coming along so those are the Fock space bases and in that basis one can write down well defined state by taking superpositions so first I will stop the discussion on scalar field here first we will try to get the similar kind of structure for other kinds of field for example a dirac field as we discussed and for photon fields as well. So we will realize the Fock-based structure for all of them and then we will, in the remaining week, remaining classes for field theory introduction, we will try to do business with states of our choices. So, this is the structure, the Hilbert space structure of a field. It is like oscillator is momentum space or Fourier space and every property of oscillator is more or less coming through with cleanliness. Only thing a $\delta(0)$ appears here and there that you have to be very careful because we are doing a continuum theory. So, I stop here for scalar field. In the next class, we will start dealing with Dirac fields and see whether the same kind of Fock space structure comes about for Dirac field or not. And once we are through with that, we will do photon fields or U1 field, electromagnetic field.
So, I stop over here.