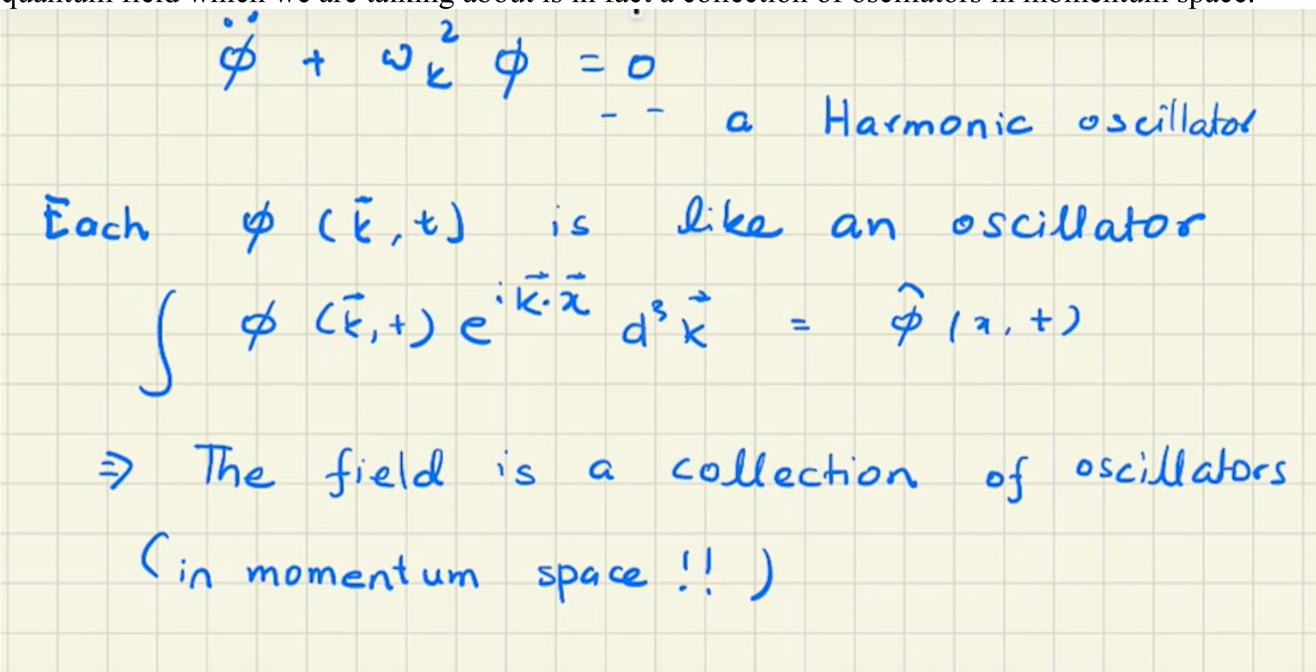
Foundation of Quantum Theory: Relativistic Approach
Quantum Field Theory 1.3
Prof . Kinjalk Lochan
Department of Physical Sciences
IISER Mohali

Quantum Scalar Field 1
Lecture- 17

So let us resume our discussion from the realization which we had in the previous class that is the quantum field which we are talking about is in fact a collection of oscillators in momentum space.



$\ddot{\phi}+\omega_k^2\phi=0$
---- a Harmonic oscillator.

Each $\phi(\vec{k},t)$ *is* a collection of oscillators.
$\int \phi(\vec{k},t) e^{i\vec{k}.\vec{x}} d^3\vec{k}$
⇒The field is a cllection of oscillators (n mommentum space!!)

And these operators or oscillators in momentum space together combine in a particular Fourier transform manner to generate quantum fields in space time. That is what we realized in the last class. We move forward in this class to see more clearly the oscillator structure and the quantization of oscillators as we used to know. the same machinery will be employed here and we will get a bunch of oscillators with their quantum properties collectively making up for the properties of ϕ . Before moving to that, let us see a previous thing which we had. Remember the Hamiltonian which we had was a quadratic structure thing. There was a Π 2 which was like p square. There was a ϕ_2 which was x^2. But there was a gradient of ϕ_2 which normally we do not see in ordinary harmonic oscillator case. So therefore we went to the Fourier basis where this gradient term became k^2 actually which combined with this $m^2c^2/\hbar^2$ giving rise to the frequency ω^2 . So that is the benefit of going to the Fourier space was.

Now one thing I should make clear here. In this discussion we had traded delta 0 ϕ. For momentum Π that is a choice we chose in fact we could have chosen not to trade $\partial_0\phi$ but to $\dot{\phi}$ there is a one over c difference between $\partial_0\phi$ and $\dot{\phi}$ so $\partial_0\phi$ is actually equal to $\dot{\phi}/c$ because remember zero means x_0 means ct so if I define my pi as a conjugate momentum with respect to $\dot{\phi}$ that means I try to trade $\dot{\phi}$ rather than $\partial_0\phi$, then I will earn up an extra c factors here and there.
For example, with respect to $\partial_0\phi$, there was no c here. But if I define my conjugate momenta by the legendary exchange of ϕ, I will end up getting an extra c factor. Therefore, $\Pi\phi$, which we computed previously, we had taken pi $\partial_0\phi$. If I do $\Pi\phi$,I will again end up getting an extra c factor as well from before compared to the previous case. So ultimately, what it does, if I stick to the definition of trading ϕ, with Π that is to say define my Hamiltonian as $\dot{\phi}\Pi$– L while Πitself is defined as $\partial L/\partial\dot{\phi}$ Then what happens that this extra $1/c^2$ factor which I was getting in the definition of the Hamiltonian exactly gets cancelled out andI will get only $\Pi^2/\hbar^4$ + gradient of ϕ^2+$m^2c^2/\hbar^2$ power square ϕ^2. This is a useful exercise for you to try. And that is what we will proceed with thatI will not use $\partial_0\phi$ anymore to do the legendary transform, butI will just use ϕ. They are just 1/c apart and therefore that simplifies the Hamiltonian because that is quadratic in either $\dot{\phi}$ or in Π. So, extra $1/c^2$ factor which was appearing in the definition of Hamiltonian can be reabsorbed in the definition new definitions of a Π. So, that is what we will do that simplifies just notational convenience there is no conceptual point about it. So, let us move ahead with this realization that now on my pi is the conjugate momenta which is obtained by taking the derivative of the Lagrangian with respect to $\dot{\phi}$ and not with respect to $\dot{\phi}/c$. Okay so again as we discussed i'm going to write down the quantum field $\phi(x,t)$ which is given over here in terms of quantum fields in momentum space which is given over here and then take the fourier transform that means we combine them using e^{ikx} where this k and this x are just real numbers real vectors not operators. Okay, so just to recapitulate the points we had learned, if I use this definition, ultimately I'm going to get oscillator kind of equations of motion for these variable $\hat{\phi}_k$.

Quantum field as oscillators

$$\hat{\phi}(x,t) = \int d^3\vec{k}\ \hat{\phi}(\vec{k},t)\, e^{i\vec{k}\cdot\vec{x}}$$

$$\ddot{\hat{\phi}}_{\vec{k}} + \omega_k^2 \hat{\phi}_k = 0 \qquad \left\{ \omega_{\vec{k}} = \sqrt{k^2c^2 + \frac{m_0^2c^4}{\hbar^2}} \right.$$

$$\hat{\phi}_{\vec{k}} = \sqrt{\frac{\hbar}{2\omega_k}}\left(\hat{a}_{\vec{k}}\, e^{-i\omega t} + \hat{a}^\dagger_{\vec{k}}\, e^{+i\omega t}\right)$$

In that case

$$\hat{\Pi} = \frac{\hbar^2}{c}\partial_t\phi = \frac{\hbar^2}{c}\int d^3k\,(\partial_t\phi)\, e^{i\vec{k}\cdot\vec{x}}$$

$$= (\hbar^2)\sqrt{\hbar}\int \frac{d^3k}{\sqrt{2\omega_{\vec{k}}}}\, e^{i\vec{k}\cdot\vec{x}}\left(-i\frac{\omega_k}{c}\hat{a}_{\vec{k}}\, e^{-i\omega_{\vec{k}}t} + i\frac{\omega_k}{c}\hat{a}^\dagger_{\vec{k}}\, e^{+i\omega_{\vec{k}}t}\right)$$

$$\hat{\phi}_{\vec{k}} = \sqrt{\frac{\hbar}{2\omega_{\vec{k}}}}\left(\hat{a}_{\vec{k}}\, e^{-i\omega_{\vec{k}}t} + \hat{a}^\dagger_{\vec{k}}\, e^{i\omega_{\vec{k}}t}\right)$$

$$\frac{\hat{\Pi}_{\vec{k}}}{c} = \frac{\hbar^2}{ic}\sqrt{\hbar}\cdot\sqrt{\frac{\omega_{\vec{k}}}{2}}\left(\hat{a}_{\vec{k}}\, e^{-i\omega_{\vec{k}}t} - \hat{a}^\dagger_{\vec{k}}\, e^{i\omega_{\vec{k}}t}\right)$$

Quantum field as oscillators

$\hat{\phi}(x,t) = \int d^{3k}\hat{\phi}(x,t)\, e i\vec{k}\cdot\vec{x}$

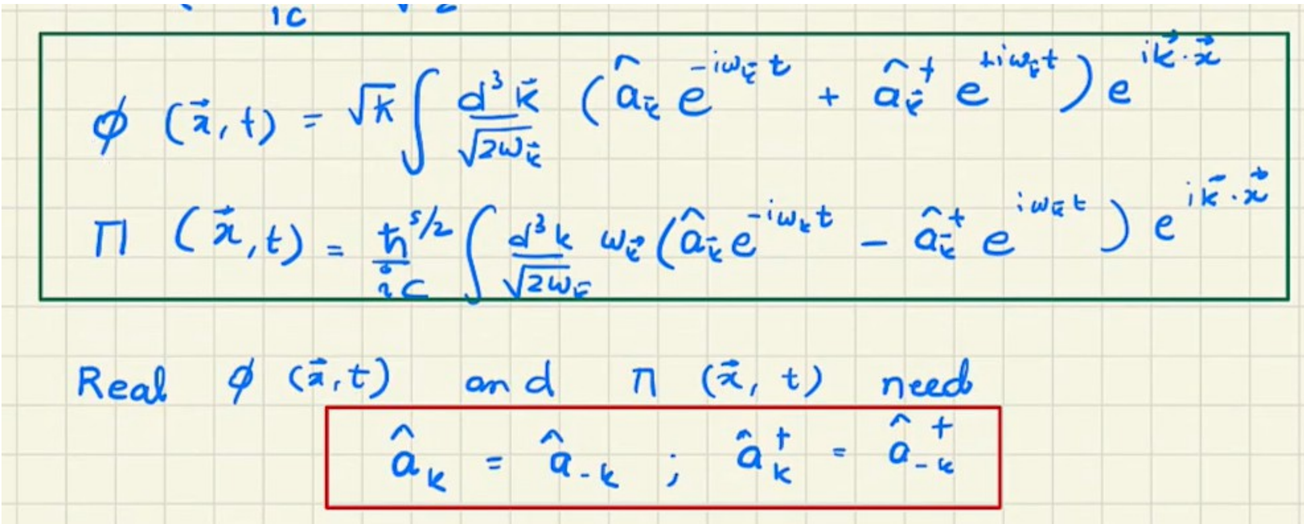
$\ddot{\hat{\phi}}_{\vec{k}} + \omega_k^2\hat{\phi}_k = 0$

$\left(\omega_{\vec{k}} = \sqrt{k^2c^2 + \frac{m_0^2c^4}{\hbar^2}}\right)$

In tha case

$$\hat{\Pi} = (\hbar^2)\sqrt{\hbar}\int d^{3k}\, e^{i\vec{k}\cdot\vec{x}}\left(\frac{-i\omega_k}{c}\hat{a}_{\vec{k}}e^{-i\omega t} + \frac{i\omega_k}{c}\hat{a}^{\dagger}_{\vec{k}}e^{i\omega t}\right)$$

$$\hat{\phi}_{\vec{k}} = \sqrt{\frac{\hbar}{2\omega_{\vec{k}}}\left(\hat{a}_{\vec{k}}e^{-i\omega t} + \hat{a}^{\dagger}_{\vec{k}}e^{i\omega t}\right)}$$



Quantum field as oscillators

$$\hat{\phi}(x,t) = \int d^{3k}\,\hat{\phi}(x,t)\,e\,i\,\vec{k}\cdot\vec{x}$$

$$\ddot{\hat{\phi}}_{\vec{k}} + \omega_k^2\hat{\phi}_k = 0$$

$$\left(\omega_{\vec{k}} = \sqrt{k^2c^2 + \frac{m_0^2c^4}{\hbar^2}}\right)$$

In tha case

$$\hat{\Pi} = (\hbar^2)\sqrt{\hbar}\int d^{3k}\, e^{i\vec{k}\cdot\vec{x}}\left(\frac{-i\omega_k}{c}\hat{a}_{\vec{k}}e^{-i\omega t} + \frac{i\omega_k}{c}\hat{a}^{\dagger}_{\vec{k}}e^{i\omega t}\right)$$

$$\hat{\phi}_{\vec{k}} = \sqrt{\frac{\hbar}{2\omega_{\vec{k}}}\left(\hat{a}_{\vec{k}}e^{-i\omega t} + \hat{a}^{\dagger}_{\vec{k}}e^{i\omega t}\right)}$$

$$\frac{\pi_{\vec{k}}}{c} = \frac{\hbar^2}{ic}\sqrt{\hbar}\sqrt{\omega_{\vec{k}}}\left(\hat{a}_{\vec{k}}e^{-i\omega t} + \hat{a}^{\dagger}_{\vec{k}}e^{i\omega t}\right)$$

$$\frac{\pi_{\vec{k}}}{c} = \frac{\hbar^2}{ic}\sqrt{\hbar}\sqrt{\omega_{\vec{k}}}\left(\hat{a}_{\vec{k}}e^{-i\omega t} + \hat{a}^{\dagger}_{\vec{k}}e^{i\omega t}\right)$$

$$\hat{\phi}(x,t)=\sqrt{\hbar}\int \frac{d^3\vec{k}}{\sqrt{2\omega_{\vec{k}}}}\left(\hat{a}_{\vec{k}}e^{-i\omega t}+\hat{a}^{\dagger}_{\vec{k}}e^{i\omega t}\right)\mathrm{e}^{i\vec{k}\cdot\vec{x}}$$

$$\hat{\Pi}(x,t)=\frac{\hbar^{3/2}}{ic}\int \frac{d^3\vec{k}}{\sqrt{2\omega_{\vec{k}}}}\omega_{\vec{k}}\left(\hat{a}_{\vec{k}}e^{-i\omega t}+\hat{a}^{\dagger}_{\vec{k}}e^{i\omega t}\right)\mathrm{e}^{i\vec{k}\cdot\vec{x}}$$

Real $\phi(\vec{x},t)$ and $\Pi(\vec{x},t)$ need

$$\hat{a}_k=\hat{a}_{-k};\ a^{\dagger}_k=a^{\dagger}_{-k}$$

For this variable ϕ, we did not get oscillator equation, remember. It used to satisfy Klein-Gordon equation. But the cost of the Klein-Gordon equation is, the appearance of the Klein-Gordon equation is replaced by a harmonic oscillator equation if I do get to the Fourier mode, okay. Particularly, the gradient term is handled well in this thing. All right, so since this is a harmonic oscillator kind of a structure, I can again do the same kind of business which we used to do for harmonic oscillators. Namely, I can try to write down it in terms of ladder operators. If you remember quantization of harmonic oscillator, we used to write x as $\sqrt{\frac{\hbar}{2m\omega}}$ under and $a + a^{\dagger}$. You need to go to your quantum mechanics notes to just realize that, that this is how we used to write in terms of a and $a^{\dagger}$ the variable x which satisfied the oscillator equations of motion. This was in Schrodinger picture where operators do not change in time, but states change in if you want to know what was the heisenberg picture thing this was roughly similar to $\sqrt{\frac{\hbar}{2m\omega}}$ very similar form only thing is that this will have $ae^{-i\omega t}$ and $+ a^{\dagger}e^{+i\omega t}$ so this is how the time dependent operator would have changed in heisenberg picture Having a creation operator here $a^{\dagger}$ is a creation operator coming with $e^{+i\omega t}$ and an annihilation operator coming with a $e^{-i\omega t}$ where ω was the frequency the oscillator was satisfying. So similarly I can have the same thing done for the $\hat{\phi}_k$ this time. It has its own frequency, $\omega k^2 d$. So that ω_k will sit here. It had its own mass, which is 1. So this m will be replaced by 1. And everything else should be the same, only for one more fact that I have different different oscillators for different different k's. So frequency depends on k. So therefore this ladder operator a and $a^{\dagger}$ should know about which variable has been written as oscillator. So they should also come with a label k. So they should be functions of k as well. SoI will have a a_k and $a_k^{\dagger}$ just like the definition we have used. So this is exact analog of the position in the operator language. So far so good.

Now let us write down what is the Π corresponding to it, the conjugate momenta Π . From the new definition as we have seen, this is just $\phi\hbar^2/c$. Previously it was $\hbar^2\dot{\phi}$ because of our usage of $\partial_0\phi$ as the conjugate variable to be traded off. But now we have just rescaled the c in and then therefore I have this definition. So what should I do?

I should take this ϕ which we have written upstairs here. And then I should write down its derivative. When I write down its derivative, the ∂t part will just hit this thing. And this $\hat{\phi}_k(t)$ we have just written here. So ultimately you see the derivative of this with respect to time will just make derivative of this with respect to time which will hit the exponentials. So exponentials will give me either $-i\omega$ and the exponential back or $+i\omega$ with the exponential back and there is overall 1 over c which is coming about. So that is what we have to do. So do it. Take the derivative of $\hat{\phi}_k$ actually. And $\hat{\phi}_k$ we have written over here like this. Just let me clean it up so that we can visualize very clearly. So what we have to do? We have to take multiply $\hbar^2/c$ with the time derivative of ϕ. Time derivative of ϕ involves time derivative of $\hat{\phi}_k$. And $\hat{\phi}_k$ itself is known in terms of $e^{i\omega t}$ and $e^{-i\omega t}$. So the derivative operator will throw this and that. So, this would be my definition of pi. There will be a relative sign difference and there will be iotas coming from the exponential derivatives. So, overall therefore, I have this structure. I have this structure that $\hat{\phi}_k$ which I have here written like this which is like before I have repeated. Then the definition of pi, if I want to write as Fourier transform of some $P_{ik}e^{ikx}$, this quantity here along with the factors which are coming about from here and here will be identified as P_{ik}. So, you just take this collection of this whole thing, this by $\sqrt{\frac{1}{\omega_k}}$ and this $\sqrt{\hbar}/\hbar^2$ kind of thing. Then you will get pik like this. This is just algebraic systematic way of writing the same thing. Okay. You can verify there is $\hbar^2$and $\sqrt{\hbar}$ factor coming from because pi remember already came with $\hbar^2$factor and pik itself comes with a $\sqrt{\hbar}$. So therefore overall $\hbar^2\sqrt{\hbar}$ will be coming out. So, this is your $\hat{\phi}_k$ and this is your pik. They are analogs of x and p. x was supposed to be summation of $ae^{-i\omega t} + a^\dagger e^{i\omega t}$. So, p therefore is a $\frac{1}{ia}$ $(a^\dagger e^{-i\omega t} - ae^{i\omega t})$. So this is what exactly x and p analog are mimicked by ϕ and its conjugate momenta in Fourier space π. So neatly I can write the conjugate pair field and its momenta like this. I have just written it neatly over here in the boxed form. One more thing, we want our fields to be real. We are talking about real fields. Right now the discussion is focusing on real fields. There is no need to actually discuss real fields fields could be complex as well because remember initially we were after finding out the operator version of wave function and there is no demand on the wave function that it has to be real but for the special case let us say if we do consider special case that the field I am after is a real field okay it's a subset of all possible fields complex field could also be there but I am right now talking about only real scalar fields in that sense if ϕ is real You can prove that its Fourier conjugate should be $\hat{\phi}_k^\dagger$ should be ϕ_{-k}. This is not real. Its Fourier counterpart is not real. If $\phi(x,t)$ is equal to ϕ dagger or star, let us say. Then it translate into the dagger of $\hat{\phi}_k$ is equal to $\hat{\phi}_{-k}$. This you can prove from just a structure of ϕ. Just put $a^\dagger$ to this equation and bring it back to its own form. You will realize that you have done this exact identification. If that is true and you know $\hat{\phi}_k$ is this. So therefore, its dagger should equal to ϕ replaced with a $-k$. And that will give you conditions like this. a k should be equal to a of $-k$ and a $k^\dagger$ should be equal to $a^\dagger$ of $-k$. So that means these operators are symmetric and going from k to $-k$. All these k's are vectors here. Sometimes I write the vectors explicitly, but sometimes I have missed the notations of writing as a vector. So, ultimately $a_{\vec{k}}$ should also be equal to $a_{-\vec{k}}$ and the same thing should be true for the dagger as well. So, now let us get back to the Hamiltonian and try to see what does it look like in terms of these ladder operators or the creation and annihilation operators. We know at the classical level when we wrote down the Hamiltonian was made from quadratic terms. So, there was a $c\hbar^2/2$ outside, then there was an integration d^3x. And $\Pi^2/\hbar^4$ so first let me elaborate one more point over here when we were writing this remember this h was obtainable from the Lagrangian density this was not full Lagrangian as we had discussed in the previous classes this is rather Lagrangian density.

If one writes the Hamiltonian

$$H = \frac{c\hbar^2}{2}\int d^3x \left(\frac{\Pi^2}{\hbar^4} + (\nabla\phi)^2 + \frac{m^2c^2}{\hbar^2}\phi^2 \right)$$

$$\Pi^2 = \hbar^5 \int \frac{d^3k}{\sqrt{2\omega_k}} \frac{d^3x'}{\sqrt{2\omega_{k'}}} \left(\frac{\omega_k \omega_{k'}}{c^2}\right) e^{i\vec{k}\cdot\vec{x}} \left(\hat{a}_k e^{-i\omega_k t} - \hat{a}_k^+ e^{i\omega_k t}\right) e^{i\vec{k}'\cdot\vec{x}} \left(\hat{a}_{k'} e^{-i\omega_{k'} t} - \hat{a}_{k'}^+ e^{i\omega_{k'} t}\right)$$

$$\int d^3x \frac{\Pi^2}{c^2}(x) = \hbar^5 \int \frac{d^3\vec{k}}{\sqrt{2\omega_k}} \frac{d^3\vec{k}'}{\sqrt{2\omega_{k'}}} \delta(\vec{k}+\vec{k}') \left(a_k e^{-i\omega_k t} - \hat{a}_k^+ e^{i\omega_k t}\right) \left(\frac{\omega_k \omega_{k'}}{c^2}\right) \left(\hat{a}_{k'} e^{-i\omega_{k'} t} - \hat{a}_{k'}^+ e^{i\omega_{k'} t}\right)$$

$$= \frac{\hbar^5}{2} \int \frac{d^3\vec{k}}{\omega_k} \left(\frac{\omega_k}{c}\right)^2 \left(\hat{a}_k \hat{a}_{-k} e^{-i(2\omega_k)t} + \hat{a}_k^+ \hat{a}_{-k}^+ e^{2i\omega t} - \hat{a}_k \hat{a}_{-k}^+ - \hat{a}_k^+ \hat{a}_{-k}\right)$$

So, therefore, after the Legendre transformation what I am getting here is Hamiltonian density not the full Hamiltonian dimension wise they should be the same thing. So, this needs to be integrated over d^3x to obtain the Lagrangian. Similarly, this Hamiltonian density needs to be integrated with respect to d^3x in order to get the total Hamiltonian. So, the same thing would be done. I just took the Hamiltonian density which we obtained right now absorbing the $1/c^2$ in the redefinition of pi as we discussed. Then the total Hamiltonian will be just integration of the Hamiltonian density over the space over the volume. So, this is my true Hamiltonian. Now, here there are three pieces, one is this $\Pi^2/\hbar^4$, one is a gradient of ϕ^2 and then there is a mc^2, $m^2c^2\phi^2$. So, we need to evaluate all these terms to know what the Hamiltonian looks like.

If one writes the Hamiltonian

$$H = \frac{c\hbar^2}{2}\int d^3x \left(\frac{\Pi^2}{\hbar^4} + (\nabla\phi)^2 + \frac{m^2c^2}{\hbar^2}\phi^2\right)$$

$$\Pi^2 = -\hbar^5 \int \frac{d^3k}{\sqrt{2\omega_k}}\frac{d^3k'}{\sqrt{2\omega_{k'}}}\left(\frac{\omega_k\omega_{k'}}{c^2}\right) e^{i\vec{k}\cdot\vec{x}}\left(\hat{a}_k e^{-i\omega_k t} - \hat{a}_k^{+} e^{i\omega_k t}\right) e^{i\vec{k}'\cdot\vec{x}}\left(\hat{a}_{k'} e^{-i\omega_{k'} t} - \hat{a}_{k'}^{+} e^{i\omega_{k'} t}\right)$$

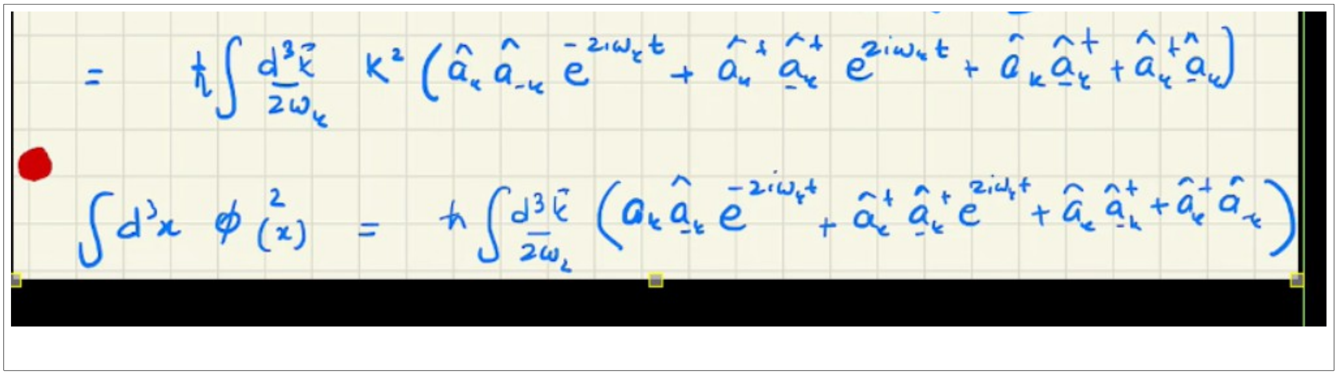
$$\int d^3\vec{x}\, e^{i(\vec{k}+\vec{k}')\cdot\vec{x}} = \delta^{(3)}(\vec{k}+\vec{k}')$$

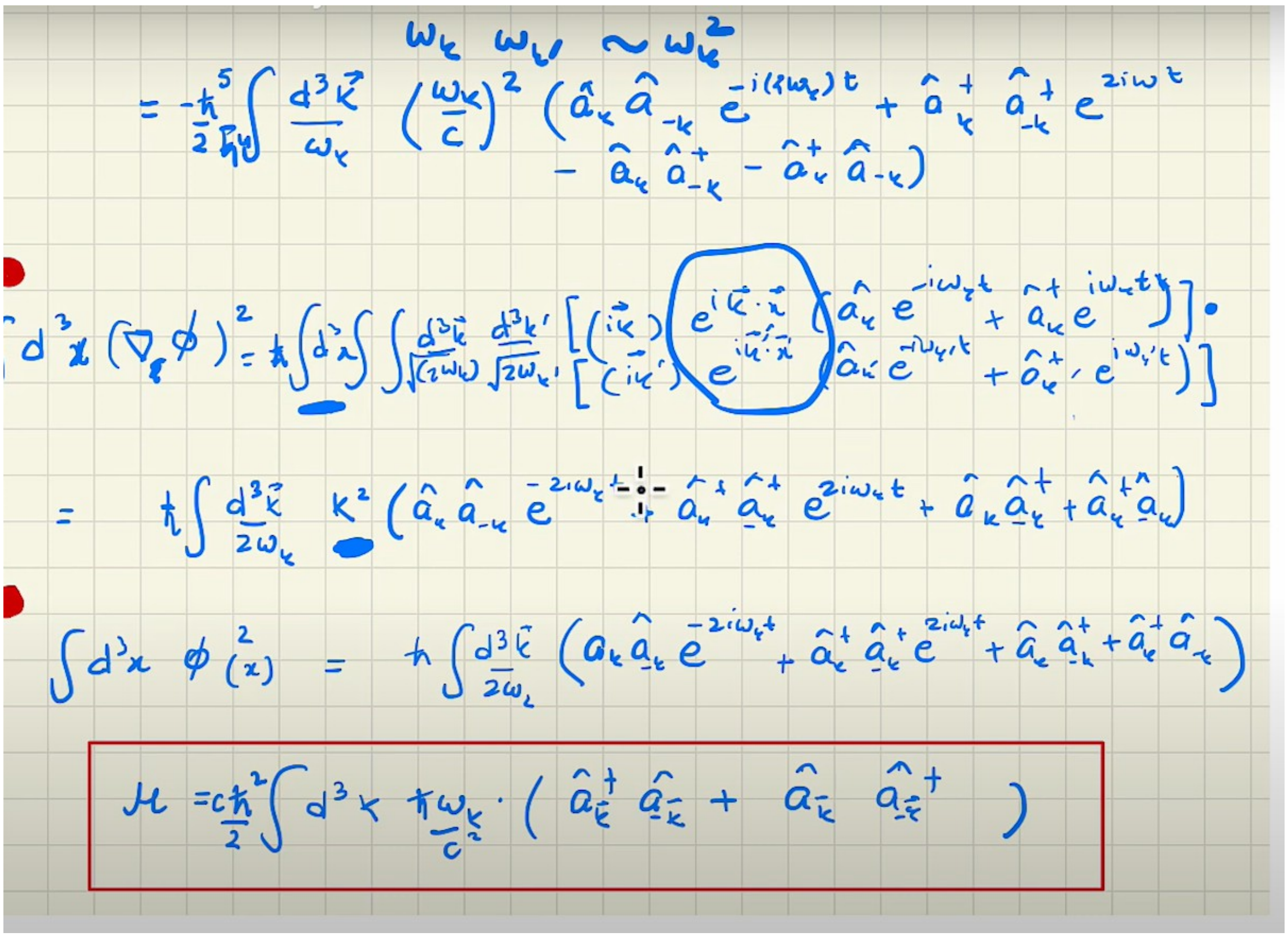
$$\int d^3x \frac{\Pi^2}{c^2}(x) = -\hbar^5\int \frac{d^3\vec{k}}{\sqrt{2\omega_k}}\frac{d^3\vec{k}'}{\sqrt{2\omega_k'}}\, \delta(\vec{k}+\vec{k}')\left(\hat{a}_k e^{-i\omega_k t} - \hat{a}_k^{+} e^{i\omega_k t}\right)\left(\frac{\omega_k\omega_{k'}}{c^2}\right)\left(\hat{a}_{k'} e^{-i\omega_{k'} t} - \hat{a}_{k'}^{+} e^{i\omega_{k'} t}\right)$$

$$\hat{a}_{k'} \downarrow \hat{a}_{-k}$$

$$= -\frac{\hbar^5}{2}\int \frac{d^3\vec{k}}{\omega_k}\left(\frac{\omega_k}{c}\right)^2\left(\hat{a}_k \hat{a}_{-k} e^{-i(2\omega_k)t} + \hat{a}_k^{+}\hat{a}_{-k}^{+} e^{2i\omega t} - \hat{a}_k \hat{a}_{-k}^{+} - \hat{a}_k^{+}\hat{a}_{-k}\right)$$

$$d^3x(\nabla\phi)^2 = \hbar\int d^3x \int\int \frac{d^3\vec{k}}{\sqrt{2\omega_k}}\frac{d^3k'}{\sqrt{2\omega_{k'}}}\left[(i\vec{k})\, e^{i\vec{k}\cdot\vec{x}}\left(\hat{a}_k e^{-i\omega_k t} + \hat{a}_k^{+} e^{i\omega_k t}\right)\right]\cdot\left[(i\vec{k}')\, e^{i\vec{k}'\cdot\vec{x}}\left(\hat{a}_{k'} e^{-i\omega_{k'} t} + \hat{a}_{k'}^{+} e^{i\omega_{k'} t}\right)\right]$$





If one writes the Hamiltonian

$$H = c\frac{\hbar^2}{2}\int d^3k\left(\frac{\Pi^2}{\hbar^4}+(\nabla\phi)^2+\frac{m^2c^2}{\hbar^4}\phi^2\right)$$

$$\Pi^2=\hbar^5\int\frac{d^{3k}}{\sqrt{2\omega_k}}\frac{d^{3k}\,'}{\sqrt{2\omega_k\,'}}\left(\frac{\omega_k\omega_k\,'}{c^2}\right)e^{i\vec{k}\cdot\vec{x}}\left(\hat{a}_{\vec{k}}e^{-i\omega_k t}-\hat{a}^{\dagger}_{\vec{k}}e^{i\omega_k t}\right)e^{i\vec{k}\,'\cdot\vec{x}}\left(\hat{a}_{\vec{k}}\,'e^{-i\omega_k\,'t}+\hat{a}^{\dagger}_{\vec{k}}\,'e^{i\omega_k\,'t}\right)$$

$$\int d^x\,\Pi^2\frac{(x)}{c^2} =$$

$$\hbar^5 \int \frac{d^{3k}}{\sqrt{2\omega_k}} \frac{d^{3k}{}'}{\sqrt{2\omega_k{}'}} \delta(\vec{k}+\vec{k}{}')\left(\hat{a}_{\vec{k}} e^{-i\omega_k t} - \hat{a}^{\dagger}_{\vec{k}} e^{i\omega_k t}\right)\left(\frac{\omega_k \omega_k{}'}{c^2}\right)\left(\hat{a}_{\vec{k}}{}' e^{-i\omega_k{}' t} - \hat{a}^{\dagger}_{\vec{k}}{}' e^{i\omega_k{}' t}\right)$$

$=$

$$\frac{-\hbar^5}{2}\int \frac{d^3\vec{k}}{\omega_k}\left(\frac{\omega_k}{c}\right)^2 \hat{a}_k \hat{a}_{-k} e^{(-2i\omega_k)t} + \hat{a}^{\dagger}_k \hat{a}^{\dagger}_{-k} e^{(2i\omega_k)t} - a^{\dagger}_k a^{\dagger}_{-k} - a^{\dagger}_{-k} a^{\dagger}_k$$

$=$

$$\hbar \int \frac{d^3\vec{k}}{2\omega_k} k^2 \left(\hat{a}_k \hat{a}_{-k} e^{-2i\omega t} + \hat{a}^{\dagger}_k \hat{a}^{\dagger}_{-k} e^{2i\omega t} + \hat{a}_k \hat{a}^{\dagger}_k + a^{\dagger}_k \hat{a}_k\right)$$

$=$

$$\int d^3x\, \phi^2(x) = \hbar \int \frac{d^3\vec{k}}{2\omega_k}\left(\hat{a}_k \hat{a}_{-k} e^{-2i\omega t} + \hat{a}^{\dagger}_k \hat{a}^{\dagger}_{-k} e^{2i\omega t} + \hat{a}_k \hat{a}^{\dagger}_k + a^{\dagger}_k \hat{a}_k\right)$$

$\omega_k \omega_k{}' \approx \omega_k^2$

$=$

$$\int \frac{d^3\vec{k}}{\omega_k}\left(\frac{\omega_k}{c}\right)^2 \left(\hat{a}_k \hat{a}_{-k} e^{-2i\omega t} + \hat{a}^{\dagger}_k \hat{a}^{\dagger}_{-k} e^{2i\omega t} - \hat{a}_k \hat{a}^{\dagger}_k - a^{\dagger}_k \hat{a}_k\right)$$

$$\int d^{3x} (\nabla_x \phi)^2 = \hbar \int d^3x \int\int \frac{d^3\vec{k}}{\sqrt{2\omega_k}} \frac{d^3\vec{k}{}'}{\sqrt{2\omega_k}} \left[i\vec{k}\, e^{i\vec{k}\vec{x}}\left(\hat{a}_k e^{-i\omega_k t} + \hat{a}^{\dagger}_k e^{i\omega_k t}\right)\right]\left[i\vec{k}{}'\, e^{i\vec{k}{}'\vec{x}}\left(\hat{a}_{k'} e^{-i\omega_k{}' t} + \hat{a}^{\dagger}_{k'} e^{i\omega_k{}' t}\right)\right]$$

$=$

$$\hbar \int \frac{d^3\vec{k}}{2\omega_k} k^2 \left(\hat{a}_k \hat{a}_{-k} e^{-2i\omega_{kt}} + \hat{a}^{\dagger}_k \hat{a}^{\dagger}_{-k} e^{2i\omega_k t} + \hat{a}_k \hat{a}^{\dagger}_{-k} + \hat{a}^{\dagger}_k \hat{a}_{-k}\right)$$

$$H = \frac{c\hbar^2}{2}\int d^3k \frac{\hbar\omega_k}{c^2}\left(\hat{a}^{\dagger}_{\vec{k}} \hat{a}_{-\vec{k}} + \hat{a}_{\vec{k}} \hat{a}^{\dagger}_{-\vec{k}}\right)$$

This is a slightly elaborate mathematical procedure whichI will just do systematically, so that we become very careful. So, just appeal to the definitions, I need to know what is Π and I need to square it, I need to know what is the gradient of Π , I need to square it and I need to know what is Π itself, I need to square that. So, let us do bit by bit. First, I look for what is the Π , Π we had just written in the boxed equation here, Π is this quantity, h power to the power Π by 2 by η by c and then this integration of the quantity which is looking like a – $a^\dagger$ the usual momentum in the harmonic oscillator kind of thing only thing is that the true pi is made from the collection of all such conjugate momenta Π's all together coming with the fourier factor e^{ikx} generate for you the conjugate momenta pi and I need to square this quantity that mean I will take this definition and multiply it with itself so therefore I would have a one pi coming with a single integral π^2will come with a two integrals so there should be two integrals which I have written as a symbol here so actually it is something like this $d^3k\sqrt{2\omega k}$ outside remember what was there if you pay attention, outside it was h to the power 5 by 2. After taking the square, it would become h to the power 5. And there was iota outside as well, whose square should give you a – 1. And there is a c^2 which should come in the denominator. So, let me do it. So, there should be a c^2 which I have put it over here. And there should be – sign. And there should be – sign. We can keep it for the time being. See, there is a – sign over here. Then inside quantities will appear twice. Once with a $\vec{k}$ and once with a $\vec{k}\,'$.

So, inside quantities where this $\omega_k a_k e^{-i\omega t}$, $a^\dagger e^{i\omega t}$ and $e^{ik\ddot{x}}$. So, which has appeared here ω_k $e^{ik'}$and this bracket which is appearing and similar thing should appear for k' as well. So, all these quantities have appeared twice and I have a Π^2. So, let me keep the – sign everywhere which I apparently had not written down. So, let me correct it. All right. Okay. Okay, fine. So, Okay, so now let us see what do I get out of it. So remember I need to do the d^3x integral on $\Pi^2/\hbar^4$. So $\Pi^2/\hbar^4$ I can write from here directly. First I need to do $d^3x\ \Pi^2$ integral and then I can divide by h to the power 4.So when I do the d^3x integral, d^3x will search for the x dependency. x dependency is coming here and here only. There is no other x dependency anywhere. So,I will get an integration like $d^3x\ e^{ik\ddot{x}}$ + e to the power i$k'\dot{x}$. So, together I can write it like this. And this I know is a definition for delta function, three-dimensional delta function, which is $\vec{k}$ + k'. So, this is what I have written over here. Okay, they should be fine. All right. And then there is a double integral which is left out d^3k and d^3k'. So, this delta function ensures that all the k' should take the value of – k okay and we know certain properties of things when they go to – k a a k goes to a of – k which is same as a k here okay this will go into a of – k but it is same as a of k as well so but let us not use that I just write as a of – k so all k' would be written as a – k. ω_k remember was supposed to be k^2c^2 + $\vec{k}^2$, this is magnitude + $m^2c^4/\hbar^2$. So, when I take ω_{-k}, this remains unchanged because it is k^2. So, $\vec{k}$ magnitude and $-\vec{k}$ magnitude are the same thing. So, when I take ω_{-k}, this remains unchanged because it is k^2 . So, $\vec{k}$ magnitude and vector – k magnitude are the same thing. So, therefore, instead of having ω_k, $\omega_k'd$, it should have gotten converted into omega of $\vec{k}$, omega of vector – k, which is same thing as ωk^2d, which is what we have written over here, okay. So, overall, I would have a – h5 prime/2, 2 will come from root 2 root 2 here and $\sqrt{\omega_k}$ $\sqrt{\omega_k\,'}$ with identification that omega – k is equal to ω_k will give me just ω_k downstairs. Upstairs, I should have an $\omega k^2/c^2$, which is over here. And then all the operators where k is identified to be –, $k^\dagger$ is identified to be – k. So, that is what I have written. So, you can again should work it out to see exactly what we have written in the line over here should come about. Okay. So, this is fine. Now, let us look at one more thing, since you remember there was a $\hbar^4$ here, $\hbar^4$ here, so I can divide it by $\hbar^4$, I do not know why I wrote a c^2 here, it should not have been, but it should have been $\hbar^4$, so that $\hbar^4$ will come in the denominator and will cancel the fifth power of $\hbar$ and ultimately the first term will just be – of $\frac{\hbar}{2}$ the whole factor here all right now let us go to the second factor the second factor is the gradient of ϕ and then square it I do the same thing when I look for a gradient the gradient is with respect to positions vector x it will search for position it will only find positions in the exponential when I take the gradient it will bring down the iota k the vector down the gradient of exponential ikx just gives back exponential ikx with iota $\vec{k}$ back and then I have to square it that meansI will write the integrals twice so here So, with a k here with the $k'd$ this is the gradient of ϕ, this is the gradient of ϕ again meaning this is the gradient of $\int \hat{\phi}_k e^{ikx} dx$ and the second thing is integration of $\int \hat{\phi}_k e^{ik'x} dx$. So, these two brackets have come from the multiplications of gradient of ϕ. Again, the same thing,I will undergo the integration in d^3x. It will identify this pair of exponentials and convert it into delta of $k + k'$ like before. The exact same thing and therefore the same game happens one of the k' is converted into $- k$ so this time there is a subtlety previously I had ω_k and ω_k' and when k' became $- k$ it became ωk^2 because these were the same thing this time I have a iota k and iota k' first iota iota will give a – sign and then k and k' is there and with the delta function when I do the d^3k' integration this k' will be converted into $- k$ so overall it will become into a + $|\vec{k}^2|$ so you see you will get a k^2 with a + sign outside. So againI will just suggest you to do this once carefully yourself that you should take the gradient multiply the two integrals do the exponential integration d^3x of this quantity this will give you delta of $k + k'$ and use that delta of $k + k'$ into one of the $k'd$ integrals you will get this result okay lastly I have a ϕ^2 which I need to evaluate so that means I have to just take twice of the same thing ϕ which was d^3k of $\sqrt{\hbar}$ was outside if you remember there was a $a_k e^{i\omega t} + a_k^\dagger e^{-i\omega t}$ and $e^{ik\dot{x}}$ and in the square twothese quantities will come so another $\sqrt{\hbar}$ will come that we make $\hbar$ then there will be double integral d three k d three k' another factor of the same thing repeated with e to the power $ik'd\dot{x}$ and do the d^3x integral to that as well which is required here and use again that there will be a delta function which you will get for k and $k'd$. You will get a delta function for k and $k'd$ and ultimately you will land up on this equation over here. Okay, so just let me clean things up once more so that we can have a quick look what do we get from the π^2 term I am going to get this here. This has to be divided by h to the power of 4 as well to get the first term. From gradient of ϕI will get this term over here. And from the mass term, I would get this term here, which needs to be multiplied with $m^2c^2/\hbar^2$. So $m^2c^2/\hbar^2$. So if I combine this quantity and this quantity here, $m^2c^2/\hbar^2$, you will see all those quantities are appearing like they are appearing here as well. So they will just add up. So k^2 will add up with this. $\hbar$ is common outside integral is common and the whole factor is common so from the last two terms you will get the same kind of integrals with k^2 + $m^2c^2/\hbar^2$the integral d^3k and the whole object here now look at the term over here which is coming from the first term it also has roughly the similar kind of structure but You see there is a sign difference all the terms appearing here are positive here some terms are negative and outside there is a negative factor so these terms will become + sign the upper terms will become – sign after you absorb the – inside and they will exactly cancel these terms These terms here, here, here are exactly cancelled by these terms. So all you are left with time independent terms. So if you just do the algebra carefully, you will be left with time independent terms only. And you would get the answer that these things will combine when this was the here the – sign will become + all these quantities are +. So, ultimately there is the $\omega k^2/c^2$ the factor coming from the last two term is also this $k^2 + m^2c^2/\hbar^2$ which is also $\omega k^2/c^2$ by the way. So, you will get twice of $\omega k^2/c^2$. But there is a 1 over ω_k outside as well. So ultimately we will get ω_k here and not $\omega k^2/c^2$. So ω_k/c^2. So only these terms will survive. So you see now the whole algebra has using the ladder operators have given me a neat structure of Hamiltonian which is made from just a and $a^\dagger$s just like a harmonic oscillator does right only thing which you have is that here I have a $- k$ and here also you have a $- k$ but this should not worry us because we already know that a_{-k} is equal to a_k and $a_{-k}^\dagger$ is also equal to $a_k^\dagger$ so ultimately this is indeed very much like harmonic oscillator So, this is what we got get for ordinary harmonic oscillators as well in position if you in position space if I write down the Hamiltonian of ordinary oscillator for single oscillator exactly this structure comes about that you will get a $\hbar\omega$ times this. Now here you are getting an extra factor of $1/c^2$ and these quantities over here. We do not have to worry about it as we will see that they will just reabsorb in the redefinitions of this. But one last stage was you know harmonic oscillator in quantum mechanics can be written there Hamiltonian can be written as number operators + ½ and $\hbar\omega$. We are almost there. So, we are two steps away. One is that proving that these things are number operators and second thing absorbing these extra constants which are appearing. So, let us go in step wise. First I would want to prove that this has a number operator structure. In order to have this number operator structure, we need to know what is the commutator between ak and $a_k^\dagger$. We have never specified, we just wrote down a_k and $a_k^\dagger$ converted into them into operators, but we do not know what their commutator is. In ordinary harmonic oscillator, a and $a^\dagger$ are supposed to satisfy commutator relation identity. So, if similar kind of thing exist here as well, then we know even the Hamiltonian becomes exactly like harmonic oscillator Hamiltonian for each k. And then it is just collection of oscillators, their energy is constituting the total energy of the field as well. So, that we will do the rescaling, getting the commutator relations between the ladder operators we will do in the next class.