Foundation of Quantum Theory: Relativistic Approach

Quantum Field Theory 1.2

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Scalar field Quantization

Lecture- 16

Demonstration

$$\mathcal{L} = -\frac{c\hbar^2}{2}\left(-\frac{1}{c^2}(\partial_t\phi)^2 + (\nabla_x\phi)^2 + \frac{m^2c^2}{\hbar^2}\phi^2\right)$$

$$\sum_{\mu=0}^{3}\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\right) - \frac{\partial\mathcal{L}}{\partial\phi} = 0 \quad ✓$$

$$\partial_{ct}\left(\frac{\partial\mathcal{L}}{\partial(\partial_{ct}\phi)}\right) + \partial_x\left(\frac{\partial\mathcal{L}}{\partial(\partial_x\phi)}\right) + \partial_y\left(\frac{\partial\mathcal{L}}{\partial(\partial_y\phi)}\right) + \partial_z\left(\frac{\partial\mathcal{L}}{\partial(\partial_z\phi)}\right)$$

$$-\left(-\frac{c\hbar^2}{2}\right)\frac{m^2c^2}{\hbar^2}\phi = 0$$

$$\left(+\frac{\partial}{\partial(ct)}\frac{1}{c}\frac{\partial\phi}{\partial t} - \frac{\partial^2\phi}{\partial x^2} - \frac{\partial^2\phi}{\partial y^2} - \frac{\partial^2\phi}{\partial z^2}\right) + \frac{m^2c^2}{\hbar^2}\phi = 0$$

$$\Rightarrow \left(-\frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} + \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2}\right) = \frac{m^2c^2}{\hbar^2}\phi$$

The momentum conjugate to Ψ

$$\Pi = \frac{\partial \mathcal{L}}{\partial(\partial_0 \phi)} = c\hbar^2 \partial_0 \phi = \hbar^2 \dot{\phi}$$

Claim:

$$\mathcal{L} = \frac{-c\hbar^2}{2}\left(-\frac{1}{c^2}(\partial_t \phi)^2 + (\partial_x \phi)^2 + (\partial_y \phi)^2 + (\partial_z \phi)^2 - \frac{m^2c^2}{\hbar^2}\phi^2\right)$$

does the job

$$L = \frac{-c\hbar^2}{2}\left(\frac{-1}{c^2}(\partial_t \phi)^2 + (\nabla_x \phi)^2 - \frac{m^2 c^2}{\hbar^2}\phi^2\right)$$

$$\sum_{\mu=0} \partial_\mu \left(\partial \frac{L}{\partial(\partial_\mu \phi)}\right) - \frac{\partial L}{\partial \phi} = 0$$

$$\partial_{ct}\left(\frac{\partial L}{\partial(\partial_{ct}\phi)}\right) + \partial_x\left(\frac{\partial L}{\partial(\partial_x \phi)}\right) + \partial_y\left(\frac{\partial L}{\partial(\partial_y \phi)}\right) + \partial_z\left(\frac{\partial L}{\partial(\partial_z \phi)}\right) - \frac{m^2c^2}{\hbar^2}\phi^2 ¿¿¿ = 0$$

$$\left(\frac{+\partial}{\partial_{(ct)}} \frac{1}{c}\frac{\partial \phi}{\partial t} - \frac{\partial^2 \phi}{\partial x^2} - \frac{\partial^2 \phi}{\partial y^2} - \frac{\partial^2 \phi}{\partial z^2}\right) = \frac{m^2c^2}{\hbar^2}\phi$$

$$\left(\frac{-1}{c^2}\frac{\partial^2 \phi}{\partial t^2} - \frac{\partial^2 \phi}{\partial x^2} - \frac{\partial^2 \phi}{\partial y^2} - \frac{\partial^2 \phi}{\partial z^2}\right) = \frac{m^2c^2}{\hbar^2}\phi$$

The momentum conjugate to Ψ

$$\Pi = \partial \frac{L}{\partial(\partial_0 \phi)} = c\hbar^2 \partial_0 \phi = \hbar^2 \dot{\phi}$$

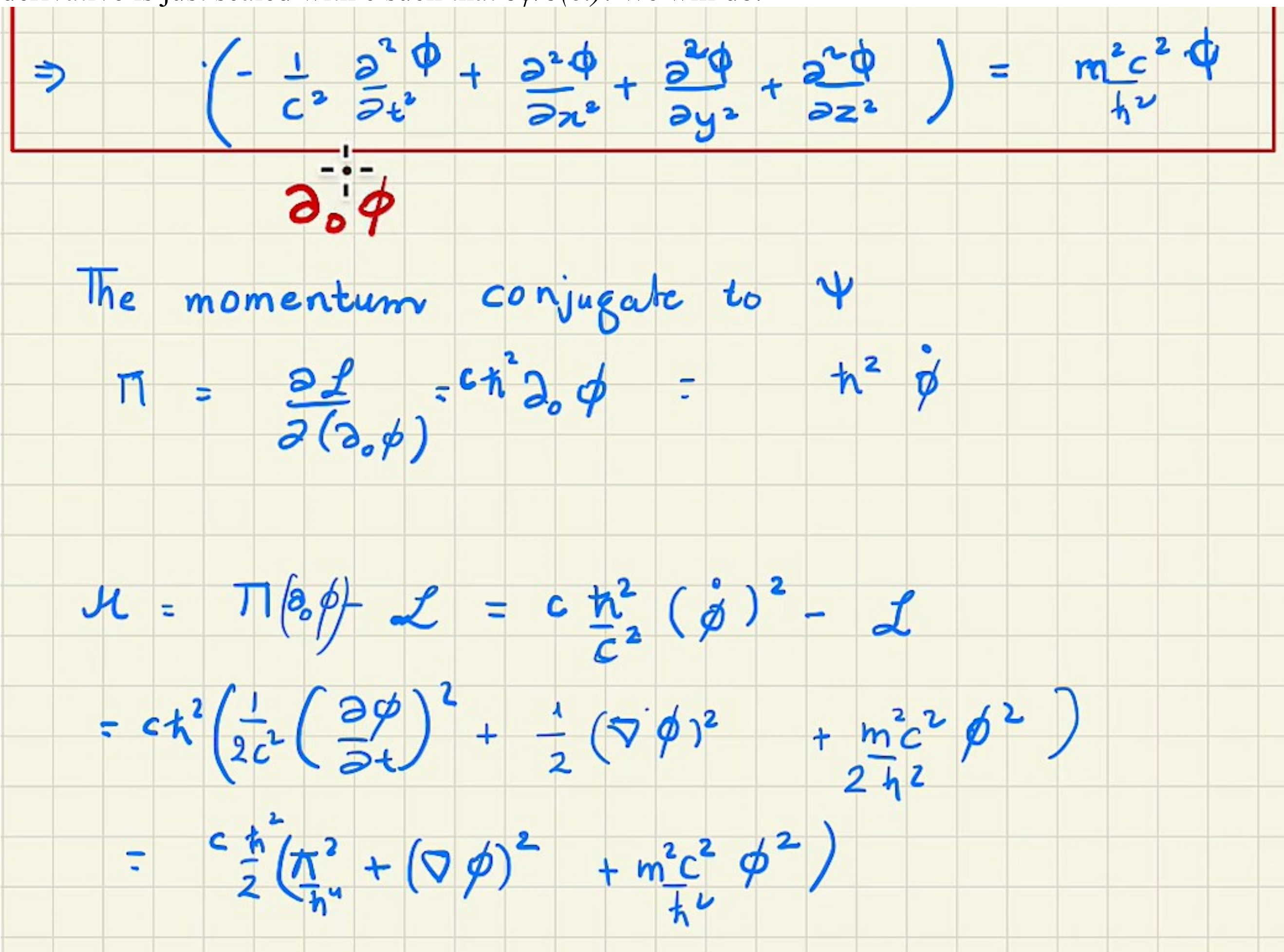
Claim :

$$L = c\frac{\hbar^2}{2}\left((\frac{-1}{c^2})(\partial_t\phi)^2 + (\partial_x\phi)^2 + (\partial_y\phi)^2 + (\partial_z\phi)^2 \right)$$

does the job

This term contains $\partial_x\phi^2$, $\partial_y\phi^2$ and $\partial_z\phi^2$. So, therefore, the partial derivative with respect to $\partial_z\phi$ will throw up twice $\partial_z\phi$ which will the twice factor will cancel with the 1/2 factor outside and then there is a another ∂_z which is outside. So, it will convert it into a $\partial^2\phi/\partial z^2$. First derivative of $\partial L/\partial(\partial_z\phi)$ will give you twice of $\partial_z\phi$ with two factor that cancels out and therefore I have a double derivative of ϕ. Similarly for yI will get a double derivative of ϕ with respect to y and for from x againI will get a double derivative of ϕ from x. So remember there are three terms $\partial_x\phi^2$, $\partial_y\phi^2$, $\partial_z\phi^2$. and the operations these these these the spatial operations give you the double derivative terms all of them will come up with a – sign because there is a – sign outside and all of them will come with a factor c $\hbar^2$so all these things came with a factor c$\hbar^2$. and this also had a factor c$\hbar^2$ so what I can do I can throw away all the $c\hbar^2$s on the right hand side and it will become zero to to the zeros side so I can just write the operational part which are just made from the ϕ's and their derivatives so from the first term I am getting the double derivative with respect to del's uh double derivative with respect to ct with a positive sign and then the three terms which are double derivative with respect to spatial coordinates which come with a negative signs and lastly I would have a + $m^2c^2\phi$ $\hbar^2$ the + sign came from the two –es and c $\hbar^2$i have thrown outside this this was a 2 ϕ and this 2 and this 2 cancelled out each other.
Okay, so ultimately you see that irrespective of the overall constants here, overall constants in equations of motions do not matter. They do matter at the level of getting the conjugate momenta as we will see, but at the equations of motion level, overall constants do not matter. And indeed from the Lagrangian which we started with, we end up getting the Klein-Gordon equation. And that is what we were looking for. So we have identified a Lagrangian from which Klein-Gotton equations can be obtained and now I can do the business of going to phase space with this Lagrangian and quantize the theory. I have not told how did I get this Lagrangian, I just told that it is the Lagrangian which does the job for us. So in this course we are not going to discuss about how to obtain a valid and good Lagrangian for theory but you can pay attention to the fact that the Lagrangian which I have written you can see that c and $\hbar^2$ over here and here are fundamental constants. This operation here ∂t ϕ^2 with a – sign $\partial_x\phi^2$ $\partial_y\phi^2$ $\partial_z\phi^2$ together remember this is an invariant operator invariant action on ϕ. mc^2 ϕ^2 $\hbar^2$is also invariant object. So it so happens that whole Lagrangian is also invariant under Lorentz transformations. So therefore in order to get a relativistic theory of ϕ fields we should write a Lagrangian which is compatible with the symmetries which we are seeking for namely the Lorentz transformation. So in this Lagrangian I have just made certain that I write down the Lagrangian which is consistent with the Lorentz symmetries. Is this the only Lagrangian which is consistent with the Lorentz symmetries? It is not. There are other many, many, many, infinitely many Lagrangians. This in some sense, in some sense is a simple most Lagrangian which you could obtain. There are other complicated Lagrangians, but they would be talking about quantizations or Lagrangians or equations of motion for all other complicated fields. Here, the Klein-Gordon equation in some sense is depicting the motion, quantum particle Motion of a quantum particle which is relativistic. It is a free particle. There is no potential in this game. So only the mass and the derivatives which are the kinetic terms are appearing. No potential is appearing. So in that sense it is trying to depict the dynamics of a simple enough system. Most simple system as you can visualize. Therefore the Lagrangian which is here is also most simple relativistically compatible Lagrangian. Okay, so the philosophy is typically to write down a Lagrangian which is consistent with the symmetries and then look for various orders of complexities. Since we are looking for simple enough systems to be described with, I have written a

Lagrangian which is simple most in that setting. Okay, fine. So, given that the Lagrangian is known to us now, which gives rise to the equations of motion. Now, it is clear. The time to go to the phase space of this Lagrangian and try to obtain the Hamiltonian and try to quantize the system. All right. So in order to go to the phase space what we have to do we have to take one of the derivatives and trade it with the momenta. So typically the time derivative is traded. So we are going to trade with $\partial_0\phi$. Time derivative is just scaled with c such that $\partial\phi/\partial(ct)$. We will do.



$$\left(\frac{-1}{c^2}\frac{\partial^2\phi}{\partial t^2}-\frac{\partial^2\phi}{\partial x^2}-\frac{\partial^2\phi}{\partial y^2}-\frac{\partial^2\phi}{\partial z^2}\right)=\frac{m^2c^2}{\hbar^2}\phi$$ The momentum conjugate to Ψ

$$\Pi=\partial\frac{L}{\partial(\partial_0\phi)}=c\hbar^2\partial_0\phi=\hbar^2\dot{\phi}$$

$$H=\Pi(\partial_0)-L=c\frac{\hbar^2}{c^2}(\dot{\phi})^2-L$$

$$=\quad c\hbar^2\left(\frac{1}{2c^2}\left(\frac{\partial\phi}{\partial t}\right)^2+\frac{1}{2}(\nabla\phi)^2+\frac{m^2c^2\phi^2}{2\hbar^2}\right)=c\hbar^2\left(\frac{\Pi^2}{\hbar^4}+\frac{1}{2}(\nabla\phi)^2+\frac{m^2c^2\phi^2}{2\hbar^2}\right)$$

So therefore this time the conjugate momenta will be defined like this. I am trying to trade off $\partial_0\phi$ with pi. Okay. Previously we used to trade off ∂t of Q with t. So the same game is there up to relativistic version. There is a c and that's what I am going to trade for.
So first I obtain the conjugate momenta corresponding to ϕ which will be identified as ∂L of $\partial_0\phi$. So again In the Lagrangian density, we will search for where is $\partial_0\phi$. So, remember here this term $1/c^2\ \partial t\ \phi^2$ can also be written as $\partial_0\phi$ whole square. And when I take the derivative with respect to $\partial_0\phi$, this term will only come into the picture and its derivative will give me twice of $\partial_0\phi$. And outside there is a – $c\hbar^2$and inside there is a – overall a + sign. And outside also there is a twice factor which is in the downstairs and $c\hbar^2$which is outside. So ultimately I am getting $c\hbar^2\partial_0\phi$. So that is what I should get c $\hbar^2\partial_0\phi$ and remember that ∂_0 is $\partial\phi/\partial$(ct). So c here and c hiding in the $\partial0$ definition cancel each other and I have just $\hbar^2\ \phi$. This dot is derivative with respect to time and not with respect to c t. c has been cancelled. Alright. So now we have the field, its conjugate momenta at hand and the Lagrangian we have identified. We can go to the Hamiltonian and the phase space as we had, we do in classical mechanics as well. So Hamiltonian in classical mechanics is obtained from qp dot – the Lagrangian. So p dot, $Q\ \dot{p}$, sorry not $Q\ \dot{p}$, $\dot{q}p$ *or* pq – L. So, p is the quantity you are trying to obtain as a new variable in the place of $\dot{q}$. This is what it is being traded for. So, Hamiltonian will be function of Q and p. Lagrangian was a function of Q and $\dot{q}$. So, $\dot{q}$ is being replaced by the new quantity p. Similarly, for fields, $\partial_0\phi$ is trying to get replaced and it is trying to get replaced by pi. So, pi $\partial_0\phi$ – Lagrangian density. That is what we would have. So again remember the momenta pi was $\hbar^2\dot{\phi}$. So I will write Π as $\hbar^2\dot{\phi}$ and $\partial_0\phi$ is $\dot{\phi}/c$ okay this would be $\dot{\phi}/c$ so effectively you will get $\hbar^2/c\phi^2$ which I have written in a funny way $c\hbar^2$divided byc^2 which will just give you $\hbar^2$c and $\dot{\phi}^2$ and – the Lagrangian okay so far so good let me just clean it up so that we can clearly see what we are headed to. So, secondly – Lagrangian. So, Lagrangian is coming with its own $\dot{\phi}$ square. So, remember Lagrangian has – $hc^2/2$ and – $1/c^2\ \dot{\phi}^2$. And there is special derivative whichI will come to in a minute. But you see Lagrangian itself is coming with $\hbar c^2/c^2\dot{\phi}^2$. So, you see the similar kind of term which is appearing here is also coming from the Lagrangian. But with a 1/2 factor. This term did not have a 1/2 factor. This has a 1/2 factor. So they will subtract out and only 1/2 of this term will survive. So that is what it is here. 1/2 of first term will survive because another 1/2 is coming from the Lagrangian which will partially cancel it. Now what about the spatial derivatives? Spatial derivatives from here, for example, we are coming with all the spatial derivatives were there, $\partial_x\phi^2$, $\partial_y\phi^2$. They will just come with negative signs in the Lagrangian. with $\hbar^2$c by 2 factor and under – 1 they will become positive signs. So, you will get a 1/2 of all the spatial derivatives squared, the gradient squared. Yeah, so overall in the Lagrangian if you look at there was a – $m^2c^2/\hbar^2\phi^2$ was there overall. with a + sign rather and then effectively with the Lagrangian outside – there was this – sign here and – L will convert into a +. So, just let me have a look at once more in the yeah. So, effectively this should have been a + only. Let me see they should have been a_+. All the things have the same sign over here. So, – k^2 and here I should get a – $c\hbar^2$. Yes, so it should have been a + over here. Fine. So, therefore, overall you will get a + from here. So, you will get total Lagrangian subtracted out what it does it makes it 1/2 of its value and the spatial derivatives comes with positive sign and the mass term comes with a positive sign and ultimately you have this as your Hamiltonian you see Hamiltonian is made up of $C\hbar^2$then 1/2 of the temporal derivative square 1/2 of the spatial derivative square and the ϕ^2 with mass square $\hbar^2$and $\hbar^2$ so which can compactly be written as a last term over here remember $\dot{\phi}$was worked.
$\Pi/\hbar^2$ so $\dot{\phi}$was $\pi/\hbar^2$ so game of going to phase space is complete when you jump to the variable in the phase space so $\dot{\phi}$will be replaced by $\Pi/\hbar^2$ so this quantity here which is appearing which is appearing is nothing but yeah so it is nothing but $\Pi/\hbar^2$so $\dot{\phi}$was $\Pi/\hbar^2$ so game of going to phase space is complete when you jump to the variable in the phase space so $\dot{\phi}$will be replaced by $\Pi/\hbar^2$ so this quantity here

which is appearing which is appearing is nothing but yeah so it is nothing butre to the power 2. So $\Pi^2/\hbar^4$. Okay. $\Pi^2/\hbar^4$. And there was a c^2 which I have probably missed out. Which is this c^2. Okay. So this is your Hamiltonian. You can see the Hamiltonian is made up of only squared quantity. This is $\Pi^2/c^2/\hbar^4$. This is Laplacian of Π^2. And then m^2c^2 $\Pi^2/\hbar^2$. So this is the Hamiltonian which is positive semi-definite. There is no quantity which is single power or odd power or no quantity which is coming with negative sign. So ultimately your Hamiltonian therefore is a positive definite quantity at least at the classical level. What it does at the quantum level we will see in a minute. Hopefully this will remain so because at the quantum level just like Q and p were converted into operators which were Hermitian, pi and ϕ and its derivatives can also be converted into Hermitian operators and square of Hermitian operators gives you positive eigenvalues. So, squared Hermitian operators have positive eigenvalues. So, none of these quantities are supposed to be negative. So, that is why by going to quantum level at least we have solved the problem of negative energy in some sense. Hamiltonian has become positive definite quantity. Let us go and see if at the quantum level we are able to see the same thing in more realistic sense, realistic sense in the dealing with when we operatorize it whether this retains the structure or not. Okay, so let us first rewrite the Hamiltonian in order to see it neatly that Hamiltonian is this hc^2 $\hbar$ by c$\hbar^2$/2 and then there will be $1/c^2$ Π^2/h to the power 4 and gradient of ϕ^2 + $m^2c^2/\hbar^2$ ϕ^2 that would be the structure which we will get Okay, so now let us go back and try to operatorize that as far as we have discussed. Everything looks consistent until the classical advent because you see all these quantities are positive semi-definite. We want to see if something similar kind of thing remains true at the quantum level as well or not and at least Hamiltonian's energy remaining positive condition would be obtained and then we will see whether we are stuck with the same problems or not. So let us go ahead. So remember we had the Klein-Gordon equation which we wanted to operate right. So we do that.

So, the Klein Gordon equation becomes an operator equation

$$\Box \hat{\phi}(x,t) - \frac{m^2c^2}{\hbar^2}\hat{\phi}(x,t) = 0$$

If we write $\hat{\phi}(x,t) = \int \hat{\phi}(k,t)\, e^{i\vec{k}\cdot\vec{x}}\, d^3\vec{k}$

$$\nabla^2 \hat{\phi}(\vec{x},t) = \int d^3\vec{k}\, (-k^2)\, \hat{\phi}(k,t)\, e^{i\vec{k}\cdot\vec{x}}$$

$$\partial_t^2 \phi(\vec{x},t) = \int d^3\vec{k}\, \frac{\partial^2 \hat{\phi}}{\partial t^2}\, e^{i\vec{k}\cdot\vec{x}}$$

Thus, the Klein Gordon equation becomes

$$\ddot{\phi}(\vec{k},t) + \left(k^2c^2 + \frac{m^2c^4}{\hbar^2}\right)\phi(\vec{k},t) = 0$$

or,

$$\ddot{\phi} + \omega_k^2\, \phi = 0$$

-- a Harmonic oscillator

Each $\phi(\vec{k},t)$ is like an oscillator

$$\int \phi(\vec{k},t)\, e^{i\vec{k}\cdot\vec{x}}\, d^3\vec{k} = \hat{\phi}(x,t)$$

So, the Klein-Gordon equation becomes an operator equation

$$\Box\hat{\phi}(x,t) - m^2\frac{c^2}{\hbar^2}\phi(x,t) = 0$$

If we write

$$\hat{\phi}(x,t) = \int \hat{\phi}(x,t)\, e^{i\vec{k}\vec{x}}\, d^3\vec{k}$$

$$\nabla^2 \hat{\phi}(x,t) = \int d^3\vec{k}(-k^2)\hat{\phi}(x,t)\mathrm{e}^{i\vec{k}\vec{x}}$$
$$\partial_t^2 \phi(\dot{x},t) = \int d^3\vec{k}\frac{\partial^2 \hat{\phi}}{\partial t^2}\mathrm{e}^{i\vec{k}\vec{x}}$$

Thus, the Klein-Gordon equation becomes

$$\ddot{\phi}(x,t) + (k^2c^2 + \frac{m^2c^4}{\hbar^2})\phi(\vec{k},t) = 0$$

Or,

$$\ddot{\phi} + \omega_k^2\phi = 0$$

---- a Harmonic oscillator.

We make the ϕ's appearing in the Klein-Gordon equations as space time dependent operators. So ϕ is a function of x and t. That means at each x and t I would have some operator which will satisfy this equation. Since I have an operator at all positions and I know any function of position can be done a Fourier transform with respect to momentum k's which is to say if I know a function *f(x)*, I can obtain it from superposition of e^{ikx} with weight factor fk. So you see the integration is over dk and e^{ikx} I put and then f of k is the weightage, which k modes comes with how much weightage is decided by fk and together they constitute f of x. Similarly, the same thing will be happening for $\phi(x,t)$ where ϕ is a function of x and t both. Out of that, the x part I am trying to expand in the plane wave basis e^{ikx} and weight factor $\phi(k)$ comma t because ϕ is also left hand side the position space time dependent is. Out of that position part, I am doing a transaction with respect to the plane waves and integrating over the k modes. So therefore, the spatial part, the space dependency, x dependency is captured by e^{ikx} integrated over all possible k's and the time dependency is captured by this $\hat{\phi}_k(t)$. So this is how a superposition I am proposing that exists to write down the operator from the position space to Fourier space, That means how many different operators at momenta k, this should be $\vec{k}$, momenta k constitute with a weight g to the power *ikx* on operator at location x. So at one location x, this is space location x, there are different different k's, k1, k2, k3. All of them combine to give you one operator at x. This is the essence. So, position operator, operator at a position is a superposition of positions at different wavelengths, different momenta and some over with the plane wave weightage. Okay, if that is the case, then I should do the Laplacian and the time derivative which appears in this box operator. Okay, so remember when I do the Laplacian, Laplacian hides three derivatives, three double derivatives in it. Double derivative with respect to x, double derivative with respect to y and double derivative with respect to z. So, for instance let us say if I act/the double derivative of with respect to x on ϕ, the double derivative will search for x dependency. x dependency is only in the exponential. So, therefore the exponential function will undergo the double derivative with respect to x and you will get a $-k_x^2$ – because iota will come down twice under double derivative and you will get a $-k_x^2$. Similarly, when I do the double derivative with respect to y, you will get a $-k_y^2$ and the exponential function back.

The whole thing will be back to itself with addition of $-k_x^2$ for the double derivative with respect to x or $-k_y^2$ if you have a double derivative with respect to y or $-k_z^2$ if you are taking the double derivative with respect to z. And when you add them all together, you will get a – of k_x^2, – of k_y^2, – of k_z^2, which is equivalent to – of the $(|\vec{k}|)^2$, meaning that magnitude of $(|\vec{k}|)^2$ with a negative sign will appear and the whole function will come back to itself. So, under the Laplacian action only the exponential part is hit/and therefore you earn a $-k^2$ inside the integral. Similarly, this box operator hides a double derivative with respect to time as well. So time part is only present in the ϕ, $\phi(k)$t. There is no time dependent in exponential neither in ks. So therefore the double derivative with respect to time will only hit this part, $\partial 2\ \phi/\partial t$ square. And remember the definition of box was $-1/c^2\partial^2/\partial t^2$ and + Laplacian. So this Laplacian will give you – k^2 this double derivative with respect to time you give you ∂2 $\phi/\partial t$ square or ϕ^2 with a – sign and this will give you a – $m^2c^2/\hbar^2$. So, ultimately if you write down the whole thing in expanded basis you will get, so let me open it up, you will get ultimately with a – sign integration of d^3k and $\phi(k)$ integration of d^3k sorry $\phi(k)$ double dot coming from the double derivative of ϕ with respect to time + k^2 which is $k_x^2 + k_y^2 + k_z^2 + m^2c^2/\hbar^2$ and this both the terms are coming with $\hat{\phi}_k$ and then $e^{ik\ddot{x}}$ is equal to 0. So the Klein-Gordon equation when opened up in the Fourier basis becomes this equation. Of course, again, if this equation has to be holding, it has to hold true, then since e^{ikx} cannot take value 0, under integration you want this to become 0 for all possible ϕ_k's which you can think about. This can only happen if your, again the integrand's first fraction goes to 0, that is to say, $\phi + k^2c^2$. So, actually you will get a k^2c^2 here and m to the power c 4. Why you will get? Because the box operator had a $1/c^2$ as well if you remember. I wanted a $1/c^2$. That c^2 if I take away from here, actually it was c^2 like this which I multiplied throughout the equation. On the right hand side since it is 0, I can multiply with c^2. You can work it out, you will get that. So ultimately you will get $\ddot{\phi} + k^2c^2 + m^2$c to the power $4/\hbar^2$ and then together multiplying the $\hat{\phi}_k(t)$. And all of these things are operators let us say.Okay. So now you see you have an equation like $\ddot{\phi}$+ somethingϕ. Let us call that something as a ωk^2 . ωk^2 is this combination. See this combination is just made from real number k real number c m c $\hbar$ everything is real number so this is just a real number so you have a equation like some operators double dot + ωk^2 and that operator is equal to zero compare it with the harmonic oscillator operator equation $\ddot{x} + \omega^2 x$ operator was supposed to be zero so therefore the five k's. It should be a level K. Let me write it down. The ϕ_ks, the ϕ_k which are appearing here, I am calling it a subscript k because I do not want to write $\phi_{,k}$ and t all the time. So this is same as ϕ_k . Either you write it in expanded version like $\phi_{(k,t)}$ or just ϕ_k . So you see these things which are appearing over here, they satisfy an operator equation which is exactly like harmonic oscillator operator equation. And from there these operators you know you have to just combine them with plane waves to get the operator in position space. So, we are after a position of space operator. The trick we played that we will write down first in momentum space operator and those momentum operators space momentum space operators combined together with plane waves generate for me the physical space operator $\phi_x(t)$. And crucial thing about this is that this $\hat{\phi}_k(t)$, these new set of operators in momentum space behave as oscillators. So, each $\hat{\phi}_k(t)$, when I tried this, I demanded that this equation should be true for all k. Here also I demanded this equation the integrand of this should vanish for all k's. So, it looks like all of the $\phi_k(t)s$ together generate $\phi(x)$, t but also this all of $\phi_k(t)$ satisfy harmonic oscillator equations. So, it looks like all of the $\phi_k(t)$'s together generate $\phi(x)$, t but also this all of $\phi_k(t)$ satisfy harmonic oscillator equations. Meaning d^3k is you can think of a different oscillators being summed over with weight factor e^{ikx}. So therefore, the field which we are really wanting to learn about, the quantum field is a set of oscillators in momentum space. It is not that in real space there are tiny, tiny oscillators. Do not confuse it with that analogy. Typically, sometimes you will hear that field is like oscillators. They are oscillators in momentum space. In adjoint space of X, these operators live, which we do not have ready access to, but we know Together, they combine in a particular fashion to generate an operator at location . The whole thing will be back to itself with addition of – k_x^2 for the double derivative with respect to x or $-k_y^2$ if you have a double derivative with respect to y or – k_z^2 if you are taking the double derivative with respect to z. And when you add them all together, you will get a – of k_x^2, – of k_y^2, – of k_z^2, which is equivalent to – of the $|\vec{k}^2|$, meaning that magnitude of the $\vec{k}^2$ with a negative sign will appear and the whole function will come back to itself. So, under the Laplacian action only the exponential part is hit/and therefore you earn a – k^2 inside the integral. Similarly, this box operator hides a double derivative with respect to time as well. So time part is only present in the ϕ, $\phi(k)$t. There is no time dependent in exponential neither in ks. So therefore the double derivative with respect to time will only hit this part, ∂2 $\phi/\partial t$ square. And remember the definition of box was – $1/c^2$ ∂2/∂t square and + Laplacian. So this Laplacian will give you – k^2 this double derivative with respect to time you give you ∂2 $\phi/\partial t$ square or ϕ^2 with a –

sign and this will give you a – $m^2c^2/\hbar^2$. So, ultimately if you write down the whole thing in expanded basis you will get, so let me open it up, you will get ultimately with a – sign integration of d^3k and $\phi(k)$ integration of d^3k sorry $\phi(k)$ double dot coming from the double derivative of ϕ with respect to time + k^2 which is $k_x^2 + k_y^2 + k_z^2 + m^2c^2/\hbar^2$ and this both the terms are coming with $\hat{\phi}_k$ and then e^{ikx} is equal to 0. So the Klein-Gordon equation when opened up in the Fourier basis becomes this equation. Of course, again, if this equation has to be holding, it has to hold true, then since e^{ikx} cannot take value 0, under integration you want this to become 0 for all possible ϕ_k's which you can think about. This can only happen if your, again the integrand's first fraction goes to 0, that is to say, $\phi + k^2c^2$. So, actually you will get a k^2c^2 here and m to the power c 4. Why you will get? Because the box operator had a $1/c^2$ as well if you remember. I wanted a $1/c^2$. That c^2 if I take away from here, actually it was c^2 like this which I multiplied throughout the equation. On the right hand side since it is 0, I can multiply with c^2. You can work it out, you will get that. So ultimately you will get $\ddot{\phi} + k^2c^2 + m^2c^4/\hbar^2$ and then together multiplying the $\hat{\phi}_k(t)$. And all of these things are operators let us say. Okay. So now you see you have an equation like $\ddot{\phi}$+ somethingϕ. Let us call that something as a ωk^2. ωk^2 is this combination. See this combination is just made from real number k real number $cmc\hbar$ everything is real number so this is just a real number so you have a equation like some operators double dot + ωk^2 and that operator is equal to zero compare it with the harmonic oscillator operator equation $\ddot{x} + \omega^2 x$ operator was supposed to be zero so therefore the five k's It should be a level K. Let me write it down. The ϕ_k s, the ϕ_k which are appearing here, I am calling it a subscript k because I do not want to write $\phi_k(t)$ all the time. So this is same as ϕ_k. Either you write it in expanded version like $\phi_{k,t}$ or just $\phi_{k,}$ So you see these things which are appearing over here, they satisfy an operator equation which is exactly like harmonic oscillator operator equation. And from there these operators you know you have to just combine them with plane waves to get the operator in position space. So, we are after a position of space operator. The trick we played that we will write down first in momentum space operator and those momentum operators space momentum space operators combined together with plane waves generate for me the physical space operator $\phi(x,t)$. And crucial thing about this is that this $\hat{\phi}_k(t)$, these new set of operators in momentum space behave as oscillators. So, each $\hat{\phi}_k(t)$, when I tried this, I demanded that this equation should be true for all k. Here also I demanded this equation the integrand of this should vanish for all k's. So, it looks like all of the $\phi_k(t)s$ together generate $\phi(x)$, t but also this all of $\phi_k(t)$ satisfy harmonic oscillator equations. So, therefore the real field which we are looking after is a peculiar combination of harmonic oscillatorssome plane wave and integration over the frequency of the all the oscillators summed over. Meaning d^3k is you can think of a different oscillators being summed over with weight factor e^{ikx}. So therefore, the field which we are really wanting to learn about, the quantum field is a set of oscillators in momentum space. It is not that in real space there are tiny, tiny oscillators. Do not confuse it with that analogy. Typically, sometimes you will hear that field is like oscillators. They are oscillators in momentum space. In adjoint space of X, these operators live, which we do not have ready access to, but we know Together, they combine in a particular fashion to generate an operator at location .