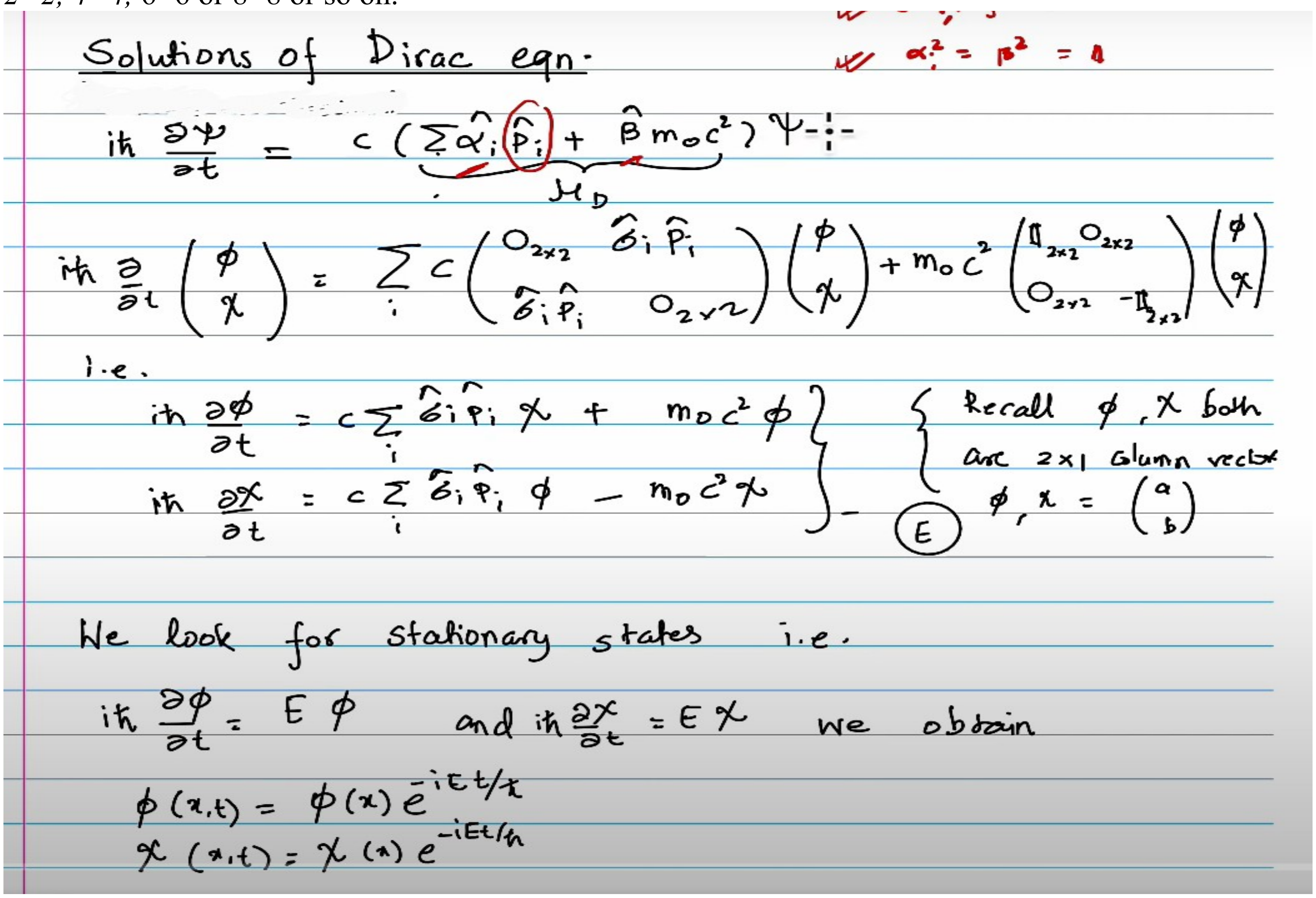
Foundation of Quantum Theory: Relativistic Approach
Relativistic Quantum Mechanics 1.4
Prof . Kinjalk Lochan
Department of Physical Sciences
IISER Mohali

Dirac Equation II
Lecture- 14

So in this discussion session, we will start looking at the solutions of the Dirac equation which we obtained in the last class, where in search of first order differential structure or first order of derivative operator, we wanted to write a low range invariant kind of differential equation which governs the wave function. we ended up getting the criteria that the equation should look like that the time the first order time derivative should equal to a Hamiltonian which is made up from $i\hbar\partial/\partial x$ which is the momentum operator multiplied with not some constants α_1 α_2 α_3 and β but matrices α_1 α_2 α_3 and β. These matrices satisfy certain conditions that αs have their anticommutator equal to twice of delta ij α and β anticommute perfectly and the squares of individual matrices of all α_i's and all the β^2 is identity and then we also realized that this can only be achieved by even dimensional matrices it can either be 2×2, 4×4, 6×6 or 8×8 or so on.



$$i\hbar\frac{\partial\psi}{\partial t} = \frac{\hbar c}{i}\underbrace{\left(\sum_i \hat{\alpha}_i \frac{\partial}{\partial x_i} + \hat{\beta} m_0 c^2\right)}_{H_D}\psi$$

$$i\hbar\frac{\partial}{\partial t}\begin{pmatrix}\phi\\ \chi\end{pmatrix}=c\sum_i\begin{pmatrix}O_{2\times2} & \sigma_i P_i\\ \sigma_i P_i & O_{2\times2}\end{pmatrix}\begin{pmatrix}\phi\\ \chi\end{pmatrix}+m_0^2c^2\begin{pmatrix}\square_{2\times2} & O_{2\times2}\\ O_{2\times2} & \square_{2\times2}\end{pmatrix}\begin{pmatrix}\phi\\ \chi\end{pmatrix}$$

$$i\hbar\frac{\partial\%\phi}{\partial t}=c\sum_i\hat{\sigma}_i\hat{P}_i\chi+m_0c^2\%\phi$$

$$i\hbar\partial\frac{\chi}{\partial t}=c\sum_i\hat{\sigma}_i\hat{P}_i\phi-m_0c^2\chi$$

Recall ϕ, χ both are 2×1 column vectors

$$\phi,\chi=\begin{pmatrix}a\\ b\end{pmatrix}$$

--------------*(E)*

We look for stationary states, ie,

$$i\hbar\frac{\partial\phi}{\partial t}=E\phi\wedge i\hbar\frac{\partial\chi}{\partial t}=E\chi$$

We obtain

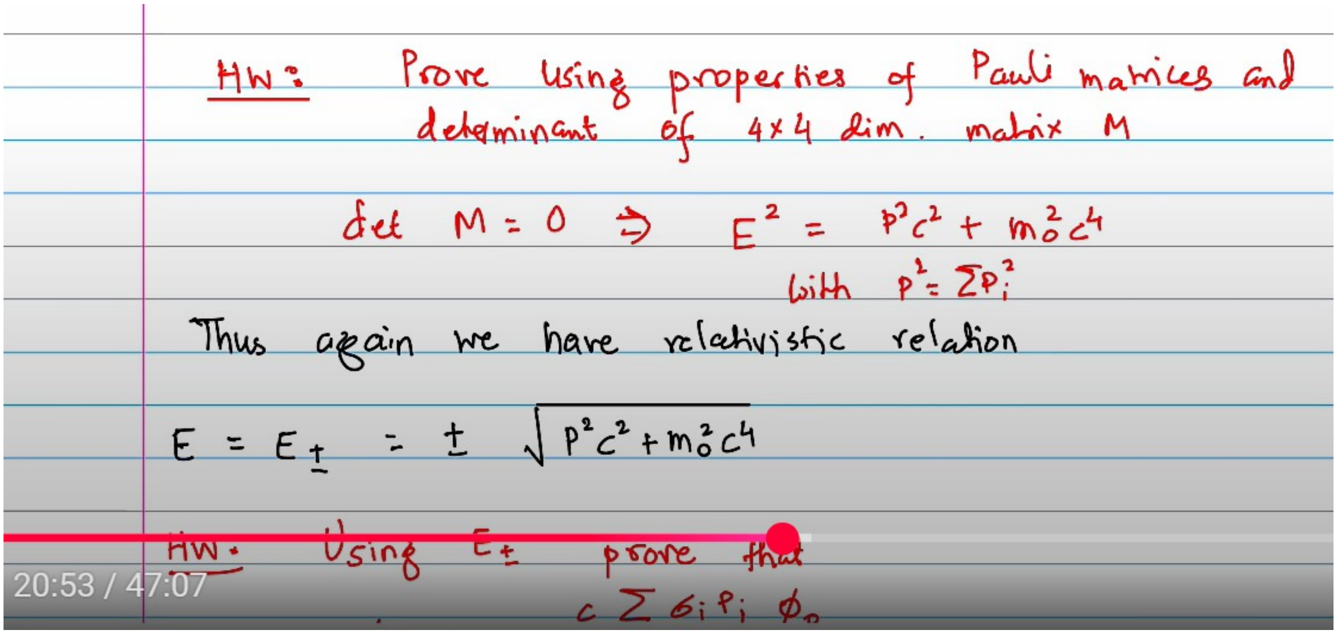
$$\phi(x,t)=\phi(x)\mathrm{e}^{\frac{-iEt}{\hbar}}$$

$$\chi(x,t)=\chi(x)\mathrm{e}^{\frac{-iEt}{\hbar}}$$

So all dimensional matrices cannot satisfy these three conditions so therefore the lowest possibility is *2×2* but in *2×2* dimensions we found out that there are no three no four independently linearly independent matrices which we want we want three of α_1 α_2 α_3 and one β overall four independent matrices in two×two dimensional matrices space there are only three such matrices which satisfy these criteria so we have to dump two×two dimensional space the next possibility is *4×4* and there indeed we have a case where we do get four linearly independent matrices which satisfy that for example one possible case of such four linearly independent matrices are this where α i's where the matrices which were *2×2* matrix in the diagonal and σ_i which are poly matrices *2×2* dimensional poly-matrices in the off diagonal σ_i σ_i when this α_i is multiply the pi's they will take all the components multiply to pi 0 matrix will remain 0 but the off diagonal block will pick up pi operators remember pi's are – $i\hbar\partial/\partial_{xi}$. That's what we are writing P I says. Similarly, the β part which is identity in the diagonal and – identity and off diagonal randomness are *0×0* can be written with multiplication of m_0 c^2 which is here. Now, we also learnt that when the operators appearing αs, βs are *4×4* dimensional matrices. The wave function ψ itself is a 4×1 dimensional vector, ψ_1, ψ_2, ψ_3, ψ_4. So, when we choose to write the whole matrix equation over here in terms of *2×2* blocks, that upper block is made up of two *2×2* matrices, one is 0 and one is σ_i and lower block is also σ_i and 0. Then what we could do can split the full wave function also into *2×1* and *2×1*. So upper block will be called ϕ and lower block will be called χ. So upper and lower blocks are doublets. ϕ is a doublet which is made up of two elements like here a, b and similarly χ is another doublet which is also made up of two elements something like a prime, b prime or c, d. So, this is the structure which will get I have written one realization of $\alpha_1\alpha_2\alpha_3$ which is of diagonal σs and β being identity – identity along the diagonal. You can in general get another combination on some other matrices which also satisfy the same relations in *4×4* dimensional matrix

space. But we also know that those matrices which you will generate are just unitary transformed version of these matrices. So, they are not really different, they are unitary equivalent. So, quantum physics which is invariant under unitary transformation should not give rise to new predictions. So, in 4×4 dimensional space, this is a choice which is unitary equivalent to all other choices one can make.
Okay all right now with this information we want to look at what kind of time evolution equation we get it so suppose this operator hits the first element the first doublet then that should be equal to the upper two rows of this equation that is means this upper two thing multiplying the whole thing over here + this upper two thing will multiplying the whole column vector here will give rise to $i\hbar\, \partial/\partial t$ of ϕ. So, therefore $i\hbar\, \partial/\partial t$ of ϕ should be equal to χ_0 matrix multiplying ϕ which is $0 + \sigma_i\phi$ multiplying χ acting/χ rather because ϕ remember is an operator. So, σ_i pi acting on χ. On the second term m_0c^2 identity multiplying ϕ which is $m_0c^2\phi + 0$ matrix multiplying χ which is nothing. So, there is this is the two terms which you will get. Similarly, for getting the lower doublet χ differential equation, I should look at this block multiplying the block here and the lower block here multiplying the full column vector here, which will give rise to this differential equation $i\hbar\, \partial\chi/\partial t$ is equal to $c\sigma_i\phi$ acting on ϕ this time and $- m_0c^2\chi$. So you see if I split the $4\times$one dimensional column vector ψ it gets differential equation of two coupled doublets ϕ and χ are written in a differential equations which are coupled differential equations. Differential equation of ϕ is not completely written in terms of ϕ alone it depends on χ. Similarly differential equation of χ does not only depend on χ alone, it also depends on ϕ. So, therefore, they are coupled equation. You might have seen equations of coupled harmonic oscillator for example, where first oscillators position determines the time evolution of second and vice versa. Similarly, here the first half of the full wave function ψ_1, ψ_2, ψ_3, ψ_4, upper two element which are ϕ. Their time evolution depends on what the values of the lower two elements ψ_3, ψ_4 are that means what is χ and vice versa. So, this is the coupled set of equations we end up with. And we want to look for stationary state solution that is to say that ψ should be coming back to itself after $i\hbar\, \partial/\partial t$ operation just like we had discussed in the previous classes as well for positive energy solutions stationary solution this equation should be satisfied. That this operator acting on ψ should give me back the ψ and the eigenvalue E. That means if I write in terms of ϕ and χ, this operator acting on this doublet, pair of doublet ϕ and χ, I should get the Ethe doublet ϕ and doublet χ back. So, upper row equation will give me this and lower row equation will give me this. So, we are looking for stationary state solution that is eigen state of $i\hbar\, \partial/\partial t$ operator acting on ψ should get me ψ back with E that is equivalent to saying that even ϕ and χ will do the same thing. And you know the solution of these kind of differential equations are exponentiala constant. So, therefore, ϕ will be written as an exponential – iota e iota e t/$\hbar$ up to a constant of time. So, this is a differential equation in time, partial differential equation in time. So, therefore, the constant you will get will be constant of time, but it may depend on other positions. It can be a function of position, but constant of time. And similarly, for χ χ also have the similar kind of equation structure so therefore even the χ would have $e^{-iEt/\hbar}$ a constant of time but that can be a function of space okay so overall therefore if I want to write down the overall equation for ψ the wave function ψ ψ would be just $\phi(x)$ and $\chi(x)e^{-iEt/\hbar}$. And remember these derivative operators are appearing in the left side of the equation set E. The equation set E here has the temporal derivative of doublet ϕ and temporal derivative of doublet χ in the left hand side.
So, therefore, if we are looking for stationary state solution, the left hand sides of this equation will be replaced by E ϕ and E kylips respectively and right hand side remains the same as it was. Now, further if we want to look for not only the stationary states of temporal derivative, but stationary states of spatial derivatives as well, that means eigen states of the momentum operator as well. We all know that the eigen states of momentum operators are $e^{ip\dot{x}/\hbar}$. So, this operator acting on this function $e^{ip\dot{x}/\hbar}$ throws up the eigen value p which appears over here with the wave function back. So, therefore, if I want to have not only stationary state in time, but eigenstate of momentum operator as well with eigenvalue p, then I can write down the $\phi(x)$ as some constant now of constant of space as well. So, some fully full

constant thing and then $e^{ip\dot{x}/\hbar}$. And similarly, χ of x will be a constant χ_0 and I of p$\dot{x}/\hbar$. So therefore, this ϕ x and χ x appearing over here can be written terms of this function and therefore, ψ *will* become a constant doublet ϕ_0 and a constant doublet χ_0. This cannot depend on time as well as space. So this is a full constant of a space time. And then $e^{-iEt/\hbar}$ was already there. And + $ip\dot{x}/\hbar$ will come, where $\vec{P}$ is the eigenvalues of momentor operator. Its component $\vec{P}$ is a vector here. It is ordinary vector, not operator anymore. It is eigenvalues. Its components are eigenvalues of different $\vec{P}$ operators. So $\vec{P}^1$ operator will give P^1 eigenvalue back $\vec{P}^2$ operator will get P^2 eigenvalue back and $\vec{P}^3$ operator will get P^3 eigenvalue back and this eigenvalue P^1 P^2 P^3 which are real numbers because their eigenstate eigenvalues of a hermitian operator will generate this p vector okay so overallI will have a constant doublet ϕ_0 χ_0 and I would have a constant doublet ϕ_0 χ_0 and this plane wave $e^{-iEt/\hbar+ip\dot{x}}$, everything divided by $\hbar$. This plane wave multiplies the pair of doublets. So, this is still 4×1 dimensional matrix, doublet here, doublet here, but they are constant doublets. And a plane wave which multiplies, these are the solutions of the Dirac equation which we are looking for. So we have solution of Dirac equation where we have a constant $4\times1$the plane wave. Now this is a very interesting structure that emerges out and we will now realize the plane wave part we understand. But what is this 4×1 dimensional part which has come about that we will need to see in more detail. So let us try to solve these equations together to realize first is there any relation between the E eigenvalues of the temporal derivatives and P is the eigenvalues of the spatial derivatives. So, in Klein-Gordon equation we knew that E^2 was equal to $p^2c^2 + m_0^2c^4$. Let us see what do we get at this end. So, if I write down at all the derivative operator $\partial/\partial t$ will give me E here and E here. This pi operators will give me pi here because of the first equation here and this pi on ϕ sorry pi on χ will give rise to p numbers real number piχ because of this equation here. And similarly the second equation the first term on the right hand side will also become pismall ϕ due to this decomposition this equation over here which is the eigen state of the momentum operator. Therefore the equation will get converted into real numbersϕ and χ all the operators have acted/and we have real numbers e pis and so on so the equation which we got down remember there was m_0c^2 appearing in both sides m_0c^2 with ϕ was here and m_0c^2 with χ was here so what we can do we can take the m_0c^2 here and m_0c^2 from here to the left hand side so what we will get is this e – $m_0c^2\phi_0$ would be equal to the cσ_i eigen value pi this time χ_0 remember ψ had become constant ϕ_0 constant $\chi_0$plane wave $e^{-iEt/\hbar + p\dot{x}/\hbar}$ everything/$\hbar$ so therefore just ϕ_0 and χ_0 are constant things we want to know what are these constants but anyway first we will write down the structure soI will get $\phi_0$in principle $e^{-iEt/\hbar}$ + all this exponential term and similarly the second term $e^{-iEt/\hbar}$ + all the exponential term, but these are common to both of them we can throw it towards the 0. So,I will get only equations for ϕ_0 and χ_0 which are coming about here, the first equation will become that. The second equation will also develop the similar feature only thing we have to care about this there was a – sign already present when I take it on to the left hand side I get E + m_0c^2 acting on χ_0 And the thing which was surviving on the right hand side was the derivative operator acting on ϕ, which itself has become momentum piconstant doublet ϕ_0.



HW: Prove using properties of Pauli matrices and determinant of 4×4dim. Matrix M.

det M = 0 → $E^2 = p^2c^2 + m_0{}^2c^4$

with $p^2 = \sum p_i{}^2$

Thus again we have relativistic relation

$E = E_\pm = \pm\sqrt{p^2c^2 + m_0{}^2c^4}$

H.W.: Using $E_\pm$ prove this

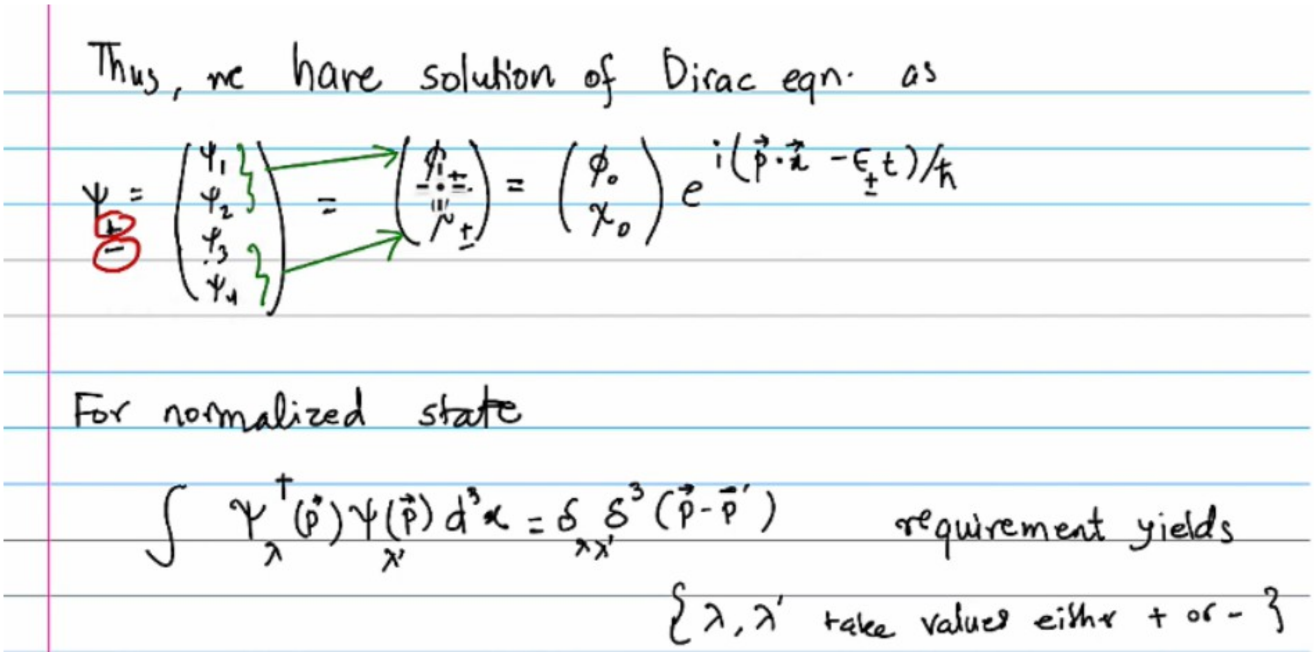
$c\sum\sigma_i p_i \phi_n$

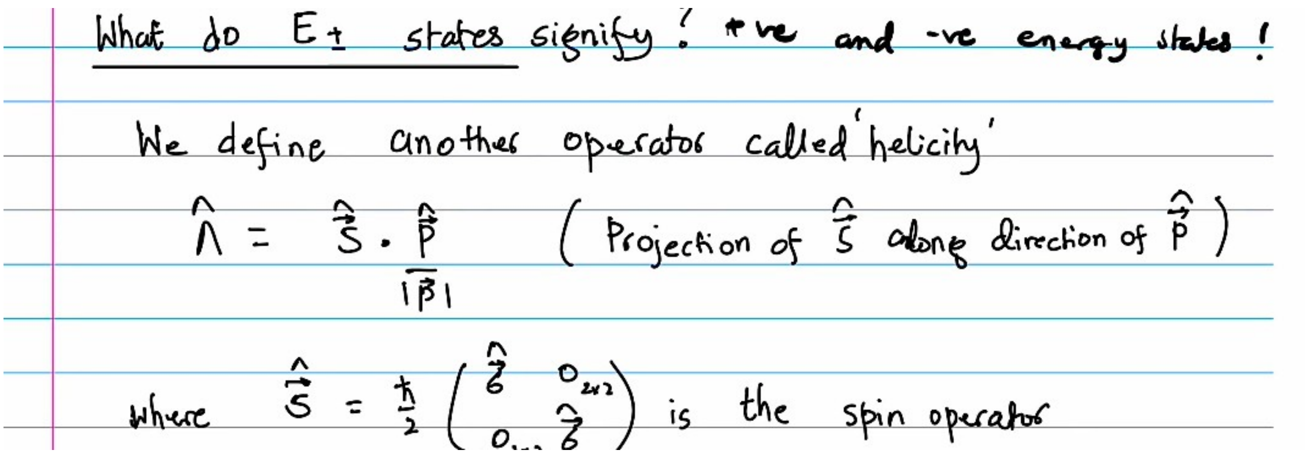
So, therefore, that can also be brought to the left hand side andI will have a matrix equation, which is somethingϕ_0 – some other things actingϕ_0. It is a doublet equation. This is 2×1 dimensional matrix equation. The lower one is also 2×1 dimensional matrix equation. So, these pairs of two doublet equation, this is 2×1 meaning some AB for ϕ_0 and some CD for χ_0 should satisfy that. We want to know what is the doublet ϕ_0 and what is the doublet χ_0. So, these equations can together be written as E_0 m_0c^2identity 2×2 identity acting on ϕ_0 χ_0. So, this identity 2×2 will hit just this doublet ϕ_0 and – csummation σ_i pi, σ_i is again 2×2 dimensional polymetrics. So, this block will hit only this doublet χ_0 andI will generate the first equation here. Similarly, the lower equation here can be generated by the lower row of this matrix equation. So, you can see it is effectively The whole equation which is obtained after all the derivatives acting is equivalent to some matrix M, 4×4 dimensional matrix M acting on these ϕ_0 and χ_0. The matrix elements are broken into blocks of 2×2 . So, this is identitysomething here, poly matricespi sum over all poly matrices and similarly here. And the other diagonal thing is identity, but with a different constant multiplied to it. So that matrix acts on ϕ_0 and χ_0 and gives me a 0. There are two possibilities. This equation can be trivially satisfied if ϕ_0 χ_0 themselves are 0. But if they become 0, this doublet becomes 0, the full wave function becomes 0. That we do not want.

We do not want the trivial solution that everything is 0, so all equations are satisfied. That we do not want. Is there a non-trivial solution? Suppose the other possibility is that there is some non-zero possibility non 0, ϕ_0 and χ_0 which do satisfy that. That can be only possible if this matrix *M* does not have an inverse. Why I am saying so? If suppose a matrix M has an inverse, I can multiply M inverse to the both side. M inverse multiplying 0 will remain 0 and M inverse multiplying M on the left hand side will become identity. So identity acting on ϕ_0 χ_0 should be equal to 0. That means ϕ_0 and χ_0 should individually be 0. This will happen if M is an invertible matrix. So therefore I do not want *M* to be invertible. M should be a matrix whose inverse should not exist. And we know from matrix theory all matrices are not invertible. There are set of matrices whose inverse does not exist. What are those matrices? The matrices whose determinant is 0. So, matrix determinant if it is 0, we know it cannot, it does not have an inverse. So, therefore, in order to have a non 0, ϕ 0 and ϕ 0, it is necessary that *M's* determinant is 0. That means determinant of this M matrix which we have written should be 0. That means this 2×2 blocks determinant multiplying this and this should be equal to 0. So, it is 4×4 dimensional matrix not 2×2 dimensional matrix. So, be slightly careful about it. So, there is a 2×2 element here, 2×2 element here, 2×2 element here and 2×2 element here. So, in principle it is 4×4 dimensional matrix and this is a good exercise for you to write down what do you get if you write down the determinant of such 4×4 dimensional matrix. If you just multiply this diagonal to diagonal, you will be left with a 2×2 dimensional matrix because it is not a number here, it is matrices here.So, you have to do it slightly carefully.

So, it is an easy exercise actually you should do and you should realize once you do that the determinant equal to 0 is exactly the same equation which we obtained for the Klein-Gordon case that E^2 is $p^2c^2 + m_0^2c^4$. So, therefore, the solutions of determinant m is equal to 0 is the same solution which we got for Klein-Gordon equation. That E which was the eigen value of temporal derivatives that was the energy comes up with two signs positive and negative. So despite we started with the first order derivative structure we hope that we will get a single signature E but the fact that Lorentz symmetry demanded that ϕ it should the wave function should be 4×1 dimensional thing not a 1×1 dimensional thing. Then it could be broken into two doublets ϕ and χ and their equations were coupled differential equation. So, two coupled differential equations of two different fields together is equivalent to a single equation a double derivative equation of a single variable in some sense. So, therefore we know that the Dirac equation manifestly satisfies Klein-Rodan equation as well because double derivative level only that is a possibility. So, therefore when we wrote down the two single ordered differential equation the differential equation are all single order so that we were hoping that due to its single order structure we will get a unique e but the fact that it is not for one one variable alone it is a two variables coupled which is equivalent to double order differential equation of a single variable so two this is what you might have seen in lagrangian approach as well lagrangian equation for motion is a double derivative equation of motion. So if you remember Lagrangian scheme of things, $\partial L/\partial\dot{q}$ and $\partial/\partial t$ was equal to $\partial L/\partial q$. So ultimately if you do this, you will get equation like $\ddot{q}$ is equal to $V'(q)$. However, if you go to phase space, the same equation is written in terms of Hamilton Jacobi equation where $\dot{q}$ is $\partial H/\partial q$and $\dot{p}$ is my $\partial H/\partial q$. So, $\dot{q}$ is ∂H/∂p and p dot is ∂H/∂q. This was + and this is –. So, double order differential equation is equivalent to two single order differential equations, which is what we are getting here as well that we ended up on the two coupled differential equations two coupled differential equations of single order which is together equivalent to one single one double order differential equation of a single variable and that is what it is reflected over here that it is not much different from Klein-Gordon equation. Our structure that it has to satisfy Klein-Gordon has manifestly put the same content in terms of energy. So, we are not getting rid of negative energy solution despite looking for a first order equation. Now, if this is true, once we have known what is E, E is + – of the same previous quantity, then it is easy to prove from here that From first equation, I know how χ_0 is related to ϕ_0.

They should be equal to this quantity on the left should be equal to this quantity on the right, which you can use the identity which we have just derived over here and write it like this, that χ_0 and ϕ_0 are related like this. This is actually coming from the second equation. That you can divide the whole thing by E + m_0c^2 such that χ_0 will become this quantity divided by $E + m_0c^2$. So, this is what the exercise I was trying to propose, but now you see what it is. Now, we knew that ultimately we have this structure at hand. The full wave function can come up with two energies, positive energies or negative energies. So, I am going to write ψ + for positive energy and ψ – for negative energy. And it is made up of two doublets, ϕ and χ. Again I am going to call it ϕ + for positive energy and ϕ – for negative energy and similarly for χ. χ + for positive energy, χ – for negative energy.





Thus we have solution of Dirac eqn. as

$$\psi_{\pm}=\begin{pmatrix}\psi_1\\ \psi_2\\ \psi_3\\ \psi_4\end{pmatrix}=\begin{pmatrix}\phi_{\pm}\\ \chi_{\pm}\end{pmatrix}=\begin{pmatrix}\phi_0\\ \chi_0\end{pmatrix}e^{i\frac{(\vec{p}\vec{x}-E_{\pm}t)}{\hbar}}$$

For normalized state

$$\int \psi^{\dagger}_{\lambda}(\vec{p})\,\psi_{\lambda}{}'(\vec{p})\,d^{3x}=\delta_{\lambda\lambda'}\,\delta^3(\vec{p}-\vec{p}\,')$$ *requirement yields*

{λ, λ' take values either + or – }

What do $E_{\pm}$ states signify? +ve and –ve energy states !

We define another operator called 'helicity'

$$\Lambda = \hat{\vec{S}} . \frac{\hat{\vec{P}}}{|\hat{\vec{P}}|} \quad \text{(Projection of } \hat{\vec{S}} \text{ along } \hat{\vec{P}} \text{)}$$

where $\hat{\vec{S}} = \frac{\hbar}{2} \begin{pmatrix} \hat{\vec{\sigma}} & O_{2\times 2} \\ O_{2\times 2} & \hat{\vec{\sigma}} \end{pmatrix}$

And ultimately I know that even those ϕ_+ or ϕ_- can be written in terms of plane waves of positive energy and plane waves of negative energy – *iEt* in the exponential or – *iEt* in the exponential. The – sign thing will be called the thing with the – sign in the exponential with positive energy. So, – sign here will come with a + and it should be – + here and + sign will be declared as a –, E – means negative energy solution. So, this is how the structure is, both positive energy and negative energy. And since We have demanded them to be a solution, we wanted to of the Dirac equation and the eigen states of momentum as well as temporal derivatives. After all, eigen state of a Dirac Hamiltonian which was appearing on the right hand side, which was made up of momentum operator and we demanded for eigen state of momentum operator that makes it eigen state of Dirac Hamiltonian as well, so they should be orthonormal. So, therefore, positive energy Eigen states should be orthogonal to each other and negative energy states should be orthonormal to each other and positive and negative energy should not talk to each other, they should be also orthonormal. So, there should be full set of orthonormality.
Different Eigen values as well as different energies, the signs are different. Remember E is completely fixed by P. Its sign is not fixed, but its magnitude is completely fixed. So, independently if I fix only P, if the magnitude of *P* is fixed, *P* and *P'* are same, then *E* and *E'* magnitude will be the same, only sign can differ. And that sign difference is captured by this delta function. The total orthonormality therefore

is equivalent to orthonormality in the eigenvalues P_1, P_2, P_3 **c**ollectively written as $\vec{P}$. They should be

the same, energy should be the same that means their magnitude as well as the sign should be the same. The magnitude sameness is guaranteed by this Dirac delta of momentum. The sign should be the same is maintained by this Kronecker delta that they should be the same sign objects otherwise they will not be normalized to one. So, that is for normalized orthonormal set of solution this should happen. And as we again signify E + E – are positive and negative energy solutions of the Dirac equation. Apart from having positive energy and negative energy, it was already there in the Klein-Gordon equation. What have we earned more in this?
So, we can see that apart from having a plane wave part which is this and positive energy solution and negative energy solution which was already there in the Klein-Gordon.

HW : Verify $\hat{\Lambda}$ commutes with the Dirac Hamiltonian $\mathcal{H}_D$

$$[\hat{\mathcal{H}}_D, \hat{\Lambda}] = 0$$

Thus, the Dirac particle can be identified with eigenvalue of helicity too.

If the particle is moving 'classically' along z-direction

$$\vec{p} = (0, 0, p)$$

<u>HW :</u>Verify Λ commutes with the Dirac Hamiltonian $\mathcal{H}_{\mathcal{D}}$

$$[\hat{H}_D, \hat{\Lambda}] = 0$$

Thus, the Dirac particle can be identified with eigenvalue of helicity too.

If the particle is moving 'classically' along z-direction.

$$\vec{P} = (0, 0, p)$$

This part was common in the Klein-Gordon equation also this exponential came about. The new thing is this pair ϕ_0 χ_0 doublets which is the 4×1 dimensional vector written in terms of two 2×1 dimensional vectors. What are these? So, now we will focus attention to what are these. So, first we define something called a helicity a helicity operator. What is helicity operator? It is an operator depicted by λ. That operator is just projection of momentum that is the normal vector of the momentum, unit normal along the momentum direction. Operator divided, the momentum vector upon the magnitude of the momentum. And dot product with S. S is some matrix valued vector that means its x component is some matrix, its y component is some matrix and its z component is also some matrix.

And the matrices which are going to be put in the x component, y component, z component are $\frac{\hbar}{2}$ σ_x σ_x in the diagonal and everything else is off diagonal, $\frac{\hbar}{2}$ here σ_y σ_y and off diagonals are 0 and $\frac{\hbar}{2}$ σ_z σ_z and off diagonals are 0 So, the components of this $\vec{S}$ are two copies in the diagonal of the poly wave matrices. So, this is therefore is a spin operator remember σ poly matrices for 2×1 dimensional case. Spin up, spin down, spin down, spin up kind of thing. σ_i s used to measure the spin for 2×1 column vectors.

So, therefore, individual σ matrices can measure the spins of either ϕ_0 or χ_0. So, these are like some spin object. It looks like it has some spin and this has some spin because poly matrices can act on this and poly matrices can act on that. But this new S operator which is the 4×1 dimensional vector which operator which is 2×2 block in upper diagonal and 2×2 in the lower diagonal is a spin operator for the 4×1 dimensional vector ψ_1, ψ_2, ψ_3, ψ_4. This can act on the 4×1 dimensional vector. So, therefore, this is the spin operator for the full wave function. So, what we are doing, the helicity which we are defining a new operator as is just the projection of the spin operator along the momentum along which the plane wave is moving. Plane wave has some momentum $\vec{P}$, eigen state of momentum operator that means it has some direction along which it is moving. And it is moving with certain spin, spin is also a vector quantity as we wrote down poly matrices some operator i some operator *j* some operator *k* so therefore it is matrix value vector so this is spin operator has spin matrices have some direction I *j k* so I take the dot product and I get a matrix which is a operator but along the direction of the momentum that thing is called helicity helicity is the projection of spin operator along the momentum with which particle is moving.

So, now this new operator helicity, first we should verify that it commutes with the full Dirac Hamiltonian and in this αs were also made up from poly matrices in the off diagonal remember. And this time the spin operator which are which we are trying to make helicity operator have spin operator σs poly matrices in the diagonal dotted with the p previously it was multiplying in the off diagonal. Now this is an useful exercise actually that this λ commutes with the previous Hamiltonian Dirac Hamiltonian that you should prove. This is a homework exercise two lines exercise one should do and you will be able to prove that this operator which is made from math $\vec{\sigma}.\vec{P}$ here and here commutes with the Dirac Hamiltonian which was Dirac Hamiltonian which was written as the first line. Just convert the momentum operator to their eigenvalues and you can verify that math- vec σ . vec P in that case previously it was appearing as $\vec{\sigma}.\vec{P}$ in the off diagonal and they commute. So if two operators commute that means they can have simultaneous eigenvectors. So that means the solutions of Dirac equation can have eigen state of helicity operator as well. So, not only the full solution of the Dirac equation which was plane wave 4×1 dimensional vector. Now we know it is a doublet of spins and that thing would be an eigen state of the full Dirac Hamiltonian as well. So the full solution of the Dirac equation is identified by actually four quantities. The three momentum $\vec{P}_{1,}\vec{P}_{2,}\vec{P}_{3}$ that will fix the total energy's magnitude then a sign whether it is positive energy or negative energy and then there is a helicity element which is also there so call it small h. So ultimately the full solution of the Dirac equation which we just wrote down here apart from having the plane wave part and three momentum which is completely fixing the magnitude of energy would have a sign of energy which is free that you have to choose and the helicity of this which is also an eigen state possible so it has some helicity. what are the possible values helicity can take so let us do an example suppose the particle is traveling along the *z* direction that means momentum has only one component along the *z* direction *x y* components are zero in that case if it is having only one momentum direction the projection of a spin along $\vec{P}_s$. $\frac{\vec{P}}{|P|}$ will just be $\vec{\sigma}z$ because magnitude of full $\vec{P}$ divided by its magnitude will just give me unit vector along z direction.

Then $\hat{\Lambda} = \hat{S}_z = \frac{\hbar}{2}\begin{pmatrix}\hat{\sigma}_z & 0 \\ 0 & \hat{\sigma}_z\end{pmatrix} = \frac{\hbar}{2}\begin{pmatrix}1 & & & 0 \\ & -1 & & \\ & & 1 & \\ 0 & & & -1\end{pmatrix}$

→ Prove : This operator has two eigenvalues

$+\frac{\hbar}{2}$ for $\begin{pmatrix}1\\0\\0\\0\end{pmatrix}$ and $\begin{pmatrix}0\\0\\1\\0\end{pmatrix}$

$-\frac{\hbar}{2}$ for $\begin{pmatrix}0\\1\\0\\0\end{pmatrix}$ and $\begin{pmatrix}0\\0\\0\\1\end{pmatrix}$

Ex : - If the particle moves in the y-direction, find out the eigenfunctions of the helicity.

An eigenstate of the Dirac eqn with particle moving in z-direction having helicity eigenvalue

$$\Psi_{p,\lambda,\frac{\hbar}{2}} = N\begin{pmatrix}1\\0\\ \frac{c\hat{\sigma}_z p}{m_0c^2+E_\lambda}\begin{pmatrix}1\\0\end{pmatrix}\end{pmatrix} e^{i(pz - \frac{E_\lambda t}{\hbar})}$$

Then

$$\hat{\Lambda} = \hat{S}_z = \frac{\hbar}{2}\begin{pmatrix}\hat{\sigma}_z & 0 \\ 0 & \hat{\sigma}_z\end{pmatrix} = \frac{\hbar}{2}\begin{pmatrix}1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1\end{pmatrix}$$

→ Prove: This operator has two eigenvalues

$$\frac{+\hbar}{2}\ for \begin{pmatrix}1\\0\\0\\0\end{pmatrix} \quad \text{and} \quad \begin{pmatrix}0\\0\\1\\0\end{pmatrix}$$

and

$$\frac{\hbar}{2}\,for\begin{pmatrix}0\\1\\0\\0\end{pmatrix}\quad\text{and}\quad\begin{pmatrix}0\\0\\0\\1\end{pmatrix}$$

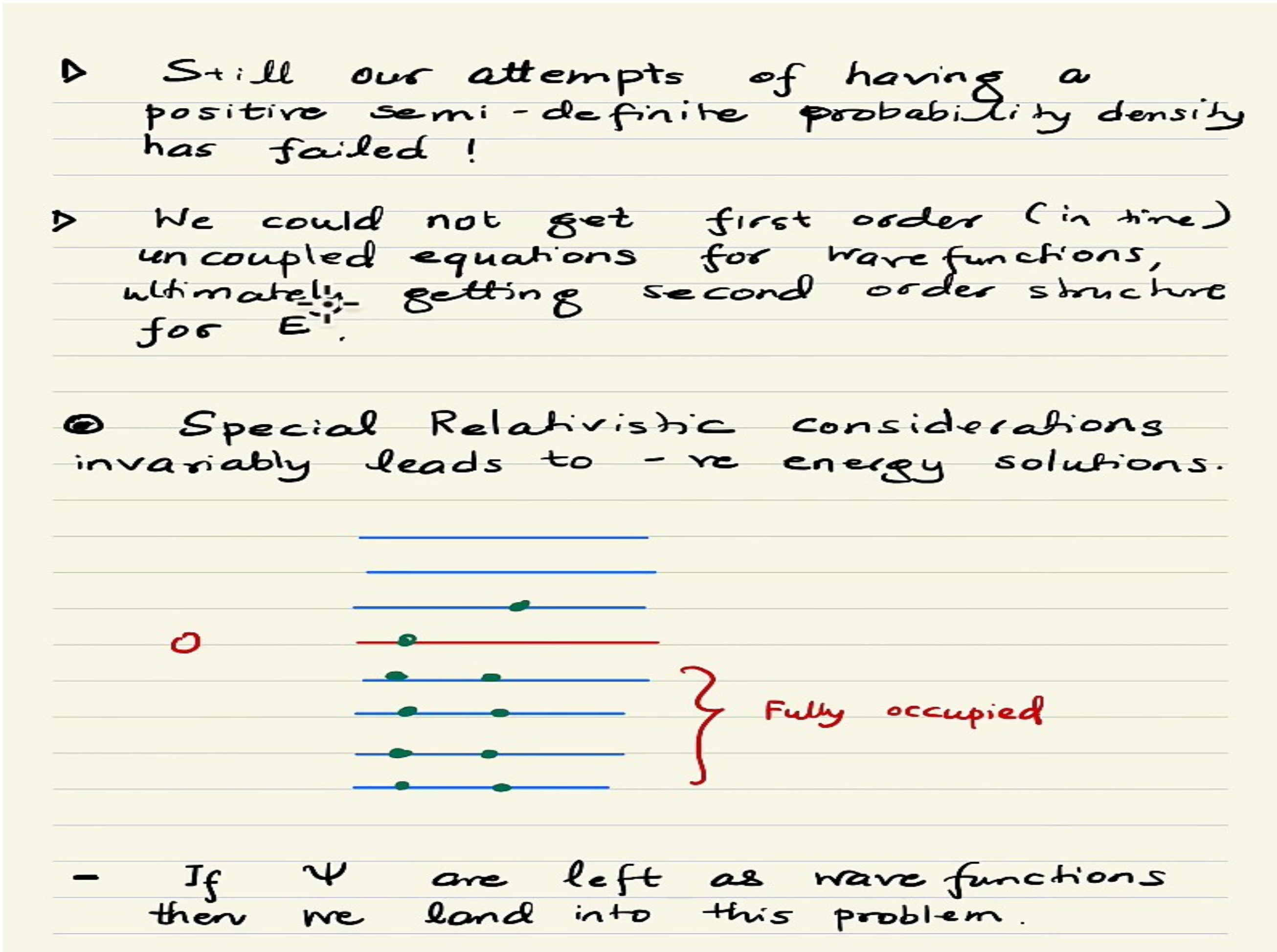
Ex: If the particle moves in the y – direction, find out the eigenfunctions of theh helicity.

An eigenstate of the Dirac eqn with the particel moving in z -direction having helicity eigenvalue.

$$\Psi_{P,2,\frac{\hbar}{2}}=N\begin{pmatrix}1 & 0\\0 & 0\\0 & 0\\\frac{c\hat{\sigma}_z P}{m_{0c}^2+E_\lambda} & \begin{pmatrix}1\\0\end{pmatrix}\end{pmatrix}e^{i(Pz-\frac{E_2 t}{\hbar})}$$

$\hat{S}_z$ will just give me S_z. And for $\hat{S}_z$ 4 + 1 dimensional space is just $\frac{\hbar}{2}$ $\sigma_z\sigma_z$ in the diagonals.

That means $\frac{\hbar}{2}\begin{pmatrix}1 & 0 & 0 & 0\\ 0 & -1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & -1\end{pmatrix}$ .You see, this is the helicity operator for a particle which is moving along the *z* direction or the plane wave which is moving along the *z* direction. This matrix is simple enough. You can see that it has four eigen value, four eigen states, two distinct eigen values. You can solve this eigen value equation. You will find out equation like this. $(\lambda+1)^2(\lambda-1)$ should be *zero.* That means it has two eigenvalues + 1 and two eigenvalues – 1. Therefore, there are two eigenstates with energy + $\frac{\hbar}{2}$ and two eigenstates with energy – $\frac{\hbar}{2}$. You can clearly verify, you can check that this state 1 0 0 0 is an eigenstate of this operator with eigenvalue $\frac{\hbar}{2}$. Similarly, eigenstate *0, 0, 1, 0,* the third element being 1 and all other elements being 0 is also an eigenstate of the same operator with eigenvalue $\frac{\hbar}{2}$. While if I put 1 in the second place, 0, 1, 0, 0, then it is an eigenstate of the operator $\hat{S}_z$ or Helicity operator with – $\frac{\hbar}{2}$. And similarly, 0, 0, 0, 1 is also an eigenstate of the helicity operator or $\hat{S}_z$ operator with eigenvalue – $\frac{\hbar}{2}$. So, this you see if particle moves in the z direction then there are two possible choices of helicity + $\frac{\hbar}{2}$ and – $\frac{\hbar}{2}$. They are degenerate states that means there are two eigenstate which share the eigenvalue $\frac{\hbar}{2}$ and two eigenstate which share the eigenvalue – $\frac{\hbar}{2}$. They are degenerate states that means there are two eigenstate which share the eigenvalue $\frac{\hbar}{2}$ and two eigenstate which share the eigenvalue – $\frac{\hbar}{2}$. Actually, if you can work out, you can prove that even if particle were moving along the y direction, then the helicity operator will become $\hat{S}_y$ and it will be $\sigma_y\sigma_y$ which will be off diagonal matrix in the diagonal. So, this 2×2 would not be 1×1, it would be off diagonal. But still you can prove that again it so happens that despite its off diagonal structure, it has only two eigenvalues, which is again + $\frac{\hbar}{2}$ and – $\frac{\hbar}{2}$. So, therefore, we saw in general you can prove that it can have only two eigenvalues in any direction it moves, $\frac{\hbar}{2}$ or – $\frac{\hbar}{2}$. So, therefore, the full eigen solution of a Dirac equation has a structure. It is identified by three momentum numbers *F*, signature of the energy λ and helicity either + $\frac{\hbar}{2}$ or helicity – $\frac{\hbar}{2}$. The full wave function will be for example + $\frac{\hbar}{2}$ is 1 0 in the in the so *1 0 0 0* can be a case or 1 0 0 1 0 it can be a case and their superposition is also an eigen state with the same eigen value so in general a ϕ_0 multiplying 1 0 and $\varkappa_0$ multiplying *0 0 1 0* together should generate for me this. So, a ϕ_0 some constant let us say multiplying *1 0 0 0* and some other constant multiplying *0 0 1 0*, they both have the same helicity eigenvalue. So, therefore, a combination also has the same helicity $\frac{\hbar}{2}$. So, upper block if you are going to call ϕ, this is ϕ_0, lower block will be called χ_0 and I know ϕ_0 and χ_0 are related as the exercise we just performed that the ϕ_0 and χ_0 are related by that because this is the lower block of the Dirac equation. So, overall the full solution therefore becomes 1, 0, this is ϕ, this is ϕ_0 let us say, then this is χ_0, the superposition of this has become χ_0 and they should be related like that, ϕ_0 and χ_0 should be this and the plane wave with positive energy. Similarly, what positive or negative energy depending upon E_λ, E_λ is positive, E_λ is positive, if λ is negative, E_λ is negative. Similarly, for negative energy eigenstate I can either have a *0 1 0 0* or a *0 0 0 1* or a superposition $c' + d'$. Again when I combine these two upper block will become something ϕ_0 and lower wall block will become something other ϕ_0 and they are should be related like that. So, upper block is ϕ_0 and lower block should be this factorϕ_0. So you see the helicity with negative helicity eigenstate is something like this. Positive helicity will be one here and one here, first and third element. Negative helicity will be second and the fourth element non-zero. And they are related by this thing because we have to satisfy the Dirac equation. So these are the eigenstates of Dirac equation. They have come up with what? Momentum which was already there and sine.This much was there for Klein-Gordon equation as well. Momentum which will determine the total magnitude and the sign of energy. The new piece is spin or a helicity structure. It comes with a new helicity term, helicity property, spin along momentum direction. Previously it has a 1×1 dimensional vector for Klein-Gordon, so there was no spiner part. Now this has a spin structure, therefore this wave function is called a spiner. So this is fine this is a new theory dirac equation which gives rise to spinal part and new properties but still we do not have a solution to negative energy solutions again our hope was if I write first order differential equationI will get a single signature energy which has not realized we have a very double signature kind of energy which was already there.



Okay so therefore we could not get rid of this problem even with the first order of differential equation structure because we could not get first order decoupled equations we have coupled equations which ultimately becomes equal to a second order differential equation so it looks like if I demand a special relativity compatibility

It would invariably lead to a negative energy solutions as well which are problematic because probability density cannot be defined so there were many attempts of understanding if special relativity has to be respected second order differential equation doesn't work first order differential equation also seems does not working because it predicts negative energy states and if particles can have negative energy their probability density will come into question how do I interpret quantum mechanics then so there were various proposals. One proposal was the negative energy by Dirac where he argued somehow nature works in a way that all the negative energy states which are allowed are already full. They are fully occupied remember in the structure each state for a spinor can be accommodated with either + or – sign spin helicity $\frac{\hbar}{2}$ or – $\frac{\hbar}{2}$ if all of them are occupied being fermion which we will see more detail in quantum field theory cannot accommodate more so Dirac's proposal for this Dirac equation was that negative energy states are completely occupied. So therefore, there cannot be any new particle with negative energy. Only thing which we can have is a positive energy. This is a working ad hoc solution being proposed. Or one has to give up the notion of probability density which was predicted by the continuity equation. So therefore, we have to interpret that whatever is coming is not probability density. Probability density is only charge density. Charge density can be positive or negative depending/*c*harge. Okay, so then probably we are talking about fields which we are talking about wave functions which have some charge and if they have some charge they will talk to electromagnetic field outside. We should write down a special relativity compatible Hamiltonian of a charged particle. Right now we have written for a free particle. We should write down for charged particle talking to electric field. That attempt we should do or the other only possibility which we are left with just like when we were doing Klein-Gordon equation we were left with that the size appearing over here cannot be taken as a numbers meaning that they have to become operators. So, that the density operator, the density probability density which we were interpreting should all become operators and then they should have act on some certain things to get rise to numbers. All this indicates that we are talking about a theory of multi-particles. It is not a theory of a single particle and there is problem if I try to enforce a single particle interpretation. We are led to either multi-particle description or wave function themselves becoming operators. All of them is hinting towards one possibility which is that we have to let go of the quantum mechanical structure at special relativity if we want to be truthful to relativity. And therefore, make this size into operators themselves. Operators dependent on position and time. Remember the wave functions were x and t. When we make them operator, they will no longer be 4×1 dimensional things as we have discussed because we are looking for a solution of operator previously. Now we will give up that notion that it they are themselves vectors they will become operators themselves in what dimension and what×water matrix that will all be known when we try to do make convert them into operators and that is the domain of quantum field theory so we see that both attempts or first order and second order invariably leads us to the structure hinting towards a quantum field theory and now it looks like there is no escape but to deal with quantum field theory develop a theory which talks about fields and not about wave functions and that is what we will try to do in the next class onwards okay so I stop here.