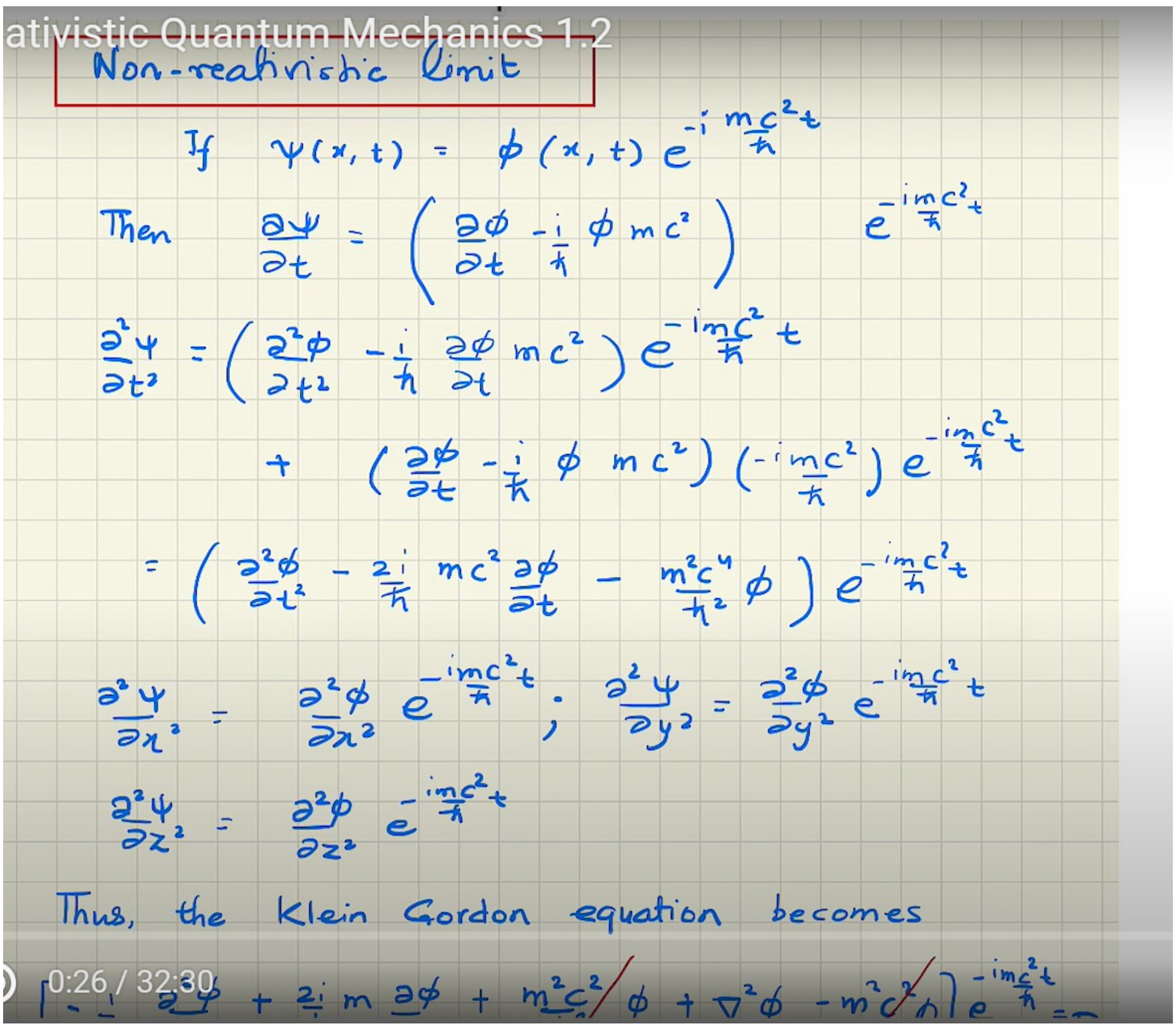
Foundation of Quantum Theory: Relativistic Approach
Relativistic Quantum Mechanics 1.2

Pr of . Kinjalk Lochan
Department of Physical Sciences
IISER Mohali

Klein Gordon equation II
Lecture- 12

So, today we will discuss the recovery of non–relativistic limit from the Klein–Gordon equation, the relativistic equation what we have learned for wave functions. So, we know that now we have to deal with the Schrodinger state of Schrodinger equation, the Klein–Gordon equation for relativistic particles.



Non-relativistic limit

If $\psi(x,t)=\phi(x,t)e^{-imc^{2t}/\hbar}$

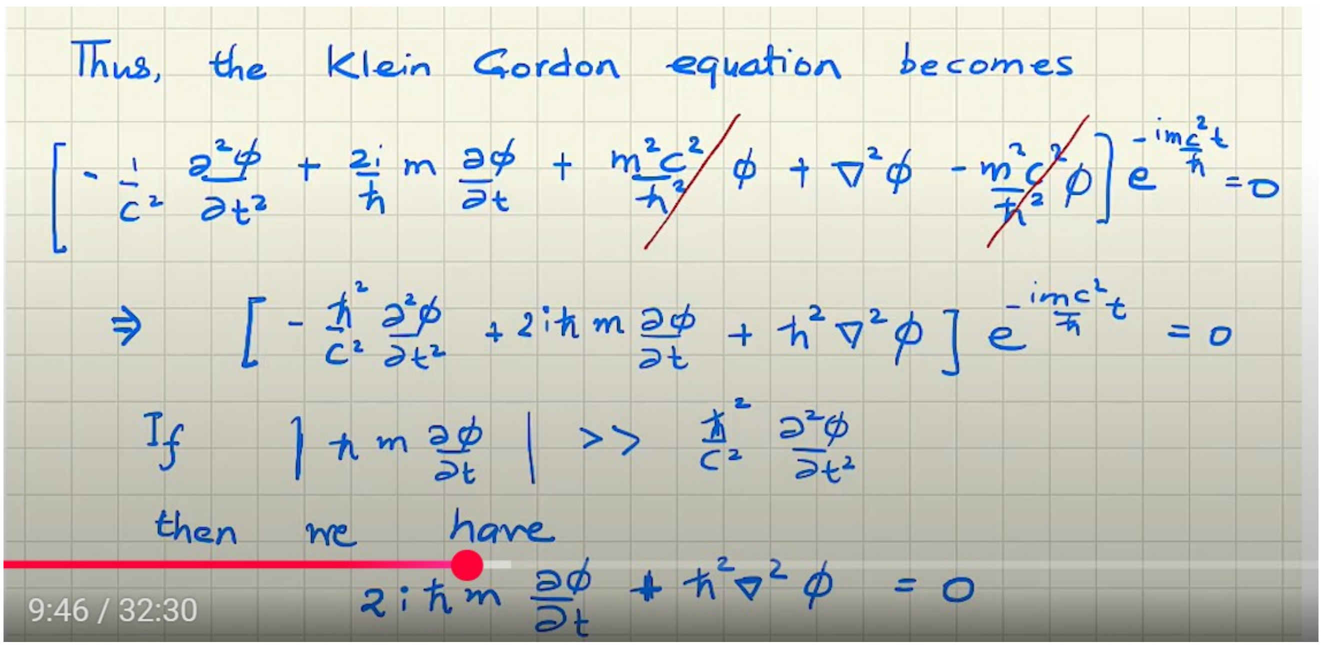
Then $\frac{\partial\psi}{\partial t}=(\frac{\partial\phi}{\partial t}-\frac{i}{\hbar}\phi mc^2)e^{-imc^{2t}/\hbar}$

$$\frac{\partial^2\psi}{\partial t^2}=\frac{\partial^2\phi}{\partial t^2}-\frac{i}{\hbar}\frac{\partial\phi}{\partial t}mc^2e^{-imc^{2t}/\hbar}+(\frac{\partial\phi}{\partial t}-\frac{i}{\hbar}\phi mc^2)(\frac{-imc^2}{\hbar})e^{\frac{-imc^{2t}}{\hbar}}$$

$$=\ (\frac{\partial^2\phi}{\partial t^2}-\frac{2i}{\hbar}mc^2\frac{\partial\phi}{\partial t}-\frac{m^2c^4}{\hbar^2}\phi)e^{-imc^2\frac{t}{\hbar}}$$

$$\frac{\partial^2\psi}{\partial x^2}=\frac{\partial^2\phi}{\partial x^2}e^{-imc^2\frac{t}{\hbar}};\frac{\partial^2\psi}{\partial y^2}=\frac{\partial^2\phi}{\partial y^2}e^{-imc^2\frac{t}{\hbar}}\quad \textit{and}$$

$$\frac{\partial^2\psi}{\partial z^2}=\frac{\partial^2\phi}{\partial z^2}e^{-imc^2\frac{t}{\hbar}}$$



Thus the Klein Gordon eqaution becomes

$$[\frac{\partial^2\phi}{\partial t^2}+\frac{2i}{\hbar}mc^2\frac{\partial\phi}{\partial t}+\frac{m^2c^4}{\hbar^2}\phi+\nabla^2\phi-\frac{m^2c^2}{\hbar^2}\phi]e^{-imc^2\frac{t}{\hbar^2}}=0$$

If

$$\left|\hbar m\frac{\partial\phi}{\partial t}\right|\gg\frac{\hbar^2}{c^2}\frac{\partial^2\phi}{\partial t^2}$$

Then we have

$$2i\hbar m\frac{\partial\phi}{\partial t}+\hbar^2\nabla^2\phi=0$$

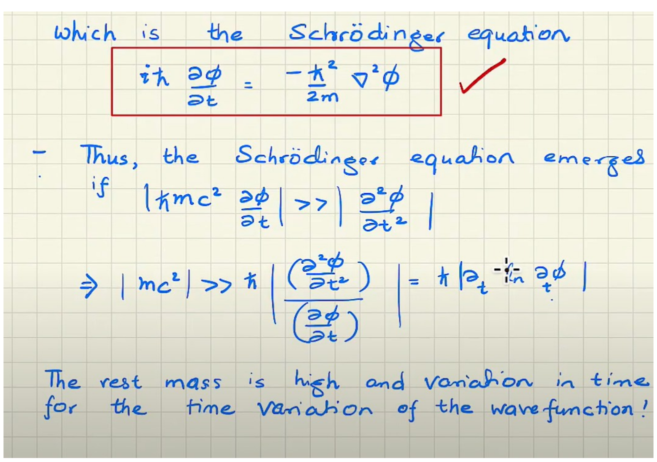
So, if I just can go back to the slide where we had written the Klein–Gordon equation is this one. that this is the equation which really governs particles moving at any speed. However, we have seen and we have examples, lab verifications of things, the Schrodinger equation is not too bad. that means if I am looking at particles which are not moving with very high velocities, for them effectively we can approximate the quantum evolution by Schrodinger equation. So, how do we recover that Schrodinger equation is also recoverable at low speeds limit from the Klein–Gordon equation. So, one thing is that when I want to go to non–relativistic limit there are essential statement is that v/c is a ignorably small quantity. When I take v/c tending to zero limit that means there is no velocity whatsoever compared to c. So, in that sense the equation will be free of c. They should not, we know in the Schrodinger equation there is no appearance of speed of light c. So, one way of doing that would be taking this v/c tending to 0 limit. that means β goes to 0, the special relativity parameter β goes to 0. In this process, wherever I see v/c in the Schrodinger equation, in the Klein–Gordon equation, I can put it to 0. But in quantum mechanics, we do not pose problems in terms of v rather than we have a p/c. Momentum/mc let us say whenever p/mc is 0 then also we can recover the non–relativistic limit in quantum mechanics this will become operator so p/mc is supposed to be very small but if I do that that means that I have to throw away the whole terms which are like p/mc, remember. The Klein–Gordon equation which we had written down is a manifestation of the statement that $E^2/c^2 - p^2/p^2c^2$ is equal to m_0^2c, $m_0^2c^2$, okay. So, sorry, so there should not be c^2 here, so there is just p^2. Now if this is the case if I divide everything by $m_0^2c^2$ or the rest mass so in that case I will get here one and here I will get $m_0^2c^2$ and here I will get $m_0^2c^4$ if I take this quantity to zero that is a valid non–relativistic limit but that will erase out all the momentum So, momentum is not appearing in the equation afterwards. So, there are all special derivative will go away if I employ this technique that I have to get this quantity down to 0. For a finite c, speed of light is finite. So, only thing is that p/mc can approach 0 is only through if momentum goes to 0 limit.

that means we will be talking about non–dynamical particles which are just located at a point not have any other distribution in space that would not be very beneficial because we want to know schrodinger equation also talks about probability of particles of being here they are moving with certain velocity momentum expectations and whatnot so I do not want p going to zero so what should I do the other way would be to take c tending to infinity limit c tending to infinity limit is also a non–relativistic limit that means any speed finite speed is practically ignorably small compared to the speed of light this is not really infinity but *v/c* is ignorably small quantity this is a dimensionless number so what it tells is that it is very large number and *v/c* can be ignorably small and I do not have to throw away the p tending to zero I do not have to take I can maintain *p* and whenever I see a appearance of *c* I will compare with terms wherever *c* are present or not and I will see that c tending to infinity limit what will happen.

So, one thing which will happen is in the relation which we wrote down *math - E^2 over c^2 – vec p ^ 2* was supposed to be $m_0^2c^2$. So, the right hand side is going to become very large that is very high rest momentum or rest energy. Rest energy or rest momentum is very high. that means compared to its rest mass, you are not moving at sufficiently fast velocity. So, this is non–relativistic limit is also high rest energy limit or rest momentum limit. that means due to your inertia, the momentum develop, inertia meaning energy due to m_0^2, $m_0 c^2$ would be the rest mass energy that is much dominant over other forms of energy like kinetic or potential. So, that is the limit we will try to use without sending momentum to 0 and then try to recover the non–relativistic limit out of the Schrodinger equation or out of the Klein–Gordon equation. When I write the stationary state $\psi(xt)$, we have seen that previously it should be some $\phi(x)$ times $e^{iEt/\hbar}$ $e^{-iEt/\hbar}$. that is what the stationary limit we have discussed before as well. Now, in this E, there are two terms hiding. E is math- `sqrt{p^2 c^2 + m_0^2 c^4}`. So, there is a, if mc^2 is very large compared to pc. I can pull out this m^2c^4 out then I will have a m c^2 and under root I will have a $1 + p^2/c^2$ and m^2c^4 this is what we had done for kinetic energy if you remember as well that we went to a frame where this ratio was very small So, that is to say the energy can be written as a Taylor expansion of this quantity as we had done mc^2 + *times* $1 + p^2c^2m^2 c^4$ with a power half + higher powers as we have seen for kinetic energy expansion. So, what we can do, we can separate out a rest mass energy factor mc^2 times 1 from the total energy and I can pull it out in the exponential. So, this $e^{-iep/\hbar}$ is made from rest mass energy + other higher order terms which depends/the ratio Pc/mc kind of things. So this mc^2 kind of thing. So this kind of expansion we can do and I can pull out this saying that rest mass energy is very high. So I can have a Taylor expansion and then I can pull out this thing. This is just simple mathematical trick which I have pulled out. So therefore I can choose to write my wave function in terms of exponential which solely depends on the rest mass energy coming from this mc^2 and the one in the Taylor expansion and all other things ikeep along with $\phi(x)$ and define it a new $\phi(x)$ of t. So, this is just rest mass pulling out, which I can do mathematically, there is no problem with that. Then I do the first derivative, I will get *a* $\partial\phi/\partial t$, when the derivative will hit this and the exponential back and when the derivative hits the exponential, I will get a $-i\phi mc^2$, because there is a *t* linearly proportional to the *t* term in the exponential and there is a ϕ outside. So, I will get $mc^2/\hbar$ with a *-i* here and the exponential back. Similarly, if I take the double derivative, you will get this kind of term, first the double derivative will hit this term and leave the exponential around, then the double derivative hit the exponential and leave the first term around. This is what you will get, you can simplify that you will get terms like $\partial^2\phi/\partial t^2$, $i\hbar mc^2\partial\phi/\partial t$ and $-m^2c^4/\hbar^2\phi$.

So the double derivative of this splitting will give you this rise to this kind of terms. Now if I want to do double derivative with respect to x, in this kind of splitting x is only appearing in this ϕ. In this second part of the exponential there is no x. So $\partial^2\psi/\partial x^2$ will just hit the ϕ part and I will have a double derivative of ϕ with respect to *x.* And similarly for y and similarly for z as well. So I would have three kinds of term for special derivatives all hitting the derivatives all the derivatives hitting the ϕ in temporal part the derivatives will hit ϕ twice one times ϕ and one times the exponential and twice the

exponential meaning double derivative on this product function can have splitting that double derivative solely acts on this double derivative solely acts on exponential or double derivative splits/one derivative on ϕ and one derivative on exponential And then we can put all these terms back into the Schrodinger equation. From the spatial derivatives, I will just generate the Laplacian part of ϕ with exponential out, so that I can write like Laplacian times exponential out. From a temporal derivative, I will get all these terms, all these terms. And remember in Klein–Gordon equation also there is a rest mass, rest momentum $m^2c^2/\hbar^2$ term, which you can see gets exactly cancelled from this term. There is a – sign here, but remember double derivative appears with a $-1/c^2$. So, therefore it will turn things into a + and therefore this will exactly cancel the rest mass term. So, I will be left with the first term, the second term, third term goes out, the fourth term which survives and this exponential. Ultimately, I have these three terms surviving times the exponential is equal to 0. Obviously, exponential cannot hit value 0. So, that means the terms in the bracket should be equal to 0. Now, look at the structure. I have this kind of thing where a double derivative of ϕ is appearing, a single derivative of ϕ is appearing and the Laplacian of ϕ is appearing. Now, in the limit when the term which is double derivative of ϕ is very very small compared to the single derivative term as well as the Laplacian term. So, suppose this is happening that this is the weakest of them all. that means magnitude wise forget about I or signature, magnitude wise $\hbar$ m times d ϕ/dt term should be much much greater than the $d^2\phi/dt^2$ term. If that happens in the temporal part if there are two terms I can say that this is much greater than that so I can maintain this and throw away this so therefore if I do that I will be left only with one term as temporal derivative which is $\partial\phi/\partial t$ we have chosen that the second term the first term is much weaker than the second term and in that case you will see the remaining two terms exactly give rise to the schrodinger equation. So in the limit when the double derivative of ϕ is much smaller than the term compared to the single derivative of ϕ. Remember ϕ was what?
ϕ was the rest mass independent part. Rest mass energy was coming in the exponential.
I have pulled out the rest mass energy. So therefore rest mass energy independent part is the ϕ. that ϕ's double derivative is much weaker than its single derivative. In that limit I get the equation which governs the ϕ which is rest mass independent part is the schrodinger equation all right so let us look at in what limit the schrodinger equation has emerged.



which is the Schrodinger equation

$$\left|\hbar m\frac{\partial\phi}{\partial t}\right| \gg \frac{\hbar^2}{c^2}\frac{\partial^2\phi}{\partial t^2} \to \left|mc^2\right| \gg \hbar\left|\frac{\left(\frac{\partial^2\phi}{\partial t^2}\right)}{\left(\frac{\partial\phi}{\partial t}\right)}\right|$$

$$= \hbar\left|\partial_t \ln \partial_t \phi\right|$$

The rest mass is high and variation in time for the time variation of the wavefunction!

As we saw first derivative is much stronger compared to the second derivative or second derivative is much weaker than the first derivative if you arrange things properly you will see the coefficient multiplying the first derivative is exactly $\hbar mc^2$ that means mc^2 rest mass energy has to be much much greater than the double derivative divided by the single derivative times i$\hbar$ so this $\hbar$ should have been in the denominator actually all right so that tells me that the rest mass energy has to be much greater than the log derivative of ϕ . ϕ is the temporal derivative of the wave functions that part which does not contain the rest mass energy. It has some time profile. It has some time derivative.

That time derivative should not change much faster. Meaning the time ϕ should remain practically constant. Meaning it should not be very high that means double derivative is much weaker than the single derivative and the scale to compare how small is small that the ratio should not come close to the rest mass energy. So, there should not be rapid variations in time for the ϕ, $\dot{\phi}$ should practically smoothly very slowly evolve in time not very fast. So, if the rest mass energy is high and variation in the time for the time variation that meaning variation in $\dot{\phi}$ over ∂t of ϕ should be much much smaller for the that part of the wave function which is not containing the rest mass energy in that limit that function $\dot{\phi}$ or ϕ behaves as the part which is schrodinger equation so the part of you can see that the probability if I used to believe in non–relativity probability density which we will see in the couple of minutes. If I take $|\psi|^2$ it is also equal to $|\phi|^2$ because exponential part of rest mass energy is just a phase. So, effectively you can say that in the non–relativistic limit where rest mass energy is 2ϕ then the actual wave function is the Schrodinger wave functions up to a phase which is very high which we can ignore as well because effectively there is no new information the phase is bringing about. So, this is the high rest mass energy not zero momentum. The high rest mass energy of the wave function is the limit in which you do get Schrodinger equation back. Fine, so once we have understanding how to get the Schrodinger equation and other things, let us look at the definitions of probability densities, the probability current and all other things which we have learned for Schrodinger equation.

Probability density and Current

The K-G equation

$$\left(\Box - \frac{m^2c^2}{\hbar^2}\right)\psi = 0$$

$$\left(\Box - \frac{m^2c^2}{\hbar^2}\right)\psi^* = 0$$

$$\Rightarrow \quad \psi^*(\Box\psi) - \psi\,\Box\psi^* = 0$$

$$\Rightarrow \quad -\frac{1}{c^2}\left(\psi^*\partial_t^2\psi - \psi\,\partial_t^2\psi^*\right) + \left(\psi^*\nabla^2\psi - \psi\,\nabla^2\psi^*\right) = 0$$

$$\frac{1}{c^2}\partial_t\left(\psi^*\partial_t\psi - \psi\,\partial_t\psi^*\right)$$

$$= \nabla\left(\psi^*\nabla\psi - \psi\,\nabla\psi^*\right)$$

$$\partial_t\left[\frac{i\hbar}{2mc^2}\left(\psi^*\partial_t\psi - \psi\,\partial_t\psi^*\right)\right]$$

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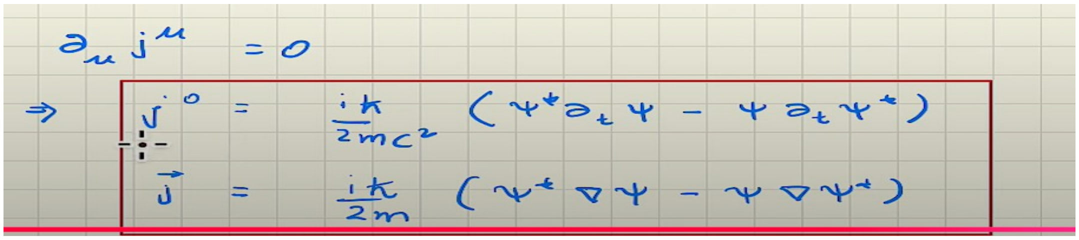
$$\partial_t\left[\frac{i\hbar}{2mc^2}\left(\psi^*\partial_t\psi - \psi\,\partial_t\psi^*\right)\right]$$

$$= \frac{i\hbar}{2m}\left(\psi^*\nabla\psi - \psi\,\nabla\psi^*\right)$$



Probably density and current densities

The K.G. equation

$$(\Box - m^2 \frac{c^2}{\hbar^2})\psi \quad = 0$$

$$(\Box - m^2 \frac{c^2}{\hbar^2})\psi^\star \psi^\star(\Box)\psi - \psi(\Box)\psi^\star = 0$$

$\rightarrow$

$$\frac{1}{c^2}\partial_t(\psi^\star\partial_t\psi - \psi\partial_t\psi^\star) = \nabla\partial_t[\frac{i\hbar}{2\mathrm{mc}^2}(\psi^\star\nabla\psi - \psi\nabla\psi^\star)] = \frac{i\hbar}{2}m(\psi^\star\nabla\psi - \psi\nabla\psi^\star)$$

$$J^\mu = \frac{i\hbar}{2\mathrm{m}}(\psi^\star\nabla\psi - \psi\nabla\psi^\star)$$

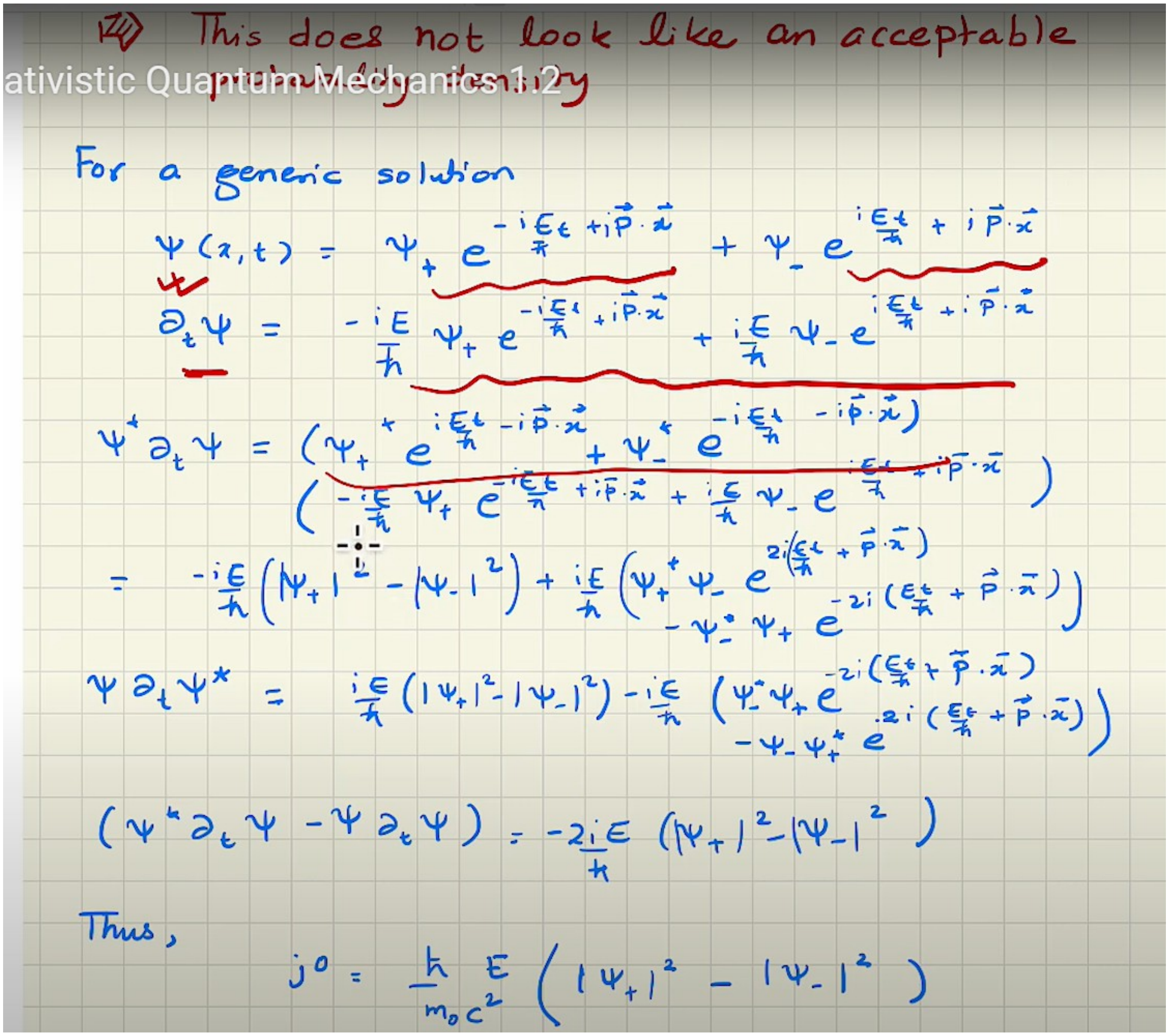
Now we want to see the analog in relativistic setting and want to see whether again it is possible to recover the usual probability density and if there is any need to modify the definitions of probability and probability densities and probability currents in the relativistic domain.
So again we start with the Klein–Gordon on equation which is this. You can see that the box operator over here is made up from the double derivatives of space and time all coming with real coefficients. And then there is a rest momentum $m^2c^2/\hbar^2$ acting on ψ. So all these quantities inside the bracket all the operators which are acting on ψ are real quantities. They are meaning real derivatives. Now, if I take

the star of this complex conjugate of the whole equation, nothing will happen to the box because it is made up of real quantities. They are meaning real derivatives. Now, if I take the star of this complex conjugate of the whole equation, nothing will happen to the box because it is made up of real quantities. Nothing will happen to the rest momentum as well because it is also made up of real quantities. Only thing that will change will be $\psi^\star$. ψ will go to $\psi^\star$ under complex conjugation. So, that means ψ and $\psi^\star$ both satisfy the Klein–Gordon equation. Remember, in the Klein–Gordon, in the Schrodinger picture as well, ψ and $\psi^\star$ were also the solution of the Schrodinger equation with a potential phase difference of I pi. There is a - sign which would come about due to flip of I of the $\hbar$. And since overall phase is not a matter of concern, then ψ *and* $\psi^\star$ is also a solution of the wave equation, of the Schrodinger wave equation. Relativistic setting also we are getting the similar kind of thing that both ψ *and* $\psi^\star$ are satisfying the wave equation. So, now exactly like we proceed for deriving the probability current equation, we just take multiplication of the first equation with $\psi^\star$ and multiply ψ with the second equation and then we subtract them out. So, I would have a $\psi^\star\Box\ \psi - \psi^\star m^2c^2/\hbar^2\psi$ and then the reversed role of that ψ and $\Box\psi^\star - \psi\ m^2c^2/\hbar^2\ \psi^\star$. So while the boxes will have $\psi^\star$ and ψ flipped across them, similarly m^2, c^2, $\hbar^2$ will have $\psi^\star$ and ψ in the first equation on the left and the right respectively, while in this lower equation ψ *and* $\psi^\star$ are coming on the opposite sides. But since this is a constant kind of thing, m^2, c^2, $\hbar^2$ is not a real time non–trivial operator. So therefore this can come out and ψ and $\psi^\star$ together can give rise to . The box cannot do that because box has derivatives which have to act on ψ or the box over here has to act on $\psi^\star$ *first*. These things can do that because they are nothing to do with derivative operation. They are just multiplication operators. So they can come to the left and both the terms will then become the mass term $c^2\hbar^2$. times $|\psi|^2$ and under subtraction they will exactly cancel out. So, I will be left with a $\psi^\star\Box\psi - \psi\Box\psi^\star$. So, let us open the boxes for both the terms and collect all the temporal derivatives and all the spatial derivatives. So, if I collect all the temporal derivatives on one side, I will get a $\psi^\star\Box\ \partial t^2\ \psi$. So, this $-1/c^2\partial t^2$ is coming from the $\Box\psi$*'s* temporal part. And then the Laplacian part, which is hiding in the box, that will also have a $\psi^\star$ and ψ. Similarly, the opposite way, this derivative over here, the $\Box\psi^\star$ will give me $-1/c^2\ \partial\psi^\star/\ \partial t^2$. And the ψ will be on the left. And there is a relative sign difference between the two terms. So here is a –. So effectively, you can see how these two brackets will emerge out. One for temporal derivative and one for spatial derivatives and that should be equal to 0 because of the identity. Now you can see the quantity appearing over here is nothing but the tempo. So, what first I can take?

I can take the temporal derivative on the other side of equality. So, both the terms should be equal to each other then.Then we can see that both the terms are actually the derivative of this quantity that temporal part here is just the time derivative of this quantity while the Laplacian part here is just the gradient of this particular scalar. So, you can see that when you do the temporal derivative, a term like derivative of $\psi^\star$ and derivative of ψ will also appear, but that will exactly cancel out from the term when the ∂t hits the second term here. So, you will have only surviving terms as this or even in gradient if you work out, you will find out only surviving terms are this. So, I can write down The equations which I have is the temporal derivative of this quantity should be equal to the gradient of that quantity. If that is the case, what I can do, I can just multiply $i\hbar/2m\ c^2$, $i\hbar/2m$ because $1/c^2$ is already there in the temporal part. So, I multiply this constant on the both sides of the equation. Then you will see, you can call this quantity as something. So, let us call it some J^0, so small J^0 and so this times 1 power of c, let us say 1 power of, so this constant $2m$c. So, with the definition, with the definition of this J^0 is i$\hbar$ $2mc^2$ times this whole derivative is J^0 and $i\hbar/2m$ with the whole thing as this as J then you can write down this this equation is nothing$^{\partial}{}_{\mu}$ $^{\partial tJ0}$ is equal to the laplacian of the or the gradient divergence of this. I should have been careful. ikept calling repeatedly at gradient. This is the divergence of this because this is a vector. Gradient of a scalar is here. Multiplying another scalar is giving me a vector. So, this should be a divergence, not a gradient.

So, just correct your notes. This should be a divergence. And ∂t this should be divergence of this quantity. So therefore, when I define this, the first quantity is the temporal derivative of J^0 and the second quantity is the divergence of the math- vec J. Together you can write down it is ∂t of J^0 – divergence of j is 0, which in the compact notation can be written like this, ∂mu j mu is equal to 0, okay. So, $\partial_\mu J^\mu$ as if you remember is just the structure that it is $\eta_{\mu\nu}$ and J^ν and it is a diagonal matrix η_{00} is –1. So, that is why a relative sign difference between these two terms will come and η_{ii}'s are just spatial derivatives of math- vec J. So, this is how you will get things. So, in this in order to obtain the continuity equation the same equation or like this was coming in the Schrodinger equation as well. Now the lesson we have learned that we have to modify the definition of J^0 to this much and J to this. J looks untouched. The math- vec J was the same definition which you used to get in the Schrodinger picture. However, the J0 has changed. In J0, this was just $|\psi|^2$. Therefore, this was called probability density previously. This time I have to call $|\psi|$ not $|\psi|^2$, but $\psi^\star \nabla \psi - \psi \nabla \psi^\star$ with a $i\hbar/mc^2$ as probability density. Now, there is apparent problem with this. First of all, it is a real quantity because this and these are complex conjugate of each other. Therefore, this I multiplication makes it real. This is fine.

However, the requirement of the probability density is not very acceptable solution. Why do I say so?



→ This does not look like an acceptable probability density.

For a generic solution

$$\psi(x,t)=\psi_{+.}\,e^{\frac{-iEt}{\hbar}+i\vec{P}\vec{x}}+\psi_{-.}\,e^{-\frac{iEt}{\hbar}+i\vec{P}\vec{x}}$$

$$\partial_t\psi=\frac{-iE}{\hbar}\psi_{+.}\,e^{\frac{-iEt}{\hbar}+i\vec{P}\vec{x}}+\frac{iE}{\hbar}\psi_{+.}\,e^{\frac{iEt}{\hbar}+i\vec{P}\vec{x}}$$

$$\psi^{\star}\partial_t\psi=\psi^{\star}_{+.}\,e^{\frac{iEt}{\hbar}-i\vec{P}\vec{x}}+\psi^{\star}_{-.}\,e^{\frac{-iEt}{\hbar}-i\vec{P}\vec{x}}\left(\frac{-iE}{\hbar}\psi_{+.}\,e^{\frac{-iEt}{\hbar}+i\vec{P}\vec{x}}+\frac{iE}{\hbar}\psi_{+.}\,e^{\frac{iEt}{\hbar}+i\vec{P}\vec{x}}\right)$$

$$= \frac{-iE}{\hbar}((|\psi_{+.}|)^2-(|\psi_{-.}|)^2)+\frac{iE}{\hbar}(\psi_{+.}^{\star}\psi_{-.}e^{2i(\frac{Et}{\hbar}+\vec{P}\dot{\vec{x}})}-\psi_{-.}^{\star}\psi_{+.}e^{-2i(\frac{Et}{\hbar}+\vec{P}\dot{\vec{x}})})$$

and

$$\psi\,\partial_t\psi^{\star}=\frac{iE}{\hbar}((|\psi_{+.}|)^2-(|\psi_{-.}|)^2)-\frac{iE}{\hbar}(\psi_{-.}^{\star}\psi_{+.}e^{-2i(\frac{Et}{\hbar}+\vec{P}\dot{\vec{x}})}-\psi_{+.}^{\star}\psi_{-.}e^{2i(\frac{Et}{\hbar}+\vec{P}\dot{\vec{x}})})$$

$$\therefore(\psi^{\star}\partial_t\psi-\psi\,\partial_t\psi^{\star})=\frac{-2iE}{\hbar}((|\psi_{+.}|)^2-(|\psi_{-.}|)^2)$$

Thus

$$J^0=\frac{\hbar}{m_0c^2}((|\psi_{+.}|)^2-(|\psi_{-.}|)^2)$$

Because if I just look for stationary solutions, iknow there are positive energy solutions or negative energy solutions. So, let us cook up a generic solution, which will be a superposition of a positive energy solution and negative energy solution. So this is a positive energy solution of Klein–Gordon. This is a negative energy solution of Klein–Gordon. So then I can do the superposition with coefficient ψ^+ and ψ^-. that is also a valid solution for the Klein–Gordon equation. Then I take the first derivative of ψ, which will give you this quantity. This is simple algebra you should follow up. Then I calculate the $\psi^{\star}\,\partial t\,\psi$. There is a bit of algebra you have to take the $\psi^{\star}$ which is the complex conjugate of the ψ and the derivative of ψ which we have just evaluated make the product then you will get various terms there will be $|\psi^+|^2$ of this mod square of that but then there will be cross terms which will be like this the cross terms will be like this that is just simple plane algebra you have to verify and then you have to take in order to get the probability density. You have to take the complex conjugate of that, subtract those things out, multiply with I times something. So that is what let us do. So I take this quantity which is obtainable over here and this quantity which is also obtainable over here. And when I do the subtraction, the crucial thing is that the cross term if you see are exactly the same, they will cancel out. But there is a relative sign difference between the $|\psi|^2$ terms. So this is is $-iE\hbar$ and this is $-iE\hbar$. So when I compute this quantity, the cross term will vanish, cancel each other. But the mod square terms will survive with a sign difference because there is a sign difference between these two. that means the probability density will survive with a sign difference. So therefore, there is a problem at hand because the definition of probability density is coming with A –B kind of structure and this is not always necessarily positive semI definite. Remember whenever ψ^+ is greater than ψ^-, it is positive. But if I am looking for a generic solution, there is no guarantee that ψ^+ has to always be greater than ψ^-. Even a ψ^- which is let us say 3/4 and ψ^+ which is 1/4 or some number which is making the normalization to 1. So, let us say this is root 3/2 and it is 1/2. In that case, ψ^- is greater than ψ^+.

It is a valid solution of the wave equation, but for that it turns out the probability density will become negative. So, this will become 1/4 and this will become 3/4 and probability density will become -one half of $\hbar$ e/m_0 c^2. So, therefore, whenever $|\psi^+|^2$ is greater than $|\psi^-|^2$, you have a problem at hand. You will get a negative probability density, which is not acceptable. For one particular case, if suppose I just look at the negative energy solution, which is this, its probability density is completely negative.

So, if we have to keep the continuity equation, J^0, which can be negative now, cannot be interpreted as probability density. Because it can be, probability density of any distribution should be positive semi–definite. So, there was a problem as far as initial Klein–Gordon equations were discussed. The probability densities replacement in the relativistic limit was not clear. $|\psi|^2$ which was being declared as probability density is not a protected quantity. Remember in the Schrodinger equation we had this structure that $\partial t\ |\psi|^2$ was equal to –of divergence of J and then if I integrate it over all space. integrated over all space then since this will take things on the boundary of this is Gauss divergence theorem divergence of a vector over all volume is equal to the vector dot the area element at the boundary of the volume that is up to in if I am talking about whole universe it is up to infinity I have to go that means I have to do the surface integral of infinity and wave function are falling faster to infinity j vanishes so the right hand side vanishes so that means $\partial t d^3|\psi|^2$ is zero that means total probability in all universe is a conserved quantity in time so this is full volume this is probability density and total integration is total probability of finding a particle in all universe that does not change in time that remains protected so that was a feature of schrodinger equation. if I want to do the similar kind of thing what is protected again j 0 will be protected so if you same thing you do $|\psi|^2$ will be written like $\psi^\star \partial t\ \psi - \psi \partial t \psi^\star$ but the algebra will turn out to be the same so this quantity integrated over all universe will be a protected quantity in the Klein–Gordon domain but this is not a positive semi–definite quantity. So I do not know what is the replacement of the probability density then and what this quantity is telling us which is J^0 . So one idea was either multiply J^0 with some charge and start calling things as charge density which can be positive or negative. Or you do not go as far as taking the wave functions directly as the solutions. They have to be replaced by certain operators in order for them to act on wave state and give you the probability densities and what not. And that approach where the solutions we fed in the solution into the probability density equation that is not allowed if in one school of thought that suggest that ψ is not a known solution should not be a known solution better they should be in operators. And that will convert, give me a operator at all x and times, ψ is a function of x and times. that will take us to quantum field theoretic domain, which we will do in coming weeks. But this is one proposal that there is no good replacement of probability density in Klein–Gordon level. I should not use the J0, which is obtained by continuity equation as a probability, but rather think carefully, Either it should be some charge density or maybe it is operator quantum field theory or maybe we can try to replace the Klein–Gordon equation with a better relativistic equation. So, the problem as we saw came about from the negative sign of e and negative sign of e was coming precisely because E^2 was something else and which was coming from the double derivative structure of ∂t that gave me E^2 kind of equation. If we have single derivative equation probably then we will be safe and we will be getting energy, energy density and other things that is the domain of Dirac equation which we will do in the next class.