Foundation of Quantum Theory: Relativistic Approach
Relativistic Quantum Mechanics 1.1

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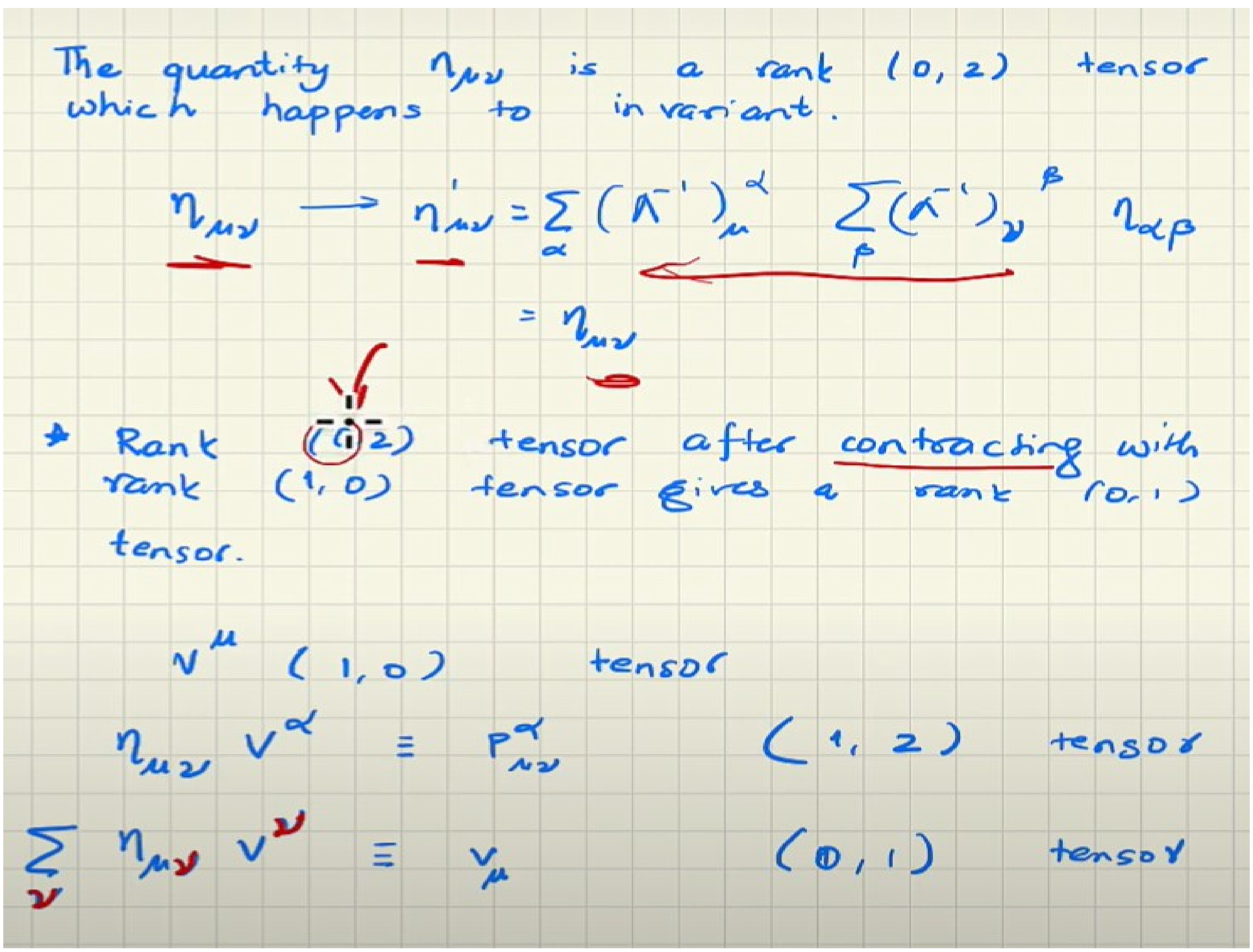
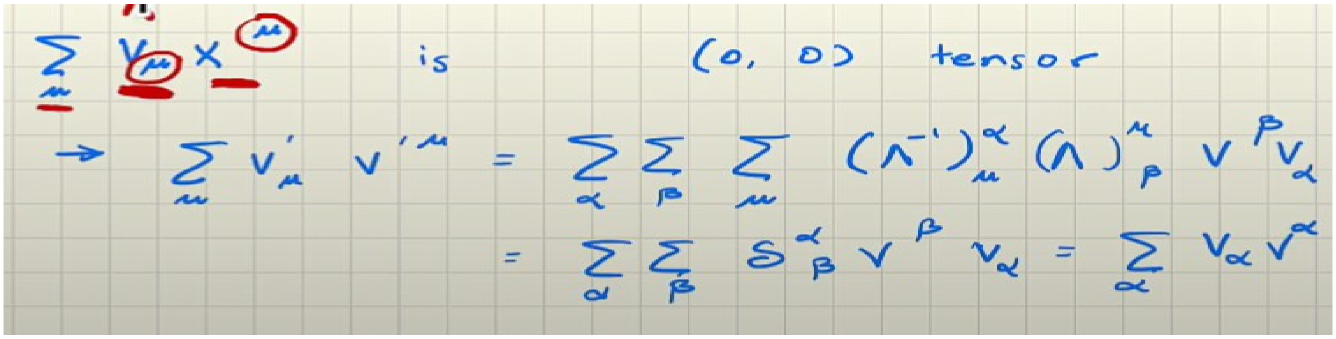
Klein Gordon Equation I

Lecture– 10

So in this discussion session what we will discuss today would be covariant laws in special relativity. The energy momentum and relation which we have seen the examples of before and the ways to derive the correct special relativistic kinetic energy where we saw that higher order momentum terms are allowed from special relativity.





The quantity $\eta_{\mu\nu}$ is a rank (0, 2) tensor which happens to be invariant.

$$\eta_{\mu\nu} \rightarrow \eta'_{\mu\nu} = \sum_{\alpha} (\Lambda^{-1})^{\alpha}_{\mu} \sum_{\beta} (\Lambda^{-1})^{\beta}_{\nu} \quad \eta_{\alpha\beta} = \eta_{\mu\nu}$$

★Rank (0,2) tensor after contracting with rank (1,0) tensor gives a rank (0,1) tensor.

V^{μ} (1,0) tensor after contracting with rank (1,0) tensor gives a rank (0,1) tensor.

$H_{\mu\nu} V^{\alpha} \equiv P^{\alpha}{}_{\mu\nu}$ *(1,2)* tensor.

$$\sum_{\nu} \eta_{\mu\nu} \quad V^{\nu} = V_{\mu} \text{ (0,1) tensor.}$$

$$\sum_{\mu} V_{\mu} X_{\mu} \quad \text{is (0,0) tensor}$$

$$\rightarrow \quad \sum_{\mu} V_{\mu} X_{\mu} V'^{\mu} = \sum_{\alpha} \sum_{\beta} \sum_{\mu} (\Lambda^{-1})^{\alpha}_{\beta} (\Lambda)^{\mu}_{\beta} V^{\beta} V_{\alpha} = \sum_{\alpha} \sum_{\beta} \delta^{\alpha}_{\beta} V^{\beta} V_{\alpha} = \sum_{\alpha} V_{\alpha} V^{\alpha}$$

And then we will use this structure, that correct definition of energy momentum, correct definitions of covariant laws to apply to quantum mechanics. So let us get going and resume our discussion from the previous discussion session which we had. Okay, so in the previous discussion session, we realized that there can be vectors, there can be co–vectors, that is rank 0 and tensors and there can be other rank tensor like *2, 0, 0, 2* and those kind of things. The properties of those things being that they transforms with various multiplications of Lorentz transformation matrix Λ. For instance, we know that the vectors transform with multiplication of one transformation matrix Λ.
The rank 1 0 vector transforms like that. Rank *(0,1)* tensor transforms like multiplication of a Λ^{-1} matrix, contraction with Λ^{-1} matrix. And therefore quantities which are contraction of rank 1 0 and rank 0 1, those quantities are invariant. That means they are (0, 0) tensor. They do not transform with any Λ in them.So those would be invariant expressions. They will remain true in all inertial frames because no Λ has to come up when we jump from one frame to another. We saw one example in the sense of a magnitude of a vector where suppose a vector is given to us that is a rank (1, 0) tensor. And we want to write down its co–vector version, which will be obtainable from contraction with $\eta_{\mu\nu}$ matrix. So by contraction, as we discussed, the second index of the rank (0, 2) tensor and upper index of rank 1, 0 tensor has to be the same. And they have to be summed over. μ has to take value 0, 1, 2, 3. Then I get a rank 0, 1 tensor. So, ultimately given a vector, I will convert it into its co

tor and then covector and the vector will be contracted together to obtain the rank 0, 0 tensor. And that rank 0, 0 tensor would be invariant across all frames. So, if I write down the magnitude in this way of one vector, it will be in prime frames. In unprime frame, the magnitude will remain same. No Λ matrices are needed to be multiplied to that. So this is reminiscent of what we have previously seen that the magnitude of a vector in spatial rotations remains the same and therefore the Newton's law under Galilean transformations were invariant. This time also the same physics or same geometrical argument are being used however only difference being the magnitude of the vector is not just the component summed square but the contraction with upper index and lower index which will introduce a – sign somewhere because remember $\eta_{\mu\nu}$ metric was – *111* in the diagonals and 0 elsewhere. So it is slightly different than the usual magnitude of a spatial vectors but ultimately magnitude in some sense and that remains invariant and that would be a protected quantity across all initial frames.

Now let us get back to the second part of the special relativity where we have decided that we would write down laws of physics in the way that they remain same across all inertial frame okay so there are two options of doing that either I write the laws which are invariant they do not change at all for example speed of light is invariant and a constant d tau is a invariant but not a constant So this is one option that I write down in log which are invariant. For example, speed of light is c is an invariant statement. It does not change from frame to frame. However, as we saw for Newton's law, when we demanded in Newton's law F was equal to ma, we do not demand under frame transformation F should not change or a should not change. We demanded that the relation should not change. f might change under rotation. Its x component can change, y component can change and z component can change. Similarly, accelerations x, y and z components can also change. Under rotations however in newton's law or galilean transformation we had demanded that this relation remains sacred that relation does not change individually they are allowed to change but the relation does not change so this is what we will go through and these kind of things where things change but the relations does not change are known as covariant laws. So, in the similar spirit, I would try to write down acceleration is equal to force.

Now, not in spatial direction, but space time dimensions.

Now, the force law (by choosing $\lambda = \tau$)

$$\frac{d^2 x^\mu}{d\tau^2} = \frac{F^\mu}{m} \quad \text{for all } \mu$$

$$F'^\mu = \sum_\nu \Lambda^\mu{}_\nu F^\nu$$

$$\frac{d^2 x'^\mu}{d\tau^2} = \frac{d}{d\tau}\left(\frac{dx'^\mu}{d\tau}\right) = \frac{d}{d\tau}\frac{d}{d\tau}\sum_\nu \Lambda^\mu{}_\nu x^\nu$$

$$= \sum_\nu \Lambda^\mu{}_\nu \frac{d^2 x^\nu}{d\tau^2}$$

$$\Rightarrow \frac{F'^\mu}{m} - \frac{d^2 x'^\mu}{d\tau^2} \Rightarrow \sum_\nu \Lambda^\mu{}_\nu \left(\frac{F^\nu}{m} - \frac{d^2 x^\nu}{d\tau^2}\right)$$

$$\therefore \frac{F'^\mu}{m} - \frac{d^2 x'^\mu}{d\tau^2} = 0 \Rightarrow \sum_\nu \Lambda^\mu{}_\nu \left(\frac{F^\nu}{m} - \frac{d^2 x^\nu}{d\tau^2}\right) = 0$$

And this should hold for any Λ

This can only be true if $\frac{F^\nu}{m} - \frac{d^2 x^\nu}{d\tau^2} = 0$ for all ν

Now the force law (by choosing λ = τ)

$$\frac{d^2 X^\mu}{d\tau^2} = F^\mu \quad \text{for all } \mu$$

$$F'^\mu = \sum_\nu \Lambda^\mu_\nu F^\nu$$

$$\frac{d^2 X'^\mu}{d\tau^2} = \frac{d}{d\tau}\left(\frac{d X'^\mu}{d\tau}\right) = \frac{d}{d\tau}\frac{d}{d\tau}\sum_\nu \Lambda^\mu_\nu x^\nu$$

$$= \sum_\nu \Lambda^\mu_\nu \frac{d^2 x^\nu}{d\tau^2}$$

$$\Rightarrow \qquad \frac{F'^\mu}{m} - \frac{d^2 x'^\mu}{d\tau^2} \to \sum_\nu \Lambda^\mu_\nu F^\nu \left(\frac{F'^\nu}{m} - \frac{d^2 x^\nu}{d\tau^2}\right) = 0$$

$$\therefore \quad \frac{F'^\mu}{m} - \frac{d^2 x'^\mu}{d\tau^2} = 0 \to \sum_\nu \Lambda^\mu_\nu F^\nu \left(\frac{F'^\nu}{m} - \frac{d^2 x^\nu}{d\tau^2}\right) = 0$$

And thiis should hold for any Λ

This can only be time if $\frac{F'^\nu}{m} - \frac{d^2 x^\nu}{d\tau^2} = 0$

So, I will define a 4 acceleration. Previously, d^2x of 3 space upon dt^2 was a correct vector acceleration because t for Galilean transformation was invariant, not anymore. So, we will define a 4 acceleration. 4 acceleration would be $d^2x_\mu/d\tau^2$, x_μ is 4 indices of *txyz* and $d\tau^2$ is an invariant quantity as we have seen. So, therefore this quantity transforms like a vector and on right hand side we should write down some 4 force that is not only spatial force we should know the temporal force in some sense as well. So this would be a covariant law. That would be a vector law in four space–time dimension. And therefore, it will remain protected if I go across different frame. That means the relation will remain protected.
We can see that since force is a four vector now in our formulation, it should transform with one Λ matrix multiplication. So in the primed frame, another inertial frame, it will be obtained from the previous f through one multiplication of Λ matrix. So you see new or new here the row and the column of these things are multiplied and therefore it is a matrix multiplication. Similarly the acceleration which is appearing on the left hand side in the primed coordinate it would be $d^2x'/d\tau'^2$ but since $d\tau'^2$ is same as $d\tau$ so I can just replace d $d\tau^2$ in the denominator and that quantity would be the double derivative of dx_μ, dx'_μ with respect to $d\tau$. And I know how x'_μ themselves or dx'_μ themselves transform. They transform like a vector. So this quantity x'^μ has been written like this and double derivative has to act on this which is an invariant derivative because d tau does not change from frame to frame. So therefore, as you can see that the acceleration itself also, since this parameters here $\Lambda^\mu{}_\nu$ matrix elements dependent upon the velocity upon time of the frame and inertial frame velocities are supposed to be constant. So they are not supposed to depend on $d\tau$. So d tau will go and hit just the x ν. So I will get the prime coordinate acceleration is obtainable from unprime frame acceleration through one matrix multiplication of Λ. So, therefore this and that force and acceleration four force and four acceleration together if I transform both of them into unprime frame they will both earn up a Λ matrix multiplication

and the law which we were looking for was that this relation was supposed to be 0 in the one inertial frame that is the primed inertial frame. If this has to be 0 then the equality between the transformation that this is 0 that means this combination should be 0 that will give me the relation over here that In unprimed coordinates, $F^{\nu}/m - d^2x^{\nu}/d\tau^2$ multiplied with a Λ matrix should be 0. And then we should solve this equation. Now, this relation has to be true. This relation has to be true not only for one single Lorentz transformation, but there are infinitely many possible Lorentz transformation. Each velocity along x direction is generating a different transformation matrix. So, this relation should be true for all inertial frames. That means all possible $\Lambda^{\mu.}$ The only way this equation is going to work out for all possible $\Lambda^{\mu}{}_{\nu}$ which are infinitely many is that when this coefficients of all $\Lambda^{\mu}{}_{\nu}$ individually are 0. That means $F^{\nu}/m - d^2x^{\nu}/d\tau^{2}$ should be 0 for all ν. And the relation which we started was that similar kind of relation was true in the prime frame for all μ, meaning all component of this vector was 0 in prime frame. And all components of this vector is still 0 in all unprime frame. So, therefore, the relation between the force and the acceleration has remained protected. So, that is one way of writing covariant laws where you write things in either tensor equations or vector equations. Because if I write in vector equations like this it has 1 μ if I transform ultimately what will happen that from prime frame to unprime frame if I go Only thing that will happen will be the same kind of equation would be there and that equation would be contracted with a Λ matrix. And the same logic we will apply that the relation has to be 0 for all possible Λ. So, therefore, this relation will become protected quantity across all inertial frame. Similarly had it been a two index relation something like some relation like $F^{\mu\nu}$ – let us say something like $dx_{\mu}/d\tau$, $dx_{\mu}/d\tau$ from this also you can see that if this is zero in one inertial frame If you go to another inertial frame, this quantity will earn up 2 Λ matrix multiplication. And this quantity will also earn up 2 Λ matrix multiplication. And ultimately, you will get a structure like 2 Λs outside f primed, dx'_{μ}/d tau and $dx'^{\nu}/d\tau$. And that has to be 0. Again, arguing that the quantity in the bracket has to be 0 in the prime frame as well. So similar kind of logic with which we can write down physical laws which maintain the same relations can be summarized as they should be either invariant equation or they should be covariant equations. By covariant I mean tensor type equation or vector type equation. Tensor meaning matrix multiplication by more than 1 Λ and vector means matrix multiplication with just 1 Λ. Together we can say that it should be just a tensor equation where rank 1 0 tensor will be vector and all other possibilities will be different rank tensor. So, ultimately the covariance means it should be a linear equation with some Λ multiplication. And if that happens, we know that in one frame, we can write down the equation that the four vector or the tensor in the four dimensional space is zero. It is all components are zero.

The corresponding statement will be, If I have a vector or a tensor whose all components are zero in one frame, then it is guaranteed that under low range transformation, all those components will remain zero in all inertial frames. So, therefore, the relation which we have written will be the same in all inertial frames. So, that is how we should write down the laws of physics. For example, correct generalization of Newton's law would be this. Let us try to see what more we can learn from invariant kind or covariant kind of equations.

Let us see what more invariant kind of laws we can write. Right now we have seen an example of a covariant kind.

Invariant law

$$\sum_{\mu} p^{\mu} p_{\mu} = \sum_{\nu} m^2 \left(\frac{dx^{\mu}}{d\tau} \eta_{\mu\nu} \frac{dx^{\nu}}{d\tau} \right)$$

$$= -m_0^2 c^2$$

Since $p^{\mu} = \left(\frac{E}{c}, \vec{p} \right)$

$$p^{\mu} p_{\mu} = \sum_{\nu} \eta_{\mu\nu} p^{\mu} p^{\nu} = -\frac{E^2}{c^2} + \vec{p} \cdot \vec{p}$$

Therefore we have

$$-\frac{E^2}{c^2} + \vec{p}^2 = -m_0^2 c^2$$

$$\therefore \quad E^2 = p^2 c^2 + m_0^2 c^4$$

Energy for no momentum is the rest-mass energy

$$E_{p=0} = m_0 c^2$$

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$$\therefore \quad KE = E - m_0 c^2$$

$$= \sqrt{p^2 c^2 + m_0^2 c^4} - m_0 c^2$$

$$= m_0 c^2 \sqrt{1 + \frac{p^2}{m_0^2 c^2}} - m_0 c^2$$

$$= m_0 c^2 \left(1 + \frac{p^2}{2 m_0^2 c^2} - \frac{p^4}{8 m_0^4 c^4} + \dots \right) - m_0 c^2$$

Invariant law

$$\sum_{\mu} p^{\mu} p_{\mu} = \sum_{\nu} m^2 \left(\frac{dx^{\mu}}{d\tau} \eta_{\mu\nu} \frac{dx^{\nu}}{d\tau}\right)$$

Since $p^{\mu} = \left(\frac{E}{c}, \vec{p}\right)$

$$\sum_{\mu} p^{\mu} p_{\mu} = \sum_{\nu} \eta_{\mu\nu} p^{\mu} p^{\nu} = \frac{-E^2}{c^2} + \vec{p} \cdot \vec{p}$$

Therefore we have

$$-\frac{E^2}{c^2} + \vec{p}^2 - m_0^2 c^2$$

$$\therefore \quad E^2 = p^2 c^2 + m_0^2 c^4$$

Energy for no momentum is the rest mass energy is

$$E_{p=0} = m_0 c^2$$

$$KE = E - m_0 c^2$$

$$= \quad \sqrt{p^2 c^2 + m_0 c^4} - m_0 c^2$$

$$= \quad m_0 c^2 \sqrt{1 + \frac{p^2}{m_0^2 c^2}} - m_0 c^2$$

$$= \quad m_0 c^2 \left(1 + \frac{p^2}{2\mathrm{m}_0^2 c^2} - \frac{p^4}{m_0^4 c^4} + \ldots\right) - m_0 c^2$$

Let us see if there is an invariant kind of relation or not.
For example we can have a four momentum previously we had a three momentum in the spatial space spatial dimension where momentum was mass times the velocity velocity was vector dx upon dt now we as we know this is not a vector in space time four dimensional and this is also not a vector in four dimension they were vectors in three dimensional space so correct generalization would be I should write down the momentum should be four vector p_{μ} and this should be obtainable from four velocity $dx_{\mu}/d\tau$ with some invariant quantity defined as a mass or the rest mass. As we have seen previously given a vector I can find out its co–version, co–vector it would be contraction with a $\eta_{\mu\nu}$ so I would obtain p_{μ} by just taking $\eta_{\mu\nu}$ summing over news as we have seen previously so then I have a vector and I have its corresponding covector and as we have seen if I take a vector and its corresponding covector then if I contract them together I generate an invariant quantity so that means I contract them P_{μ} and there is a summation over μ implied. I have not explicitly written because most of the textbook use something called Einstein summation convention. Ideally one should have written down $\Lambda^{\mu\mu}$, $p_{\mu} p_{\mu}$ and summation over μ. But most of the textbook will say if you see same index in the upper and the lower thing that means summation is already there. Okay.
So p_{μ}^{μ} up and low. It means that it is p^{μ}, p_{μ} with a summation over μ is therefore equal to Here also there is a summation over μ and I write down the two definitions of upper p_{μ}. Upper p_{μ} is this. So I write down m times $dx_{\mu}/d\tau$ and lower p_{μ} which is $\eta_{\mu\nu}$ times m $dx_{\mu}/d\tau$. So another m will make it $m^2 \eta_{\mu\nu}\, dx^{\nu}/d\tau$ and a summation over μ and summation over ν both are there. All right.
So now you see that if I do this observation that I have a double summation μ *and* ν $dx_{\mu}/d\tau$ *is* there $\eta_{\mu\nu}$ is there $dx^{\nu}/d\tau$ is there so remember we had a we have a structure $\mu\nu$ $\eta_{\mu\nu} dx_{\mu}/d\tau$ and $dx^{\nu}/d\tau$ this quantity is appearing m^2 is a constant so it can go out of the summation. And recall this $\eta_{\mu\nu}$ dx_{μ} dx^{ν} summation over $\mu\nu$ was nothing but $d\tau^2$ itself. Ultimately I get a dt^2 divided by $d\tau^2$ in the denominator. So that gives me

a –. So this quantity $\eta_{\mu\nu}\, dx_\mu\, dx^\nu$ *was* rather $-d\tau^2$. So $d\tau^2$ divided by $d\tau^2$ will give you – 1. And okay I have written in c is equal to one unit otherwise it would be $c^2 d\tau^2$ here also $c^2 d\tau^2$ so I should get a $-c^2$ so therefore I will get an answer $-m^2c^2$.
Okay sometimes people write with a dot here also dot here also a dot such that it is rest mass it is clear that it is a rest mass so I do earn up an invariant quantity which is rest mass square times speed of light square with a negative sign So, this relation as you see is an invariant relation. It does not come up with any Λ in it. It is a scalar equation. So, therefore, I have an invariant expression that $p_\mu p_\mu$ should be $-m^2c^2$, okay. This relation is an invariant equation and therefore, we can use it in any inertial frame. So, if I take four vector p_μ, whose zeroth component is energy upon c or the frequency if you like. Then and the spatial components are just spatial momentum vector. Then you can write down that $p^\mu p_\mu$ contraction that means summation over μ is implied again. You will get if you open it up $\eta_{\mu\nu}$ so this this quantity this summation here means $\eta_{00} = p^0p^0$, $\eta_{11} = p^1p^1$, $\eta_{22} = p^2p^2$ and $\eta_{33} = p^3p^3$ And no other components are generated because $\eta_{\mu\nu}$ is a diagonal thing. Only diagonal elements survive. η_{00} was -1. p^0 was E/c. So you will get a $-E^2/c^2$. And from these quantities, these are just 1, 1, 1. So p_x^2, p_y^2, p_z^2. That gives you just the magnitude of the spatial vector. So therefore, I will generate a relation which is $p_\mu p_\mu$ which is $-m_0^2c^2$ as we have seen in the boxed equation is also equal to $-E^2/c^2$ + spatial momentum magnitude square. So therefore the invariant relation which is obtained gives me a relation between energy and momentum. That means E^2/c^2 with a – sign + momentum magnitude square should better sum up to an invariant quantity. Right hand side is an invariant therefore left side combination should be invariant. So therefore I have a this as an invariant relation. And that invariant relation just tells me that energy of a particle should be p^2c^2. E^2 of a particle is $p^2c^2 + m_0c^4$. So this is a total energy of a particle moving in free space with momentum p. Previously, in non–relativistic systems, we knew that total energy is just kinetic energy + potential energy, if there is any potential energy. If you remove the potential energy, it is just kinetic energy, which we used to know in non–relativistic physics was $p^2/2m$. Now in absence of any potential that relation is modified to this relation that E^2 is $p^2c^2 + m_0c^4$. this is a new relation where even if I have a particle which is not moving with any speed that means spatial momentum is zero still its energy will not be zero in non–relativistic physics In absence of any potential if I have a particle which is not moving its kinetic energy will be zero and since there is no potential its total energy will also be zero not in a special relativity in special relativity you would have In special relativity, you would have energy of a particle with zero momentum. p is equal to zero is equal to m_0c^2.
That is the rest mass energy. This quantity is coming from just the rest mass, not due to any movement. So therefore, special relativity accounts for a rest mass energy of the particle. And therefore, if I have to get down the energy contribution in this equation due to its motion, That is the kinetic part. I should better remove the rest mass energy from the total energy. So even if particle were not moving, they contain some energy. So therefore, for moving particles, I should get the correct estimate of their kinetic energy from removing the rest mass energy. So total energy – rest mass energy should be the kinetic energy. And the total energy I know is $\sqrt{p^2c^2 + m_0^2c^4}$. So, that quantity is total energy – rest mass energy is this. Okay, so thus we get a nice relation for kinetic energy. From this, if I assume that its rest mass $m_0^2c^2$ energy is much much larger than p^2c^2. If its rest mass energy is very large, it is moving with small velocity. That would be the non–relativistic limit where you are not moving very fast. In that case, you can see that I can pull out $m_0^2c^4$, outside it will come with m_0c^2 factor and inside I will get $(1 + p^2c^2\,)/m_0^2c^4$ which one of the c^2 cancels with the c^4 and I will have a^2 root of $1 + p^2/m_0^2c^2$ and from this relation we know so actually this should be square of this should be greater much much greater than p^2c^2 so this quantity therefore is a very small quantity its momentum is much smaller than its rest mass energy divided by c^2 If this quantity is very small, I am getting a structure like $1 + x$, where x is very small. And then we can expand it for a small axis. So, if I tailor expand this square root function, I will get 1 + first order term will be x, second order term will be x^2 by factorial 2 and so on. So, I would

have a m_0c^2 outside. And I have a 1 + first x which is $m_0^2/(x/2)$ rather because half will come from the square root factor. So, that would be $(1 + x/2 - x^2)/4$ or something like that sorry $x^2/8$ you will get from this because $1/2$ factor times $1/2 - 1$ and then a 2 factorial. This kind of expansion you will get and a – $m_0^2c^2$ is coming because we have removed the rest mass energy okay so from there you can see whatever we previously wrote about the kinetic energy comes through you open this bracket up m_0c^2 from here from this bracket will cancel with this and first term will start with a $p^2/2m_0^2c^2$ and outside there is a m_0^2/m_0c^2 so they will cancel each other and I will get a first term which is $p^2/2m_0$ this times that. Then the second order term will be p^4 with a – sign rather p^4 divided by $8m_0^3c^2$ and then there will be higher order terms. So, this is the term remember which we had introduced in the perturbation theory when we were discussing the correct structure of kinetic term in the time independent perturbation theory approach. So you can see this is not the only term there are higher order terms which we have not written. So as you increase your velocity those higher velocity terms will start appearing and in the full non relativistic in the full relativistic limit all the terms in this series expansion should be not ignorably small. So they should be accounted for. So whatever we are doing in Schrodinger equation or Schrodinger mechanics is actually the case where the rest mass wins over the kinetic term handsomely. So that if I totally ignore even this term, I am at the usual unperturbed Schrodinger equation kind of structure. The ideas which we have learnt that there are invariant laws, there are covariant laws.

Now, the force law (by choosing $\lambda = \tau$)

$$\frac{d^2x^\mu}{d\tau^2} = \frac{F^\mu}{m} \quad \text{for all } \mu$$

$$F'^\mu = \sum_\nu \Lambda^\mu{}_\nu F^\nu$$

$$\frac{d^2x'^\mu}{d\tau^2} = \frac{d}{d\tau}\left(\frac{dx'^\mu}{d\tau}\right) = \frac{d}{d\tau}\frac{d}{d\tau}\sum_\nu \Lambda^\mu{}_\nu x^\nu$$

$$= \sum_\nu \Lambda^\mu{}_\nu \left(\frac{d^2x^\nu}{d\tau^2}\right)$$

$$\Rightarrow \frac{F'^\mu}{m} - \frac{d^2x'^\mu}{d\tau^2} = \sum_\nu \Lambda^\mu{}_\nu \left(\frac{F^\nu}{m} - \frac{d^2x^\nu}{d\tau^2}\right)$$

$$\therefore \frac{F'^\mu}{m} - \frac{d^2x'^\mu}{d\tau^2} = 0 \Rightarrow \sum_\nu \Lambda^\mu{}_\nu \left(\frac{F^\nu}{m} - \frac{d^2x^\nu}{d\tau^2}\right) = 0$$

And this should hold for any Λ

This can only be true if $\frac{F^\nu}{m} - \frac{d^2x^\nu}{d\tau^2} = 0$ for all ν

$\therefore \frac{d^2x^\mu}{d\tau^2} - \frac{F^\mu}{m} = 0$ is a covariant law

* Physical laws / principles should be either invariant or covariant.

Now for the force law (by choosing $\lambda = \tau$)

$$\frac{d^2 X^\mu}{d\tau^2} = \frac{F^\mu}{m} \quad \text{for all } \mu$$

$$F'^\mu = \sum_\nu \Lambda^\mu_\nu F^\nu$$

$$\frac{d^2 X'^\mu}{d\tau^2} = \frac{d}{d\tau}\left(\frac{dX'\mu}{d\tau}\right) = \frac{d}{d\tau}\frac{d}{d\tau}\sum_\nu \sum_\nu \Lambda^\mu_\nu x^\nu$$

$$= \sum_\nu \Lambda^\mu_\nu \frac{d^2 X^\mu}{d\tau^2}$$

$$\Rightarrow \frac{F'^\mu}{m} - \frac{d^2 X'^\mu}{d\tau^2} \quad \Rightarrow \quad \sum_\nu \Lambda^\mu_\nu \left(\frac{F'^\nu}{m} - \frac{d^2 X'^\nu}{d\tau^2}\right) = 0$$

And this should hold for any Λ

This can only be true if $\frac{F^\nu}{m} - \frac{d^2 X^\nu}{d\tau^2} = 0$

$$\therefore \quad \frac{d^2 X^\mu}{d\tau^2} - \frac{F^\mu}{m} = 0 \quad \text{is a covariant law.}$$

⋆ Physical laws/ principles should be invariant under or covariant.

Covariant laws are changing component wise, but the relation remains the same. Invariant laws are the things which remain the same across all frames. And we saw some example where $F^{\mu\nu}$, the force law can be written as four vector law. That would be a covariant law and the momentum relations or e^2 is equal to $p^2 c^2 + m_0^2 c^4$ is a invariant equation that remains same across all frame. However, unfortunately the Schrodinger equation which we have dealt in the standard quantum mechanics is neither of them, is neither invariant nor covariant.

Schrödinger equation is none !!

because

$$\left[i\hbar \frac{\partial}{\partial t} + \frac{\hbar^2}{2m}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) \right] \psi = 0$$

is not a vector or scalar equation !!

Thus this will not remain same across different inertial frames.

⇒ So will the concepts built upon it.

⇒ $|\psi|^2$ being the probability density

$$i\hbar \frac{\partial |\psi|^2}{\partial t} = -\frac{\hbar^2}{2m} \nabla \cdot \left(\psi^* \nabla \psi - \psi \nabla \psi^* \right)$$

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \vec{J} = 0$$

$\rho = |\psi|^2$ probability density

$$\vec{J} = -\frac{i\hbar}{2m} \left(\psi^* \nabla \psi - \psi \nabla \psi^* \right)$$

Schroodinger equation is none!!

because $\left[i\hbar\frac{\partial}{\partial t}+\frac{\hbar^2}{2\text{m}}(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial x^2})\psi \right]=0$
is not a vector or scalar equation !!
Thus this will not remain same across different inertial frames.

⇒ So will be the concepts built upon it.

⇒ $|\psi|^2$ being the probability density.

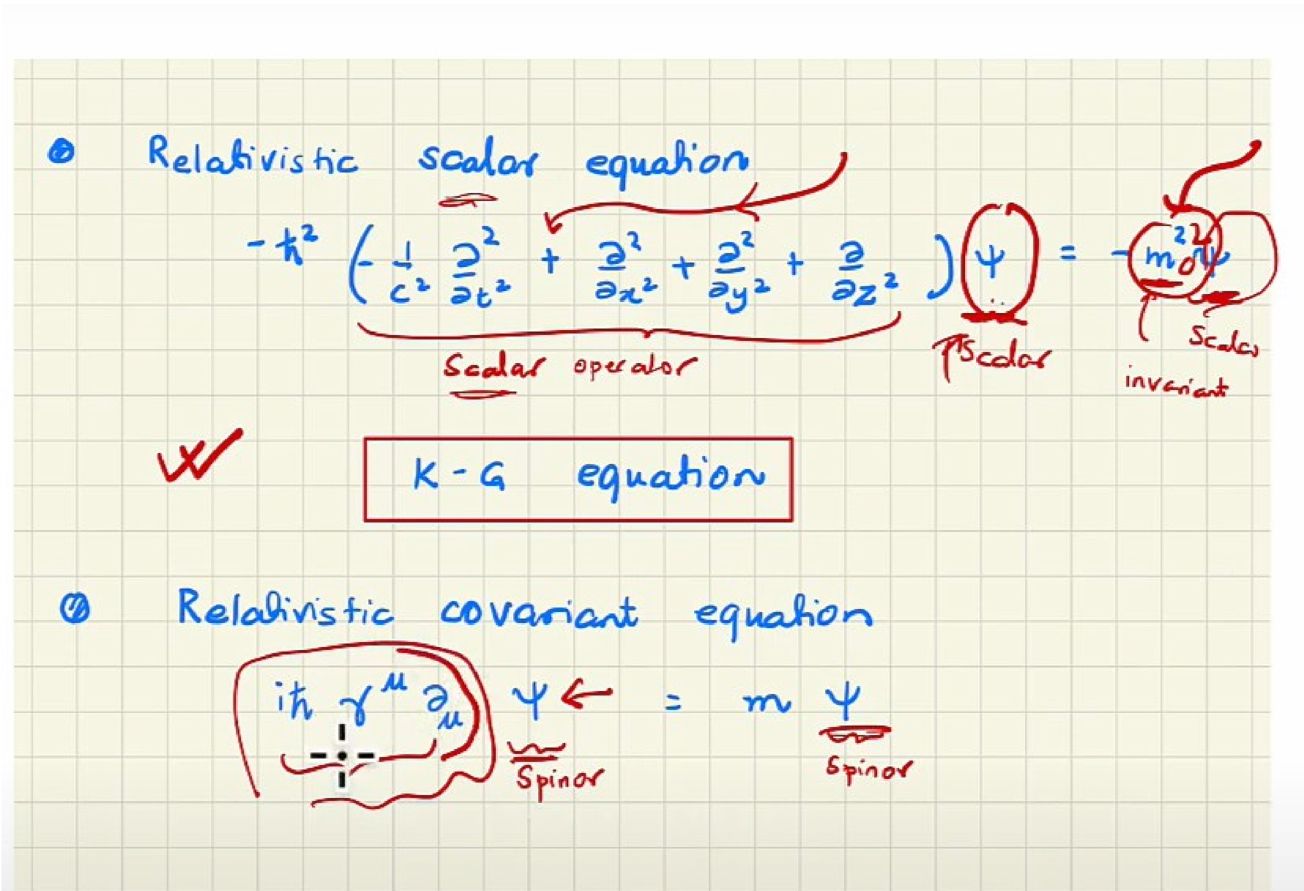
$$i\hbar\frac{\partial|\psi|^2}{\partial t}=\frac{-\hbar^2}{2\text{m}}\nabla\cdot(\psi^{\star}\nabla\psi-\psi\nabla\psi^{\star})$$

$$\frac{\partial\rho}{\partial t}+\nabla\cdot\vec{J}=0$$
$\rho=|\psi|^2$ is probability density
$$\vec{J}=\frac{-i\hbar}{2\text{m}}(\psi^{\star}\nabla\psi-\psi\nabla\psi^{\star})$$

As you can see Schrodinger equation can be written in this form $i\hbar\partial/\partial t$ acting on ψ should be equal to $-\hbar^2/2m$ the Laplacian acting on ψ in absence of any potential let us say for the simplistic setting. Now you can see if I do Lorentz transformation this quantity is not an invariant operator because if I do remember this these are co–vectors. $\partial/\partial t$ the derivative with respect to coordinates are rank *01* tensor and the double derivatives are rank 02 tensor. So, right now the way we are writing Schrodinger equation is giving me on the left hand side $i\hbar\frac{\partial\psi}{\partial t}$ is rank *01* tensor should be equal to $-\hbar^2/2m$ Laplacian of ψ that is rank zero two tensor. This equation cannot be true because rank zero one tensor cannot be equal to rank zero two tensor. This transform with one Λ^{-1} this transform with two Λ^{-1}. So this equation will not remain the same across all inertial frame. So therefore Schrodinger equation is not relativistically acceptable equation it is neither invariant equation nor covariant equation so we have to drop this equation in favor of a better equation that we will see what kind of better equation should be used you can see the problem is coming from the spatial derivatives are appearing twice and the temporal derivative is appearing only once and in special relativity both time and space are just coordinate differentials coordinate themselves So therefore, one coordinate should not be given a prestigious position compared to all others. And that is what is happening in Schrodinger equation.
Spatial derivatives are being appearing twice while temporal derivatives only appearing once. So they are not at equal footing. So they will not transform in the same way.
And therefore, they will not be a covariant or invariant equation. So we have to find a replacement of Schrodinger equation which is either a vector equation or a scalar equation or some tensor equation. We will see in this course examples of this thing and how to do those. However, as a consequence, when we dump the Schrodinger equation in favor of a better covariant or invariant equation, there will be certain problems or certain concepts which we had developed with Schrodinger mechanism that has to be rechecked and re–evaluated. Okay so we have used various concepts of schrodinger equation to learn about predictions of quantum mechanics some concepts were built upon using schrodinger equation for example one of the crucial concept which we had learned from schrodinger equation was $|\psi|^2$ was declared to be a probability density and if this was a probability density Then we knew that

from the Schrodinger equation the following equation can be written. That $i\hbar\partial|\psi|^2/\partial t$ is equal to $-\hbar^2/2m$ a gradient of a particular quantity. This you might have done in ordinary quantum mechanics course as well. This can be written as a continuity equation form. $\frac{\partial\rho}{\partial t} = \nabla J$, is equal to 0. So, this quantity is rather a divergence. So, $\nabla\cdot\vec{J}=0$. And we concluded out of it that this is talking about some charge flow and a flux flow across a volume or surface containing a volume where the quantity being flowing out and in and is something declared as a probability density, $|\psi|^2$. And $\vec{J}$ was built up was something like this. And this was given in definition of probability current. So the statement was from this continuity equation, if a particle is the probability of a particle in a volume, if it is changing, that particle becomes more likely or less likely to be in this volume content, that change is accounted for the flow of the current across its surface. This is true for any current equation and this time only the current was given an interpretation of a probability. Now that we know that this equation is not true anymore for special relativistic setting or is not true for all inertial frames as to say. The interpretation or even the establishment of such a equation is under question. So, we should worry about what would replace this equation and whether we will be able to adopt this kind of interpretation that the probability is being lost or change inside a volume and that is accounted for by the current crossing the surface containing that volume. Those concepts have to be renegotiated once we change this equation. Fine so let us go ahead and try to see what kind of equations are candidate equations which can replace the schrodinger equation so we want certain operators acting on psI is equal to zero and we want that operator to be either invariant operator or covariant operator so do we know such differential operators which change covariantly or invariantly under frame transformation we do because we have seen already that this operator which we have seen previously – $1/c^2\partial^2/\partial t^2$ + Laplacian square that is an invariant operator equation. So, this is an invariant operator called a scalar operator and if that hits psI And let us say if I put some quantity which is appearing on the right hand side as an invariant quantity. So, it should it could be given some invariant quantity acting on the side. I am just writing as m actually should be
m^2c^2 for dimensionally consistent structure, but some invariant quantities. Remember we know certain invariant quantity m,be
m^2 which was obtainable from $p_\mu p_\mu$. We also know of speed of light. So all these things can be put together and dimensionally consistent invariant should be put in. So this looks like an invariant operator equation. We have left hand side, I have an invariant operator acting on a scalar. That should be equal to some invariant quantity times the wave function. Actually, as we will see in the coming classes, this is a particular realistic scenario and it indeed governs a large set of particles moving relativistically and this is known as something called a Klein–Gordon equation. And therefore, this is at the 0th order, this is a scalar equation. Actually, this can be generalized to higher order spin 1, spin 2, integer spin objects as well. And correspondingly, it will become vector equation, tensor equations as well. But the simplest setting which we are writing is a scalar equation, where here this quantity is a scalar. So Klein–Gordon equation Klein–Gordon operator rather is an invariant operator. The quantity it acts upon remains the same because it is an invariant quantity acting on some wave function. Then the same wave function should come up with an invariant object multiplication. So that is a Klein–Gordon equation that we will see in the coming classes that these are useful operators and we will learn to deal with these operators more systematically. Secondly, we can also have a structure like some invariant operator acts or let us say covariant equation where things are not scalars. Previously you saw we are acting the invariant operator on scalar quantity that ψ also does not change. This time probably we can do something better and we can demand that maybe an invariant operator acts on certain quantity which is not necessarily scalar, but then I generate the same quantity back up to an invariant quantity. So, just like here a double derivative structure which we were demanding, I know the Klein–Gordon operator is an invariant operator and gives me something invariant quantity times the wave

function back that invariant quantity is dimensionally be m^2c^2.



Eqn

- Relativistic scalar equation

$$-\hbar^2\underbrace{\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2}+\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial x^2}\right)}_{scalar\,operator}\underbrace{\psi}_{scalar} = \underbrace{-m_0^2}_{scales\,invariant}\psi$$

K.G. Equation

- Relativistic covariant equations

$$i\hbar\gamma^\mu\partial_\mu\underbrace{\Psi}_{spinor} = \underbrace{\Psi}_{spinor}$$

I try to obtain similar kind of thing with first derivative is there any first derivative operator which is also invariant and we the answer as we will learn in the course is yes potentially I can have a first derivative operator which is invariant so these are given the name Dirac kind of equations where something called γ_μ. γ_μ is some matrices as we will see this is not a simple equation as that a vector contracted with a $\underline{\underline{\partial_\mu}}$. These quantities are rather matrices. These matrices contracting with the rank 0 one denser acting on again some matrices. So this will be a matrix equation acting on certain matrix giving you invariant quantity times a matrix. So this as you can see it looks like an eigenvalue equation now.
A matrix acting on some wave function has to give me the wave function back and therefore since the operator is matrix kind of operator this side the wave function would be should have been a matrix or a column vector itself previously it was a scalar over here okay so therefore if I demand existence of some invariant structure the relativistic approach allows for Equations where these wave functions are not just a wave function in the position space, but they have a matrix structure. And therefore it has a more structure than the ordinary quantum mechanics and this guy give rise to species. New species with some new quantity. That means scalar over here is a again can be written a one cross one dimensional matrix.
Here we have an M cross one dimensional matrix. The additional degrees of freedom are generated a matrix like more entries than one with one entry we have this equation wave function which we know already if you have more than one entries we have to interpret what those more entries mean and these give rise to spin structure so how many entries will be there would be known from what spin you are coming up with and therefore those spin quantities give you something called a spinar equation
Okay, so we will see that new concepts like spins which I am writing as species structure will be coming about and as we have anticipated once a new equations are operating we will end up getting new structure of this continuity equation where something else would be appearing in row because this equation here where p was $|\psi|^2$ and $\vec{J}$ was this quantity was obtained from the schrodinger equation now we have done away the schrodinger equation with so we will have a new definition of probability density and we will have a new definition of a probability current as well and then again we have to see the structure which we previously know the schrodinger equation approach or non–relativistic approach to exist that if there is a region. The probability of finding a particle in that region is obtainable from $|\psi|^2$ integrated over this volume. And if the probability in this volume decays, continuity equation tells me that the probability decay is equal to the flux crossing through the surface. That is the information of the continuity equation. Now we have to see what kind of new continuity equation comes about and what kind of new structure do we earn up. if we take up one of these equations to be our guiding equations.
Okay so these will be the structure which we will take from special relativity into the quantum mechanics domain and the implications of that we will see in the next class onwards all right fine so I stop over here and in the next class we will start discussing these klein gordon equations and thereafter the data kind of equations okay all right so I stop here.