FOUNDATIONS OF QUANTUM THEORY: NON-RELATIVISTIC APPROACH

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Week-08

Lecture-21

Measurements: State Discrimination

The problem we are addressing in this lecture is the following. If we are given two states, psi and phi, let us say they are qubit states, so they belong to the Hilbert space H2, and they are not orthogonal, means psi phi inner product is not zero. Let us call it to be alpha, which is a complex number. Now, can we design a setup in which such that we can distinguish between the state psi and phi. So, the problem can be stated like this.

We have an ensemble of quantum systems and each of that quantum system is either in the state psi or in the state phi. And when the quantum system is given to us, we need to tell whether the state of the system was psi or phi. If they were orthogonal states, then we could have chosen a measurement basis as psi and psi orthogonal and then we could have easily told whether the input state was psi or it is orthogonal. But here it is not the case. So, the state of the ensemble which has a 50% probability of getting state psi and 50% phi, the initial state of that can be written as a mixed state half times psi outer product psi and phi phi.

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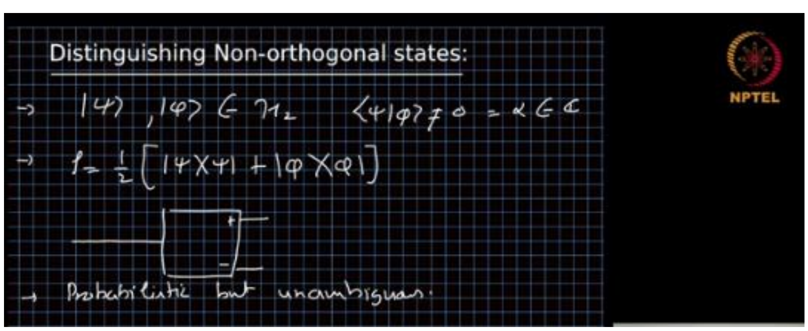

This is the initial state of the ensemble. Now we need to find an experimental setup such that we perform measurement and we get a click in the plus side or minus side and by looking at those clicks we can tell whether it's a psi state psi or phi. There are some restrictions and some conditions of this whole measurement setting that the result or the outcome or the distinguishability whether the input state was psi or phi can be probabilistic. But it should be unambiguous. So, it means for a given input quantum system, we can say the state is either psi or phi or the results are inconclusive. It should not be that we call the state to be psi, but it was not psi.

So, that is ambiguity. So, the result should be unambiguous, but it can be probabilistic. So, the first attempt can be the following. Let us design an experimental setting such that the basis in which we are performing measurement is the psi and psi orthogonal. So, if the input state is psi, then it will click in the psi in the plus click and if the input state is phi then it can click in plus or minus.

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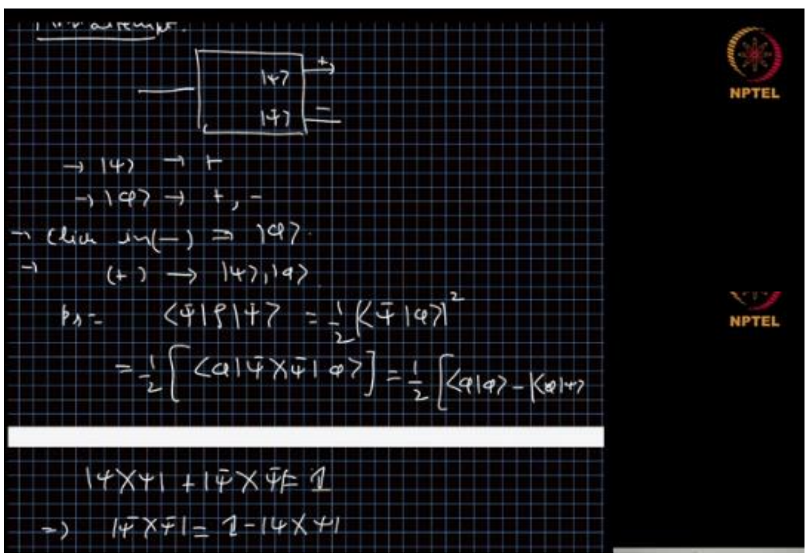

So, from here we can see that whenever we get a click in minus, this is the minus output for whenever we get a click in minus, then we know for sure it was state of phi but when we get click in plus Then it can be psi or phi. So, from this measurement setting, we can be sure when the click is in the minus, so, we know it is in state phi, but when the click is in plus, then we do not know whether it was in the state phi or psi. And the success rate will be the psi orthogonal rho psi orthogonal. Whenever there is a click in psi bar, psi orthogonal, then that is a successful case. This will be psi orthogonal phi mode square.

And this we can write as half, phi psi orthogonal, psi orthogonal phi. We know that psi outer product psi plus psi orthogonal psi orthogonal is identity. It means we can write psi bar psi bar as identity minus psi and this can be written as half times phi phi minus psi phi mod square. So, we get ps to be half times one minus phi psi mod square which was alpha mod square. If we assume psi to be cos of theta over 2 sine of theta over 2 and phi to be cos of theta over 2 minus sine of theta over 2 this is just single parameterization of the two states psi and phi they are you can see they are not orthogonal in fact psi phi is cos of theta, so, alpha is cos of theta and with this choice we get sine theta over two as our success probability. We will get the same result if we choose a measurement basis, which is phi and phi orthogonal.

So, because there is no difference between psi and phi and the success probability we get is sine squared theta over two in such kind of situations. Now, till now we have employed only the projective measurement in the basis, which will give us unambiguous result, but only with sine squared theta over two success rate. Now, can we do better than this when we go to POVM? This is what we will explore next. Let us say we have an ancilla and initially it is in the state 0 and the total state of a total Hilbert space is system tensor ancilla.

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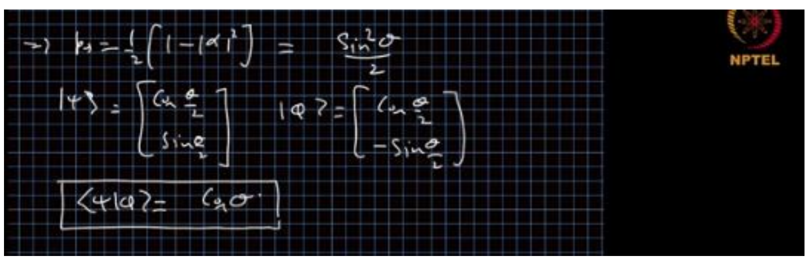

So, our total state is psi tensor 0 or let me call psi tensor 0 or it is phi tensor 0. So, the state is cos theta over 2 0 0 plus minus sin theta over 2 1 0 because we can write psi and phi as cos theta over 2 0 plus minus sin theta over 2 1. So, this is the psi initial of the system and ancilla. Now, let us say we can find a unitary which acts as follows. If we take it acts as cos theta over 2 0 0, then it will give us sin theta over 2 0 0 plus root cos theta 1 1.

And if it acts on 1, 0, it will give us 1, 0. If it acts on 0, 1, it will give us 0, 1 and when it acts on 1, 1, we will get something orthogonal to this state. So, let us say such unitary exists. We know it will work. That is why I am giving you this unitary.

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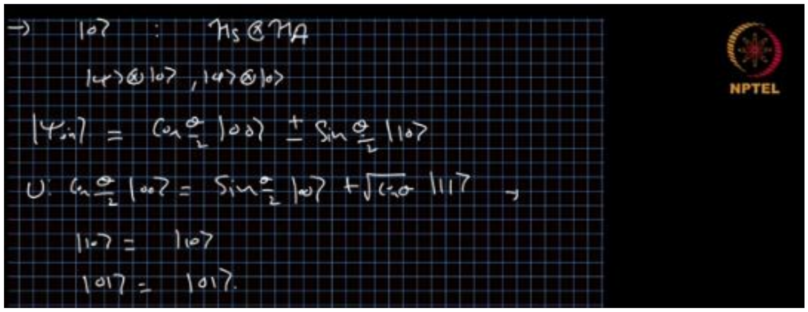

Then we will examine this unitary later on. So, what we have here is, when we apply U on psi initial, we get cos theta over 2 0 0 will go to sin theta over 2 0 0 plus root cos of theta 1 1 and plus minus sin theta over 2 will go to 1 0 will go to 1 0. We can write it as sine theta over 2, 0 plus minus 1, tensor 0. These are the first two terms, first and last term and the middle term, cos of theta, 1, 1, which can be written as 1, tensor 1. This 0 and this 1 is the state of ancilla and this is the state of system. Now, if we perform measurement on ancilla, then we can get 0 or 1. If outcome is 1, then the state of the system collapses to 1 and we have no idea whether this 1 came from the psi or phi.

If the ancilla is in 1, then system also goes to 1. But if the ancilla is in 0, then the system goes to sin theta over 2 is the overall factor. So, we multiply with root 2 and divide it by root 2. So, then the system collapses to 0 plus minus 1 over root 2. This is normalized state.

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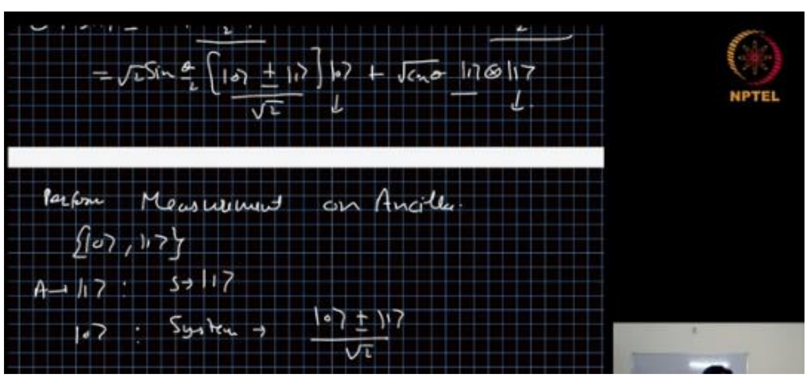

And the plus sign is for psi and the minus sign is for phi. Now, if we perform, once we have 0 in the ancilla, then the system, we can perform the measurement on the system in the 0 plus minus 1 over root 2 basis. These are orthonormal basis. And then, if we get 0 plus 1, then we know it was psi. And if we get 0 minus 1, then it was phi.

So, in that way, once the ancilla collapses to 0, we have already succeeded. Then we can figure out what was the state of, what was the input state. So, here the probability that, the probability of failure is when we get 1 in the ancilla. So, the probability of failure is cos of theta. And probability of success will be 1 minus cos of theta, which becomes 2 sin squared theta over 2.

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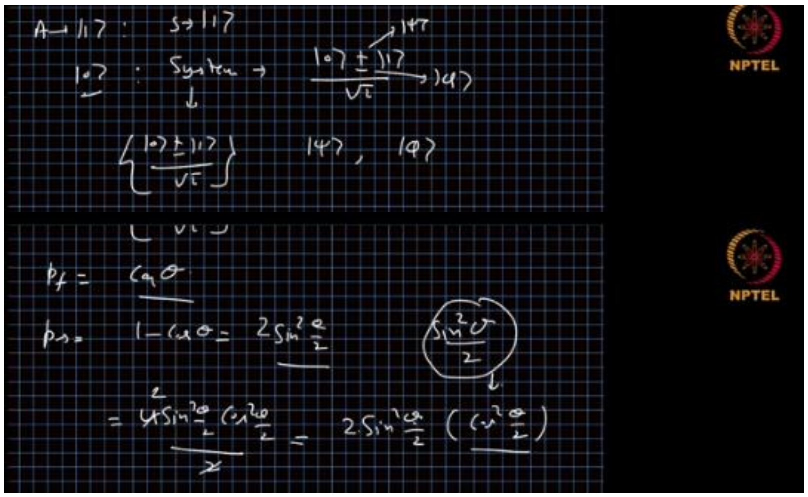

If you remember that the probability of success from the projective measurement was sine square theta over two and that can be written as four sine square theta over two cos square theta over two over two that becomes two sine square theta over two times cos square theta over two. So, the probability of success the sine square theta over two which we got from the projective measurement that is a factor cos square theta over two smaller than the probability of success we got in this new method of performing measurement. So, if we look at it carefully what we have done is we had a system and we wanted to distinguish between the two non-orthogonal states of that system. We couple it with another ancilla and the coupling unitary we found by some hit and trial method to be some strange unitary and that unitary is giving us the probability of success to be a significantly larger number than the one we got from the projective measurement. It doesn't matter how we got that unitary but that unitary is showing us that generalized measurements can give us results which are much better than the projected measurements. This is one example of showing the power of the projective measurement.

Now, how to arrive at this thing? We can look at the, we can revisit our concepts of the POVMs. So, let us say we consider a POVM with a first effect. to be some number a times psi orthogonal psi orthogonal, because we saw that psi whatsoever gives us the information about phi and phi whatsoever gives us the information about psi so we and there is no difference between psi and phi for us they are on equal footing for us then our we can assume the first effect as some number a times psi bar, psi bar, we will figure out what this a is. And the second POVM, second effect is a times phi orthogonal, phi orthogonal. And third is identity minus E1 minus E2.

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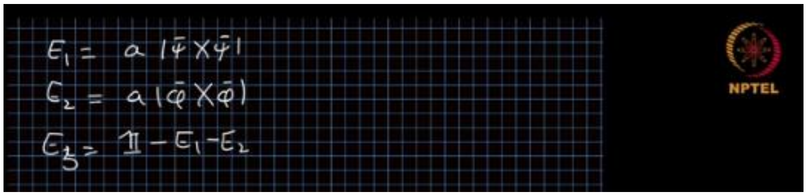

Now we have to choose a in such a way that E3 is positive it is E3 is rank one and if we have those things then we can figure out the success probability and stuff like that. Now psi, we have chosen to be cos of theta over two sine of theta over two. From here, we can write psi bar which is orthogonal to psi to be sine of theta over two and minus cos of theta over two. Similarly, phi is cos of theta over 2 minus sine of theta over 2. From here, we can write phi bar to be sine of theta over 2 minus cos or cos of theta over 2. you can check that psi bar is orthogonal to psi and phi bar is orthogonal to phi. So, from here we can write E1 as a times sine squared theta over two cos squared theta over two sine theta over two cos theta over two minus and sine theta over two cos theta over two is minus sign and two will be with plus sign same just that these off diagonal elements are having different signs. So, E3, which is identity 1, 0, 0, 1 minus E1 minus E2. So, plus or minus A times where we add these two, we get 2 sine square theta over 2 to cos square theta over 2, 0, 0.

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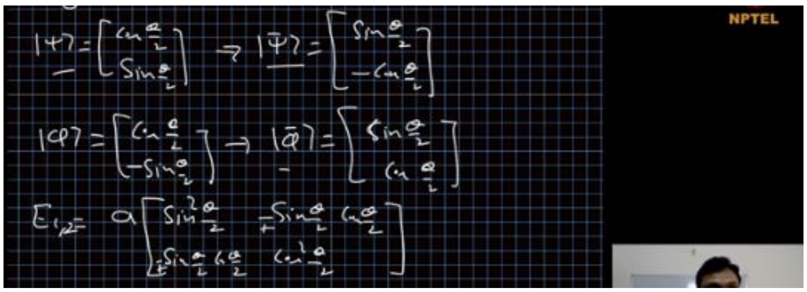

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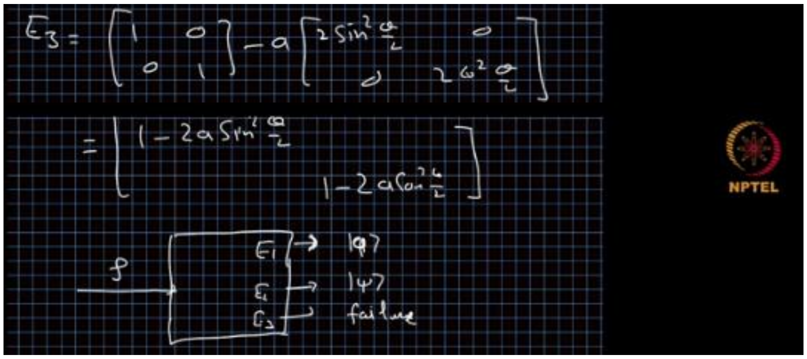

So, we get 1 minus 2A sine square theta over 2 and 1 minus 2A cos square theta over 2. Now hypothetically our setup is such that, formally our setup is such that we have our density matrix going in and output are the three measurement outcomes that is E1 E2 and E3. We know when E1 click and E1 was just psi orthogonal, then the input state was phi. When E2 clicks, which is the phi orthogonal, then the input state was psi. And when E3 click, that is the failure.

So, the probability of E3, that is expectation value of E3 will be, so, trace of E3 times rho and rho is half times psi plus phi and it turns out to be, so, the p3 will be trace of identity minus a psi orthogonal psi orthogonal minus a phi orthogonal phi orthogonal times rho. That will give us 1 minus a times half times psi orthogonal phi mod square minus a times half times phi bar psi mod square and these two numbers are same and these two numbers are sine squared theta one minus a sine squared theta over two minus a sine squared theta over two, so, total is one minus a sine squared theta over two So, this is the failure probability. Now, from here, we see that we can make E3 rank 1 by making either this element 0 or this element 0. So, it means if we make A to be 1 over 2 sine squared theta over 2, then we get a E3 to be rank 1 and if we choose A to be 1 over 2 cos square theta over 2, then E3 is rank 1.

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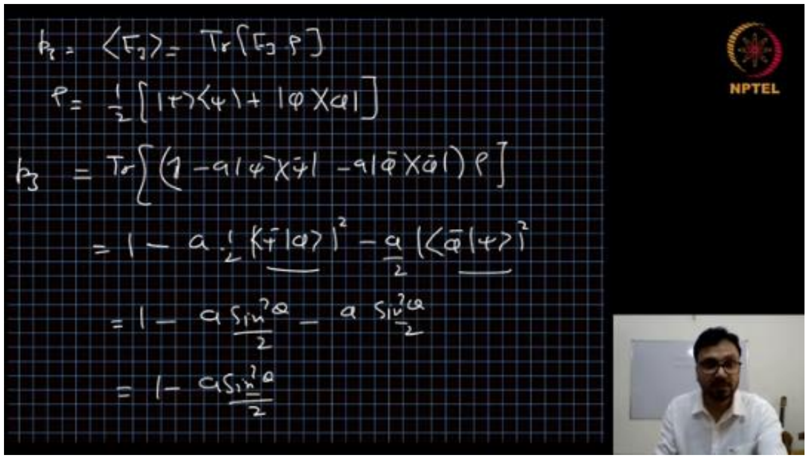

We substitute both of them in this, then p3 becomes 1 minus sin square theta over 4 sin square theta over 2. There is no over 2 here because there are half half, they cancel. So, 2 and that is 1 minus sin square theta can be written as 2 sin square theta over 2 or cos square theta over 2 over sin square 2 sin square theta over 2. This is 1 minus 2 cos square theta over 2. If we choose A to be 1 over 2 cos square theta over 2, then p3 becomes 1 minus sine square theta over 2 cos square theta over 2.

That will give us 1 minus 2 sine square theta over 2. That will be cos theta and that will be minus cos theta. So, from here we can see that the choice a to be 2 cos square theta over 2 gives us a valid probability of, because probabilities cannot be negative and we are choosing theta to be in the range that cos theta is positive. So, this is not a valid probability. So, we get the probability of failure P3 to be cos theta for a choice of A equals 1 over 2 cos square theta over 2.

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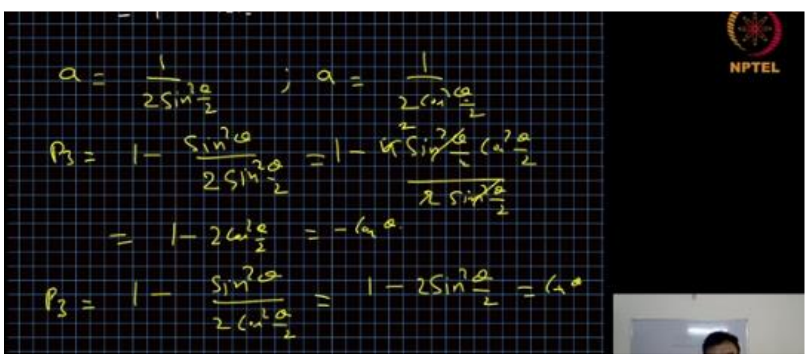

So, the probability of success becomes 1 minus P3, which is 1 minus cos theta, which is 2 sin square theta over 2. This is precisely what we got when we chose that unitary, which looks strange at a moment, but it seems like that gives us the same result as we get when we use the POVM technique. So, in that way using POVMs that too, three outcome POVM, we can find better distinguishability method of two non orthogonal states and the results are strikingly different from the one we had from projective measurement. We can implement here, if we just look at this POVM what we used, it is very interesting that E1, which is A, which was 1 over 2 cos square theta over 2, psi bar, psi bar, E2 which is 1 over 2, cos square theta over 2, phi bar, phi bar, E3 which is 1 minus 1 over 2 cos square theta over 2, psi bar, psi bar, minus phi bar, phi bar, that is just 0, 0 with some coefficient and that coefficient is that will be 1 minus 10 square theta over 2. So, from here we can see that we have three unit vectors which are not normalized.

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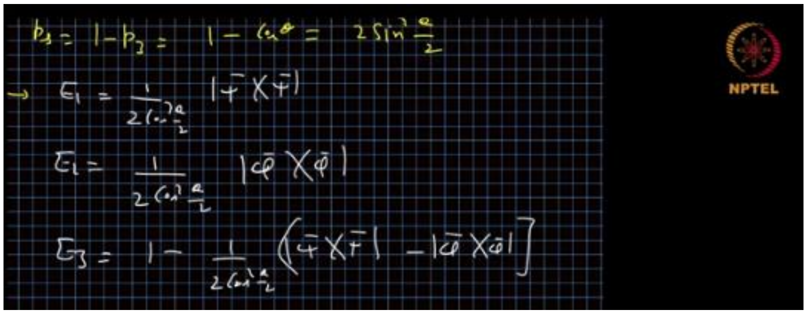

One is u1 which is 1 over root 2 cos of theta over 2 psi bar, u2 which is 1 over root 2 cos of theta over 2 pi bar and u3 which is square root of 1 minus tan theta over 2, 0. With these three, we have these three vectors which are not orthogonal. But they have the property that sum over i ui ui equals identity, so, they form a POVM. This ui ui now by using Neumark's dilation theorem, we should be able to construct three three-dimensional vectors which are extension of the u1, u2, u3 in such a way that they form an orthogonal basis. So, omega i we can make as ui direct sum ci, ci will be just a number because we already have two-dimensional vector, we want to make it three-dimensional vector and in such a way we choose ci in such a way that this ui, wi are orthogonal basis. And this remains an exercise. For this given set of ui's find ci's in such a way wi becomes orthonormal basis.

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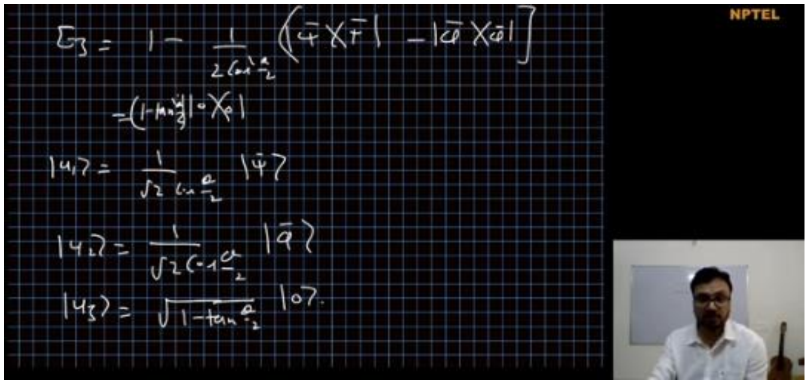

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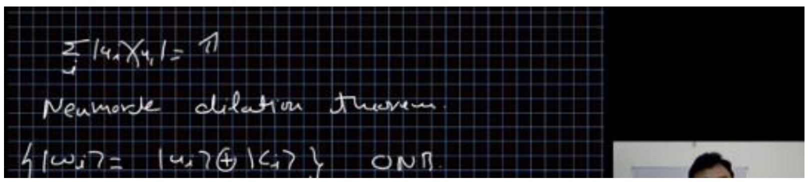

Now if we extend our Hilbert space of two vectors two dimensions to three dimensions and perform measurement with wi in wi basis then we will get the same result what we discussed in two different cases here. One with the unitary acting on the system and ancilla and other in terms of the POVM. And in this way, we can use generalized measurements to improve the distinguishability of non-orthogonal state.