Radio Astronomy

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Review of Electromagnetism Part 02

So, after the plane wave in one dimension, we consider the next case of non-conducting medium. So here, assuming there is no source like almost like in free space, and so these are the Maxwell's equations, takes these forms,

$$\nabla \cdot \mathbf{D} = 0, \quad \nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0,$$
$$\nabla \cdot \mathbf{B} = 0, \quad \nabla \times \mathbf{H} - \frac{\partial \mathbf{D}}{\partial t} = 0.$$

And so if we start from here, and we use Fourier transforms, so there's a special e x t, e is electric field function of space and time, and it gives you a Fourier conjugate of e x of omega, and there is a Fourier transform,

$$\mathbf{E}(\mathbf{x},t) = \frac{1}{\sqrt{2\pi}} \int \mathbf{E}(\mathbf{x},\omega) e^{-i\omega t} \, d\omega$$
$$\mathbf{E}(\mathbf{x},\omega) = \frac{1}{\sqrt{2\pi}} \int \mathbf{E}(\mathbf{x},t) e^{i\omega t} \, dt.$$

Now with that, you can essentially again replace in the existing Maxwell's equation, and again do those substitution,

Note - we can obtained the similar expressions for B-field. Now by use of **D = εE** and **B = μH** and the fourier transform equation we get

$$\nabla \times \mathbf{E} - i\omega \mathbf{B} = 0$$
$$\nabla \times \mathbf{B} + i\omega\mu\varepsilon \mathbf{E} = 0,$$

and finally kind of come up with the different forms of wave equation,

$$\nabla \times (\nabla \times \mathbf{E}) - i\omega \nabla \times \mathbf{B} = \nabla(\nabla \cdot \mathbf{E}) - \nabla^2 \mathbf{E} - \omega^2 \mu\varepsilon \mathbf{E} = 0,$$

Since $\nabla \cdot \mathbf{E} = 0$, we have

$$\left(\nabla^2 + \omega^2 \mu \varepsilon\right)\mathbf{E}(\mathbf{x},\omega) = 0$$

Similarly one can get

$$\left(\nabla^2 + \omega^2 \mu \varepsilon\right)\mathbf{B}(\mathbf{x},\omega) = 0.$$

The above equation is so called **wave equation.**

both for E as well as for the B. So both for electric field and magnetic field, you can come up with two different forms of wave equation. And there are certain, this thing, the refractive index (n) is kind of defined in terms of this ratio of mu and epsilon, and that's how they are defined over the free space permittivity and the electric constant, $k = \omega\sqrt{\mu\varepsilon}$.

Since, $\varepsilon = \varepsilon_r \varepsilon_0$ and $\mu = \mu_r \mu_0$ defining **n** also known as refractive index

$$n = \sqrt{\frac{\mu}{\mu_0}\frac{\varepsilon}{\varepsilon_0}},$$

Now by use of **k** and **n** we can calculate the phase velocity (i.e, velocity with which the phase of a wave travels) as

Since, $$v_p = \frac{\omega}{k} = \frac{1}{\sqrt{\mu\varepsilon}} = \frac{c}{n}, \qquad c = \frac{1}{\sqrt{\epsilon_0 \mu_0}}$$

If you know that the c, the speed of light is one over square root of epsilon zero mu zero. And the phase velocity is defined by omega over k, so that is c over the refractive index.

$$\mathbf{E}_k(x,t) = \mathcal{E}_1(k)e^{ik(x - v_p t)} + \mathcal{E}_2(k)e^{-ik(x + v_p t)}$$

$$\mathbf{E}(x,t) = \mathbf{E}_1(x - v_p t) + \mathbf{E}_2(x + v_p t)$$

Continuing further, you finally get the solution, the harmonic solution of both the electric field and magnetic field,

$$\mathbf{E}(\mathbf{x},t) = \mathcal{E}e^{i(k\mathbf{n}\cdot\mathbf{x} - \omega t)}$$
$$\mathbf{B}(\mathbf{x},t) = \mathcal{B}e^{i(k\mathbf{n}\cdot\mathbf{x} - \omega t)}$$

and you have the k vector as the wave vector, which k is the amplitude is pi over lambda, and n is the direction of propagation. So ultimately you will, if you proceed further in this calculation,

$$(\nabla^2 + \omega^2 \mu\varepsilon)\mathbf{E}(\mathbf{x},\omega) = 0 \qquad (\nabla^2 + \omega^2 \mu\varepsilon)\mathbf{B}(\mathbf{x},\omega) = 0.$$

$$k^2 \mathbf{n}\cdot\mathbf{n} = \omega^2 \mu\varepsilon,$$

$$\mathbf{n}\cdot\mathcal{E} = 0$$
$$\mathbf{n}\cdot\mathcal{B} = 0$$

This implies that the electric and magnetic induction fields associated to a plane wave are perpendicular to the direction of propagation **n**.

it has to be, it is very cumbersome, but I mean finally we get that proof that E and B fields are actually perpendicular to each other,

$$\mathbf{n}\times\mathcal{E} = \frac{\omega}{k}\mathcal{B}$$

$$\mathbf{n}\times\mathcal{B} = -\mu\varepsilon\frac{\omega}{k}\mathcal{E},$$

Using this we can get the relation

$$\boxed{\mathbf{n}\times\mathcal{E} = \frac{1}{\sqrt{\mu\varepsilon}}\mathcal{B}}$$

This relation shows us that E and B field are perpendicular to each other.

and that's their direction of propagation. So these are two fields which are orthogonal to each other, and that's how the magnetic waves propagate.

In free space $1/\sqrt{\mu\varepsilon} = c$

Thus, $c|\mathbf{B}| = |\mathbf{E}|$

Now, by defining a term called **impedance Z** of a medium as $Z = \sqrt{\frac{\mu}{\varepsilon}},$ we will study this in detail in transmission lines.

So, we can easily obtain $\mathbf{n}\times\mathcal{E} = Z\mathcal{H}.$

where, $\mathbf{H} = \mathcal{H}e^{i(k\mathbf{n}\cdot\mathbf{x}-\omega t)}$

And we also can define plane waves, this impedance of free space, and that is defined as z naught, which is the ratio of epsilon zero over mu naught,

$$\boxed{Z_0 = \sqrt{\mu_0/\varepsilon_0} = 377 \ \text{ohms}}$$

sorry, and that is roughly 377 ohms. If you define e and h, if you finally solve for e and h, then you can also derive what is the poynting vector, and from there you can derive the average energy density u.

By use of poynting vector/flux we and by time averaging it we get

$$\mathbf{S} = \frac{1}{2}\mathbf{E}\times\mathbf{H}^{*} = \frac{1}{2}\sqrt{\frac{\varepsilon}{\mu}}|\mathcal{E}|^{2}\,\mathbf{n}$$

This shows the energy flowing (flux) out of the system (i.e, energy per unit area per unit time)

And the corresponding time averaged energy density **u** can be given by

$$u = \frac{1}{4}\left(\varepsilon\mathbf{E}\cdot\mathbf{E}^{*} + \frac{1}{\mu}\mathbf{B}\cdot\mathbf{B}^{*}\right) = \frac{\varepsilon}{2}|\mathcal{E}|^{2}.$$

Okay, that's how it's related. Now that was in the nonconducting medium, of course it is, real media is lossless, sorry, lossy, and so we, here we consider lossless media, and again the different wave equations takes place,

$$\nabla^{2}\vec{E} = -\omega^{2}\mu\varepsilon\vec{E} = -\beta^{2}\vec{E}$$

$$\nabla^{2}\vec{H} = -\omega^{2}\mu\varepsilon\vec{H} = -\beta^{2}\vec{H}$$

We define a propagation constant, which is complex, which was alpha and beta, beta is the complex term, and alpha is the real term,

$$\gamma = \alpha + j\beta$$

So if you solve for the waves equation, you can actually derive the value of alpha and beta.

$$\vec{E} = Ae^{-j\beta\vec{r}} + Be^{j\beta\vec{r}}$$

$$or, \vec{E}(\vec{r},t) = \left\{Ae^{-j\beta\vec{r}} + Be^{j\beta\vec{r}}\right\}\times e^{-j\omega t}$$

$$\vec{H} = \frac{(\vec{r}\times\vec{E})}{Z}$$

$$\nabla^{2}\vec{E} = -\omega^{2}\mu_{0}\varepsilon_{0}\vec{E} = -\beta_{0}^{2}\vec{E}$$

$$\nabla^{2}\vec{H} = -\omega^{2}\mu_{0}\varepsilon_{0}\vec{H} = -\beta_{0}^{2}\vec{H}$$

The third and the last example, we did nonconducting medium, free space, nonconducting medium, and then lossless medium, and finally the real world is lossy. So if you do that, the alpha beta values becomes quite extensive, you know, and quite cumbersome, but that's how things are done. We just show you, there is nothing to worry about, we will never be using this for this particular course at least, but that's how we are done,

$$\nabla \times \vec{H} = \sigma\vec{E} + j\omega\varepsilon\vec{E}$$

$$and, \nabla \times \vec{E} = -j\omega\mu\vec{H}$$

and since there is no source we will have

$$\nabla^2\vec{E} = j\omega\mu\sigma\vec{E} - \omega^2\mu\varepsilon\vec{E}$$

$$or, \nabla^2\vec{E} = j\omega\mu(\sigma + j\omega\varepsilon)\vec{E}$$

Now, defining propagat

$$\gamma = \sqrt{j\omega\mu(\sigma + j\omega\varepsilon)} = \sqrt{-\omega^2\mu\varepsilon + j\omega\mu\sigma} = \alpha + j\beta$$

Thus, we have $\nabla^2\vec{E} = \gamma^2\vec{E}$

we are only considering **E**-field as H-field can be derived from E-field. This will have only one solution. That is $\vec{E}(r) = E_0 e^{-\gamma r}$

This is true because **$\gamma = \alpha + j\beta$** . So, if we consider the positive exponential term then solution will blow up. Thus, we can safely neglect that term. other way of thinking is that "How can E-field increase if there is no source". If we solve the expression of **γ** for **α** and **β** we will get the expression-

$$\alpha = \omega\left(\sqrt{\frac{\mu\varepsilon}{2} \times \left(\sqrt{1 + \frac{\sigma^2}{\omega^2\varepsilon^2}} - 1\right)}\right) dB/m \qquad \beta = \omega\left(\sqrt{\frac{\mu\varepsilon}{2} \times \left(\sqrt{1 + \frac{\sigma^2}{\omega^2\varepsilon^2}} + 1\right)}\right) rad/m$$

So, velocity of propagation in lossy medium (phase velocity) will be

$$\upsilon = f\lambda = \frac{\omega}{\beta} = \frac{1}{\sqrt{\frac{\mu\epsilon}{2}\left(\sqrt{1 + \frac{\sigma^2}{\omega^2\epsilon^2}} + 1\right)}}, (m/s)$$

The impedance **Z** will be

$$\frac{E}{H} = \sqrt{\frac{j\omega\mu}{\sigma + j\omega\epsilon}}\ \Omega$$

This can be found by plugging in the expression for E-field in either in Ampere's law or in Faraday's Law equation in Maxwell's equation.

Okay, so that brings us to the end of the second topic of the electromagnetic waves. So the fourth, third topic, sorry, is the polarization of electromagnetic waves. As you know, as we learned, that electromagnetic and magnetic fields are always propagating but orthogonal to each other. That gives rise to something called polarization and different polarization, linear polarization, circular polarization, and elliptical polarization. And in the circular, there is left circular, left-handed circular polarization or LHCP, or commonly called left circular polarization and right circular polarization. And then again, similar to that is RHEP, right-handed elliptical polarization and left-handed elliptical polarization.

So let's consider the, let's consider the X is a linearly polarized wave with complex amplitude of EX and Y along Y is a linearly polarized wave with complex amplitude of EY. But they both travel in the positive Z direction. Note that EX and EY may be varying with time, that's a general case. The complex, for simplicity, we ignore the time varying part and just concentrate on the spatial term for a little bit.

So E vector is composed of both EX, which is along X and EY along Y. And they are also substituted further. And so we can consider the equation of this. And so if you have the, so both EX0, EX has a component of EX0 times e to the power of j phi of X and j, EY has a j phi of Y.

$$\vec{E}(z) = \left(E_x \hat{x} + E_y \hat{y}\right) e^{-j\beta z} = \left(E_{x0} e^{j\phi_x} \hat{x} + E_{y0} e^{j\phi_y} \hat{y}\right) e^{-j\beta z}$$

Now if the two linear polarization, the time phase difference with the two components is either 0 or multiple, integer multiple of pi, then this is called the linear polarization,

1. **Linear polarization:** For the wave to have linear polarization, the time-phase difference between the two components must be

$$\Delta\phi = \phi_y - \phi_x = n\pi, \quad n = 0, 1, 2, 3, \ldots$$

If this condition satisfies then the **E(z)** will be linearly dependent on **Ex0** and **Ey0** i.e, **E(z) = (Ex0**x **+ Ey0**y**)exp(-j**β**z)**

Okay? And if they are not in pi, but the amplitude of EX0 and EY0 are the same, but the phase difference is either plus of 2N plus half pi or minus 2N plus half pi. Then they are called circular polarization,

$$\mathbf{E(z) = Ex0(x + jy)exp(-j\beta z)exp(-j\omega t)}$$

The additional term of **exp(-jωt)** provides the time dependency

Then there is left circular and right circular. And that is linked here. And if the EX0 and EY0, the amplitudes are not equal, then as well as the phi phase difference is like this, goes from either plus 2N plus half pi or minus 2N plus half pi, then these are basically called the elliptical polarization,

Elliptical Polarization: Elliptical polarization can be obtained only when the time-phase difference between the two components is odd multiples of π/2 and their magnitudes are not the same That is,

$$\Delta\phi = \phi_y - \phi_x = \begin{cases} +(\frac{1}{2} + 2n)\pi & \text{for CW} \\ -(\frac{1}{2} + 2n)\pi & \text{for CCW} \end{cases} \quad n = 0, 1, 2, \ldots \quad \text{when} \quad E_{xo} \neq E_{yo}$$

Or when the time-phase difference between the two components is not equal to multiples of π/2 (irrespective of their magnitudes). That is

$$\Delta\phi = \phi_y - \phi_x \neq \pm\frac{n}{2}\pi = \begin{cases} > 0 & \text{for CW} \\ < 0 & \text{for CCW} \end{cases} \quad n = 0, 1, 2, 3, \ldots$$

Okay? CW stands for left-handed circular polarization. And CCW is counterclockwise or right-handed circular polarization. Similarly, for the circular polarization also, there is left-handed and right-handed properties. We will come to all of this when we learn about antenna radiation and stuff. Another way of looking into polarization is by measuring Stokes parameters.

Following Stokes 1852, the definition of four-Stokes parameter.

$$\begin{aligned} I &= E_1^2 + E_2^2 \\ Q &= E_1^2 - E_2^2 \quad = I\cos 2\varepsilon \cos 2\tau \\ U &= 2E_1E_2\cos\delta = I\cos 2\varepsilon \sin 2\tau \\ V &= 2E_1E_2\sin\delta = I\sin 2\varepsilon \end{aligned}$$

$$I^2 = Q^2 + U^2 + V^2.$$

I is the quadrature sum of both the amplitudes, E1 square plus E2 square. Q is the difference of that. U is the cross product to E1 E2 times cosine of delta and V is the other cross product to E1 E2 sine delta. Delta is representative of the phase difference, which we just concluded in the previous slide.

And angular polarization is given by half of tan inverse U over Q term,

$$\tau = \frac{1}{2}\tan^{-1}\frac{U}{Q}$$

So this is very standard way of understanding the different types of polarization by quantifying the IQUV Stokes parameters. For radio astronomy also you will see that Stokes I is basically the ignorant of the polarization and but the V is the differential. And so we will see the Stokes I image, Stokes V image and so on and so forth when we do imaging of the radio data. Beside this there is another way of by following Henry Poincare to essentially represent the state of polarizations in a sphere.

So if the polarization is lying near the equator, the vector, it basically means linearly polarized. If they are lying in the north pole, it is left circular and in the south pole it is right circular polarization. There are more details to it, we will deal with it later but it is a very nice way to visually represent the state of polarization of electromagnetic wave. So thanks a lot, I mean this probably has come to you little bit intense. We will also follow this up with a kind of easy tutorial which will explain this stuff and make it more easier to understand but nevertheless we will, for people who are registered for the exams, do not worry much about it.

It is just for completeness and thorough understanding of subjects which are going to come up. We thought better to introduce all this in the first week itself. You can go and just do a revision of your basic electromagnetics from Griffiths or any other book which you like and we hope this will help. We will also take up individual concepts again as a part of that week's lecture and we again revisit those when time comes. So do not worry, stay tuned for the next week's lecture.

We start with brightness, temperature, different radiation fundamentals. It will be fun. So see you soon. Bye.