Radio Astronomy

Prof. Abhirup Datta

Department of Astronomy, Astrophysics, and Space Engineering

Indian Institute of Technology Indore

Lec-02

Review of Electromagnetism Part 1

Welcome back. This is the second lecture for this course of radio astronomy. In the first lecture we introduced the field of radio astronomy, why we are so passionate and so excited to do radio astronomy in today's world, what are the futures even for radio astronomy in India. We also showed lots of motivation in different branches as radio astronomy plays a very important role in the overall astronomical findings. That was the first lecture. In this lecture what we plan to do is step off it back and review some basic electromagnetic theory which is essential to go ahead.

As you understand, we are talking about radio astronomy, means radio signals which are getting received by the telescope, emitted from cosmic sources distance away. Now of course it involves wave electromagnetic wave propagation, it involves certain electromagnetic wave dispersion, reflection, refraction, several other things. It is governed by the basic Maxwell's equation. All those we have learned mostly as an undergraduate.

Now we could have skipped this portion and gone ahead and just assume that you know but people come from different backgrounds and have different retention of what they have done in their undergraduate. So we thought as a completeness it is better to spend one lecture in reviewing certain basics. We will not go into details. This will also follow up with a tutorial which will cover some solving numerical examples that will also revise your already learned concepts a bit more which will help us to go forward in this course. So with that let's go and start reviewing the electromagnetics.

The basic classical electromagnetic books are by J. D. Jackson but my favorite is the last one, Electromagnetics by D. J. Griffiths which is more than sufficient for most of the purposes to do.

So with that we just and of course the antenna theory we will also come separately in week number 3 and discuss more. Starting with Maxwell's equation. Now it is very important we will not go into details of the historical development of all these equations but they are fascinating nevertheless.

So, there are four equations. $\nabla \cdot D=\rho$, the charge density, D is the displacement current linked with electric field; $\nabla \cdot B=0$ (B, the magnetic field), $\nabla \times H=J$, the current density and $\nabla \times E=-\partial B/\partial t$, where B is the magnetic field. So if you perform $\nabla \cdot J= \nabla \cdot (\nabla \times H)=0$ and from there the continuity equation also gives you $\nabla \cdot J+\partial \rho/\partial t=0$. Now, if you club both of them you can definitely write $\nabla \cdot [J+\partial D/\partial t]=0$

So that is very nice. So we have now then the modified equations as $\nabla \cdot D=\rho$ which is already there, $\nabla \cdot B=0$, $\nabla \times H=J+\partial D/\partial t$ and the final equation $\nabla \times E=-\partial B/\partial t$. Now these sets of equations are also called Maxwell's equation.

In fact the first one you can relate to the Gauss's law. The second one is non-existence of magnetic monopole. The third equation is the modified Ampere's law satisfying conservation of charge and the fourth equation is the Faraday's laws of induction. Now if we consider Maxwell's equation in free space it refers to like vacuum where no dielectric and magnetic media is present. So ε and μ are all equal to zero, dielectric constant and permittivity. No charge density and no current density, so ρ and J are also equal to zero (ρ = J =0) and no means of conduction. That reduces the Maxwell's equation to $\nabla \cdot E=0$, $\nabla \cdot B=0$ and $\nabla \times E=-\partial B/\partial t$ and $\nabla \times B= \mu_0\varepsilon_0\frac{\partial \vec{E}}{\partial t}$ .These are the Maxwell's equations in free space.

Now let's take a look and revise a little bit more about Poynting's theorem. What is Poynting's theorem and what is Poynting's vector? The conservation of energy in the electromagnetic field is often referred to as Poynting's theorem. So there is charge Q subjected to a electromagnetic field of E and B. The Lorentz force which acts on the charge is given by $F=q(E+v\times B)$ (charge times "electric field plus the cross product of velocity and B"). This is very well known. So, that is the Lorentz force. Now if that from there we want to equate what is the mechanical work done by the fields on the charge Q then that rate of change of energy $\frac{dE_{mech}}{dt} = \mathbf{F}\cdot\mathbf{v} = q\mathbf{E}\cdot\mathbf{v}$. The magnetic field does no work. We can easily generalize the above equation to an arbitrary distribution of charge density ρ using the fact that $J=\rho V$. Then the above equation takes the shape of the rate of change of this mechanical energy $\frac{dE_{mech}}{dt} = \int_V \mathbf{J}\cdot\mathbf{E}\, d^3x$.

So by using Maxwell's equation again you can write it in terms of the $\nabla \times H$, $\int_V \mathbf{J}\cdot\mathbf{E}\, d^3x = \int_V\left[(\nabla\times\mathbf{H})\cdot\mathbf{E} - \frac{\partial\mathbf{D}}{\partial t}\cdot\mathbf{E}\right]d^3x.$ so you replace that and use the vector algebra form that, $(\nabla\times\mathbf{H})\cdot\mathbf{E} = (\nabla\times\mathbf{E})\cdot\mathbf{H} - \nabla\cdot(\mathbf{E}\times\mathbf{H})$, and if you replace that finally you get the shape $\int_V \mathbf{J}\cdot\mathbf{E}\, d^3x = -\int_V\left[\nabla\cdot(\mathbf{E}\times\mathbf{H}) + \mathbf{E}\cdot\frac{\partial\mathbf{D}}{\partial t} + \mathbf{H}\cdot\frac{\partial\mathbf{B}}{\partial t}\right]d^3x.$ If now you assume that the medium is linear and isotropic then the displacement current becomes just a scaling of the electric field and same as the magnetic field and intensity and if you do that then finally you manage to get the electromagnetic energy in terms of E, D, B and H, $u=\frac{1}{2}(\mathbf{E}\cdot\mathbf{D}+\mathbf{B}\cdot\mathbf{H})$. And then you also have putting into the expression you have $\int_v \mathbf{J}\cdot\mathbf{E}\,d^3x=-\int_v\left[\frac{\partial u}{\partial t}+\nabla\cdot(\mathbf{E}\times\mathbf{H})\right]d^3x.$ .Sorry. Since the volume integration V is completely arbitrary we can write this last equation as a continuity equation $\boxed{\frac{\partial u}{\partial t}+\nabla\cdot\mathbf{S}=-\mathbf{J}\cdot\mathbf{E}}$ and that one in the box is now called the Poynting's theorem or follows the law of conservation of energy.

Okay. Now what is a Poynting vector? Defining Poynting theorem, now you come to Poynting vector or Poynting flux, is defined as this vector S which is nothing but the cross product of E and H, S=E×H. Okay. And it is a measure of the rate of energy flow per unit area at that point. Relationship between a vector and a scalar product, you know that the vector product, vector potential is defined as A where del cross A, curl of A gives you the magnetic field. So if you replace B in the Faraday's law in the next first equations, you get this $\nabla\times\mathbf{E}+\frac{\partial\mathbf{B}}{\partial t}=\nabla\times\left(\mathbf{E}+\frac{\partial\mathbf{A}}{\partial t}\right)=0.$. Hence you arrive at the relationship of $\mathbf{E}=-\nabla\Phi-\frac{\partial\mathbf{A}}{\partial t}$, where phi is the scalar potential and A is the vector potential. Very good. By using Faraday's law from Maxwell's equation we get, if you just follow this expression,

$$\begin{aligned}\frac{1}{\mu_0}\nabla\times\mathbf{B}-\varepsilon_0\frac{\partial\mathbf{E}}{\partial t}&=\frac{1}{\mu_0}\nabla\times(\nabla\times\mathbf{A})+\varepsilon_0\left[\nabla\left(\frac{\partial\Phi}{\partial t}\right)+\frac{\partial^2\mathbf{A}}{\partial t^2}\right]\\&=-\frac{1}{\mu_0}\nabla^2\mathbf{A}+\varepsilon_0\frac{\partial^2\mathbf{A}}{\partial t^2}+\nabla\left(\frac{1}{\mu_0}\nabla\cdot\mathbf{A}+\varepsilon_0\frac{\partial\Phi}{\partial t}\right)\\&=\mathbf{J},\end{aligned}$$

we will be sharing the slides and then by setting $c^2=(\mu_0\varepsilon_0)^{-1}$, you finally get to derive this huge expression,

$$\boxed{\nabla^2\mathbf{A}-\frac{1}{c^2}\frac{\partial^2\mathbf{A}}{\partial t^2}-\nabla\left(\nabla\cdot\mathbf{A}+\frac{1}{c^2}\frac{\partial\Phi}{\partial t}\right)=-\mu_0\mathbf{J}}$$

And eventually if you use the Gauss's law from Maxwell's equation, instead of Faraday's law we get,

$$\begin{aligned}\varepsilon_0\nabla\cdot\mathbf{E}&=-\varepsilon_0\nabla\cdot\left(\nabla\Phi+\frac{\partial\mathbf{A}}{\partial t}\right)\\&=-\varepsilon_0\nabla^2\Phi-\frac{\partial}{\partial t}(\nabla\cdot\mathbf{A})\\&=\rho,\end{aligned}$$

, and finally,

$$\nabla^2\Phi + \frac{\partial}{\partial t}(\nabla \cdot \mathbf{A}) = -\frac{\rho}{\varepsilon_0}$$

This is all the basic electromagnetic theory and so that's what relates the phi and A if you replace the basic identities in the Maxwell's equation.

So far what we have done is we have reviewed a little bit of Maxwell's equation in different form. You can keep replacing one with the other. Remember things do change, the expressions do change. Mostly in the radio astronomy we will be using free space equations but sometimes things do pass, radio waves do pass through the lossy medium like ionosphere and things changes. But mostly it will come when we discuss the different wave propagation properties and also during the antenna radiation pattern study.

The antenna is a very vital thing for radio astronomy that is the basic core behind creating a radio telescope. So we need to understand more how the antenna radiates, what are the antenna's far field radiation pattern etc. So some of those cases we will see use of very genuine use of tricks of replacing A with B and different identities and so getting a primer in the electro

magnetics is essential so that you can follow those. Otherwise those steps won't be easy to follow. So far we have just finished the Maxwell's equation and introduced Poynting's theorem and the Poynting's vector.

So next is a wave equation. Wave equation is very well known. You have double derivative with respect to X the space vectors on one side linked with one over the V square the velocity times the double derivative with respect to time on the other side,

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{v^2} \cdot \frac{\partial^2 u}{\partial t^2}$$

So we have then the function which has to be solved is a function of both X and T,

$$u(x, t) = X(x)T(t)$$

So we do a kind of separation of variables. We assume that there are two different functions, one just spatially related and a function of space and one function of time. So then if you do that you get two separate sets of equations one with respect to time and one with respect to space,

$$\frac{d^2X(x)}{dx^2} - KX(x) = 0$$

and,

$$\frac{d^2T(t)}{dt^2} - Kv^2T(t) = 0$$

If you keep solving that by just following this simple solutions assuming then finally we arrive at by doing the standard separation of variable technique we finally arrive at the X the spatial component being nothing but a superposition of sine and cosine function with two constants A's and B's and two more constants parameters small a and small b.

So the general solution to the differential equations can be given by -

X(x) = A.cos(ax) + B.sin(bx) and **T(t) = C.cos(ct) + D.sin(dt)**

Then by use of appropriate boundary condition one can find out the constants - **A, B, a, b, C, D, c** and **d.**

The point to note here is **X(x)** and **T(t)** both varies sinusoidally. That means our function **u(x,t)** will also vary sinusoidally. Since the sinusoidal variation represents a wave. Thus, the different equation

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{v^2} \cdot \frac{\partial^2 u}{\partial t^2}$$

Is called wave equation. *As the general solution to this leads us to a solution where **u(x,t)** behaves like a wave.*

The previous equation can be simplified to

$$\frac{1}{X(x)} \cdot \frac{\partial^2 X(x)}{\partial x^2} = \frac{1}{T(t)v^2} \cdot \frac{\partial^2 T(t)}{\partial t^2} = K$$

where, **K** is the separation constant thus we have,

$$\frac{d^2X(x)}{dx^2} - KX(x) = 0 \qquad \frac{d^2T(t)}{dt^2} - Kv^2T(t) = 0$$

Now by considering the differential equation on the left we get -

$$X(x) = A \cdot \cos(ax) + B \cdot \sin(bx)$$

By using the boundary conditions: $X(x = 0) = 0$

$X(x = \ell) = 0.$

Similarly the other one the time variable also will come with a few more constants and basically one cosine function and one time function they're both harmonic functions.

We get,

$$X(x) = B \cdot \sin\left(\frac{n\pi x}{\ell}\right)$$

where ℓ is the length of the boundary, **n**=1,2,3,...∞ , and **B** is a constant. And by substituting the value of **X(x)** in its differential equation one can easily get

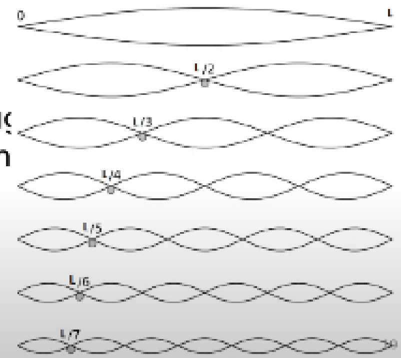
$$K = -\left(\frac{n\pi}{\ell}\right)^2$$

The solution of **X(x)** can be visualised as the imag given on the right. Now considering the equation

$$\frac{d^2T(t)}{dt^2} - Kv^2T(t) = 0$$

and by plugging in the value of **K**. we get,

$$T(t) = D\cos\left(\frac{n\pi\nu}{\ell}t\right) + E\sin\left(\frac{n\pi\nu}{\ell}t\right)$$



Now, let us consider **D = A cos(ϕ)** and **E = A sin(ϕ)**. Thus, we have

$$T(t) = A\cos(\phi)\cos\left(\frac{n\pi\nu}{\ell}t\right) + A\sin(\phi)\sin\left(\frac{n\pi\nu}{\ell}t\right)$$

By using the trigonometric property we can easily arrive at

$$T_n(t) = A_n\cos\left(\frac{n\pi\nu}{\ell}t + \phi_n\right)$$

where ϕ_n is known as phase number and A_n is known as the amplitude.

Okay so ultimately the U can be written as a cosine function of time and a sine function of X and that's the final solution,

$$u_n = A_n\cos(\omega_n t + \phi_n)\sin\left(\frac{n\pi x}{\ell}\right)$$