Radio Astronomy

Prof. Abhirup Datta

Department of Astronomy, Astrophysics, and Space Engineering

Indian Institute of Technology Indore

Lec-13

Signal Processing and Receivers

Hello, welcome to this fourth week of lectures on radio astronomy. This week particularly we will be focusing on signal processing and receivers. Now if you remember we discussed in the previous weeks like the nature of the radiation coming from the sky, thermal and non thermal origins. How do we detect and measure emissions from the sky, particularly in radio band using radio telescopes. In this context we have done the antenna radiation fundamentals. We also studied key radiation fundamental quantities like specific intensity, flux density of the cosmic sources.

In particularly we understood black body radiation and the Rayleigh-Jens limit. We also defined a few specific temperatures relevant to this particular course. One of them is the brightness temperature which is a product of the Rayleigh-Jens limit, directly proportional to the specific intensity. The sky temperature in the same line, the antenna temperature and the noise temperature.

The noise temperature here quantifies the amount of noise in a system and the sky temperature or the brightness temperature of the source quantifies the amount of signal present and is the interplay between signal to noise. And if you remember that because we are dealing with very very faint signals in case of radio astronomy, in fact we define the limit of one Jansky to be 10 to the minus 26 watts per meter square per Hertz. And the brightest radio source like the Sun in radio is about 10 to the power 6 and 8 Jansky, which still brings the amount to 10 to the power minus 20 to the minus 18 watts per meter square per Hertz. So the strength of the radio signal which we are looking for from the sky are extremely faint compared to any other electronic devices which we are using day to day life like our mobile phones or laptops etc. So the principle criteria over here is to detect the signal coming from the sky in this radio band with a significant signal to noise ratio with the amount of confidence that we can demarket that as a real signal and not just another interference.

So in order to do that we have to build larger telescopes, more sensitive telescopes with sufficient resolution and collecting area. We need to ensure that the maximum power transfer happens by impedance matching, we just did in last week. And we also need to have larger bandwidth of observations that means collecting more and more signals and that means signal noise ratio goes up. It also puts the design specification on the antenna that it should be broadband. This is what we have done so far.

What is this week's plan? This week's plan is to take you through a very basic blocks of signal processing. We know that in the beginning there is an antenna called feed here but then there are several stages of amplification, mixing IF amplifier, band pass filter, squalor detector and low pass filter and amplifier where the entire signal goes through this particular chain and then finally gets recorded in a computer. So we will deal with different kinds of signals. We will touch in a very major technique called Fourier transform. We'll talk about block diagrams of receivers, connectors, adapters, attenuators, filters, mixtures, mixtures, switches, amplifiers, etc.

How does a receiver is designed? We will talk about that and finally sampling theorem. So that's all we are going to cover this particular week. To start with the signal. A signal is basically what is coming from the sky. So why do we need to, we already have discussed the nature of the radiation by which the signals originate from the sources in the sky but we need to understand what kind of signals we are looking into.

What are the characteristics features in time and frequency so that we have to, we need to build an optimized receiver system to collect that signal. Like signal coming from, a signal may be peaking at a particular frequency. So to optimize the gain, we also need to design our telescope at that particular frequency. Suppose that spectral line is there at 1.4 gigahertz.

We need to design a telescope which will operate at 1.4 gigahertz and not 2 gigahertz. Similarly we know that the synchrotron radiation becomes more and more stronger at lower frequencies. So we need to collect that at a lower frequency. We would like to see the red shifted hydrogen signal at from a particular rate shift.

So we need to build our telescope at a particular frequency band corresponding to that red shifted frequency. Now some of this you may be familiar with, some you may not be familiar with. Don't worry, we will discuss as we go along. So anyway for this particular discussion, we need to know the nature of the signal in order to make a receiver which is candidate it more effectively. So anything we receive which can, we receive which can convey information to us with or without processing is a signal.

For example, we listen to this particular lecture. You can perceive it so this is a signal for you. More formally, we can define a signal to be an electromagnetic wave received over a period of time through an antenna which is converted into electrical time varying voltages. Ideally should only contain transmissions from the source under observation. However, there because we are receiving, we receive any signal coming in that particular band from that particular direction and it can be the sky signal which we desired to collect but as well as we can collect some interfering signal which will be detrimental to our detection and so that we have to remove from our data in order to detect the actual signal we want to collect in the first place.

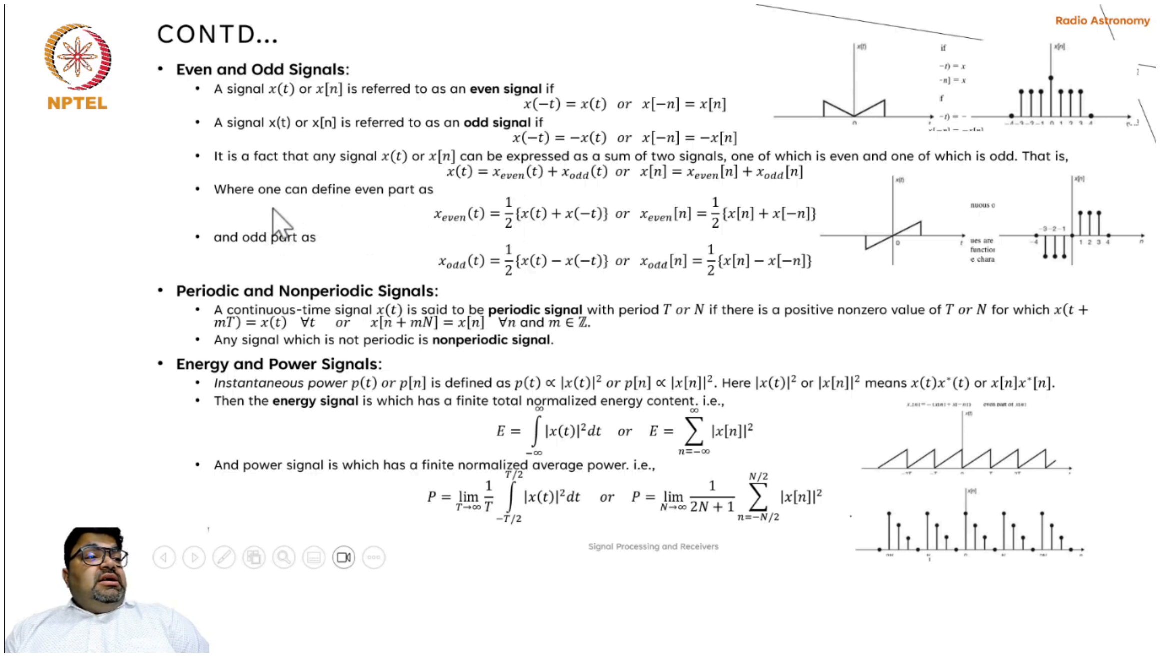
It is to be noted that it is not as simple to retrieve information from the received signal because it is contaminated by different noises, measurement errors and interferences. So like as I said, there are radio frequency interference so we are collecting a particular band for example, say 1.4 gigahertz but there can be spurious transmissions nearby which also may come in in the nearby frequencies which can spill over because of non-ideal or imperfect transmission modes and they create noise for our reception. So we receive the sky signal of course but we also receive in addition some spurious signals which are not from the sky but also but mostly from the neighborhood and man-made. We will discuss and come and discuss natures of those kind of interferences one by one in in future weeks.

We need to understand that the different types of signals and characteristics before designing a suitable receiver for it. That's the reason we need to understand nature of the signals. Broadly signals can be classified into seven different types. Continuous time and discrete time signals as the name suggests continuous time is a continuously varying with time and discrete time signal is discretized over time. Analog and digital signals if a continuous time signal x t can take on any value in the continuous interval a to b where a may be minus infinity or a finite start time and b may be plus infinity within a finite time then the continuous time signal x t is called an analog signal.

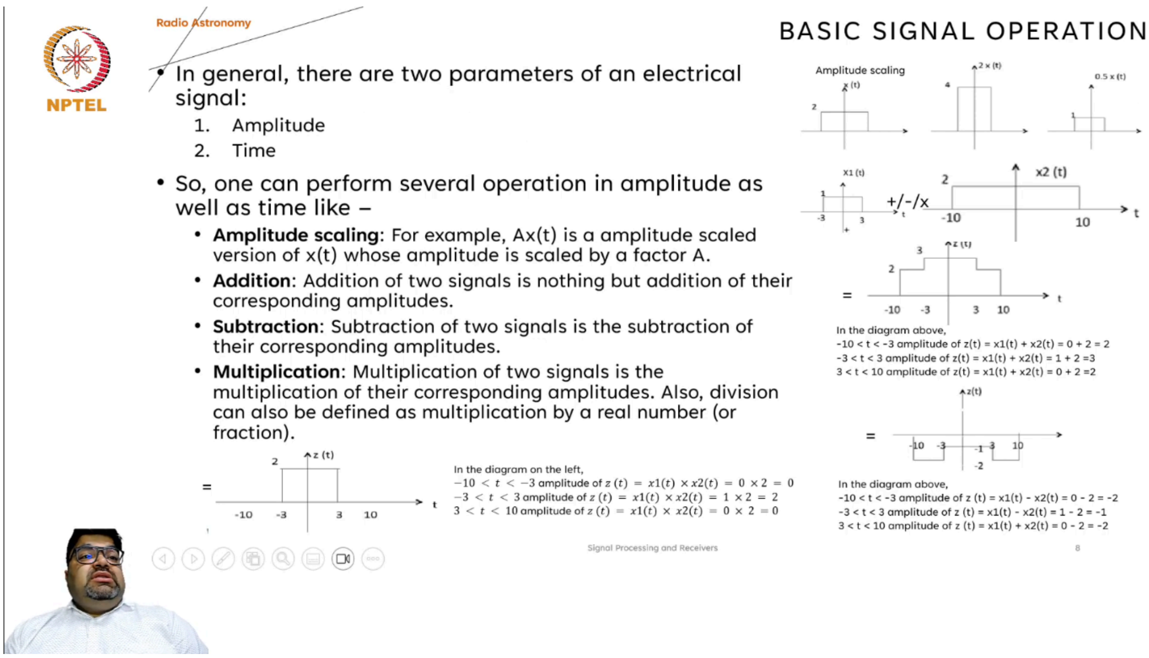
On the other hand if a discrete time signal x n can take on only a finite number of distinct values then we call this signal as a digital signal. For example a discrete signal voltage versus time can only have two volt three volt four volt values and nothing else. Real and complex signals third type a signal x t is real if it's real valued it's complex if it is composed of complex components both real and imaginary parts. Deterministic and random signals. Deterministic signal are those signals whose values are completely specified for a given time and random signals it has to be stochastically determined or statistically characterized.

Even and odd signal x t or x n is referred to as an even signal if x minus t is equal to x of t or x minus n is equal to x n. A signal is referred to as odd if x minus t is equal to minus of x t and x minus n is equal to minus of x n. It is a fact that any signal x t or x n can be expressed as sum of two signals one of which is even and one of which is odd.

So in general they are characterized by sum of odd and even signals. One can define even parts as half of x t plus x of minus t by two and in this case x n and x of minus n by two and similarly the odd part.

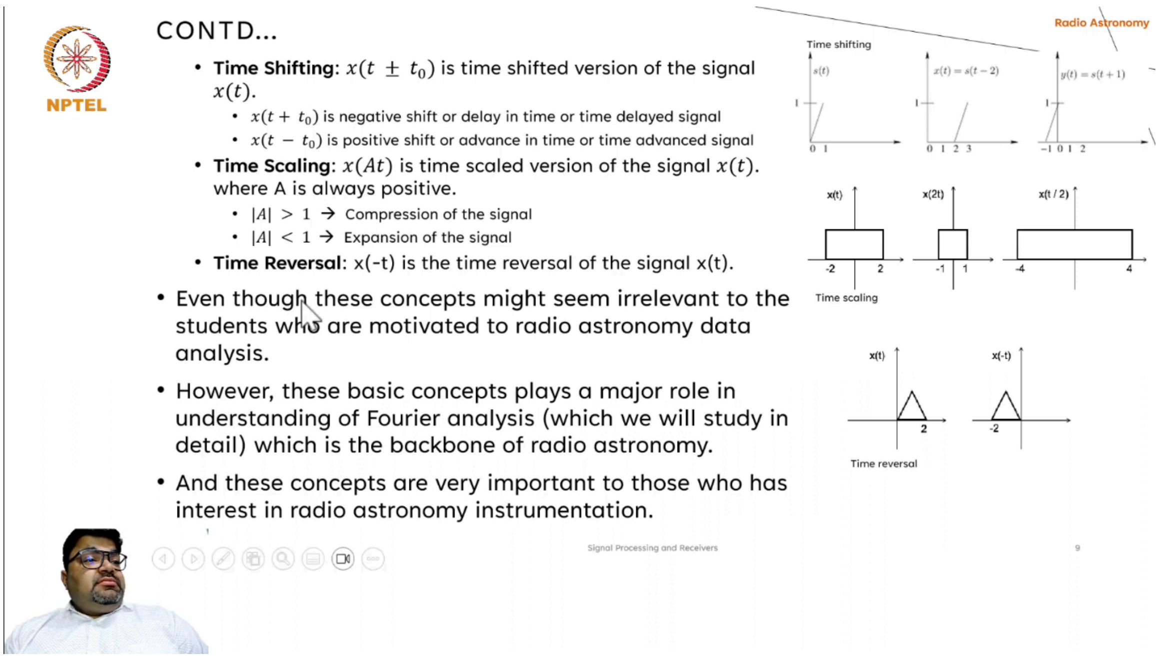


After that there can be signals which are periodic and non-periodic. A continuous time signal x of t is said to be periodic signal with period t or n if there is a positive non-zero value t or n for which x t plus m t is equal to x of t. So if the signal repeats itself in value after a certain period then it can be called as a periodic signal if not then it is a non-periodic signal. Energy and power signals instantaneous powers p t or p n is defined as p of t proportional to x t squared and p of n is x n squared. Then energy signal can be given as the total normalized energy content and power signal can be total finite normalized average power given by this.



So those are the basic signals in which we know of. So whatever we will observe from the sky will be classified in one of them and that way the operations becomes easier. So how the basic signal operations define in general there are two parameters for electrical signal amplitude and time. So one can make perform several operation in amplitude as well as time like amplitude scaling addition of two signals with corresponding amplitude subtraction of two signals multiplication of two signals time shifting shift the time um so x t plus t naught is negative shift or delay in time or delay time time delayed signal and x t minus t naught is a positive shift or advancement in time or time advanced signal. Time scaling x a of t is a time-scaled version of the signal x of t a can be greater than one corresponding to called compression of signal less than one corresponding to the expansion of signal.

Time reversal x of minus t is the time reversal of the signal x of t. Even though this concept might be seem irrelevant to the students who are motivated to radio astronomy data analysis this basic concepts play a major role in understanding what we teach next which is Fourier analysis that's why this basic constructs can can really help you to go forward with the Fourier series and Fourier transforms. So signals we defined now we define what is a system. A system is a mathematical model or a physical process that relates the input signal to the output signal in that case even a radio telescope is also a system because it there's an input signal which is coming in receiving in the in the front in the in the antenna and it passes through a lot of processing and finally gets out and recorded in the computers. So this whole block is a system even the individual blocks are also subsystems which play their own role.



Let x and y be the input and output signals of a system then the system is viewed as a transformation or mapping of x into y. So x is the input y is the output h is the some kind of an operator which is defining the system which operates on the signal itself either receives it amplifies it or does some operations like filtering mixing etc which will come soon. Okay the different kinds of systems deterministic and stochastic systems if x and y are deterministic or stochastic signals the system can be continuous time and discrete time systems where a and y are continuous or discrete time signals respectively. Systems with memory and without memory the system is said to be memoryless if the output at any given time depends only on the input at that particular time and it has no other memory then the system is called memoryless and otherwise it system is supposed to have a memory. So let's see what a memoryless system looks like for example ohms law where the voltage which is the output is proportional to the input current and linked with the system of the resistance so the r here is like an operator which takes the input signal i t and transforms that into a voltage v which is a function of t but the i voltage at a given time depends on the current at that given time only and not at a previous time or at later time that's why this system is called memoryless.

Correspondingly if i introduce a capacitor instead of a resistor in a capacitor we know it charges and then discharges so current flowing through it charges and then it doesn't flow it discharges so that itself has a memory which is a memory from the past so it it depends on the time other also in addition to the current time it also depends on the time which has passed so that's why it is a system which refers to having a memory. Causal and non-causal systems a system is called causal if its output at the present time depends on only the present and or past values of the input thus in a causal system it is not possible to obtain an output before an input is applied to the system. A system is called non-causal if its output at the present time depends on future values of the input example of non-causal systems are y t is equal to x t plus one or y n is equal to x of minus n so all memoryless systems are causal but not vice versa that's the definition of all the systems we will be coming across continuing that uh we define something called linear and non-linear systems if h where h is the transforming the input signal of x into output signal of y if h satisfied the following two conditions like additivity h of x plus x 1 plus x 2 is equal to y 1 plus y 2 and homogeneity of scaling h of alpha times x is equal to alpha y then h of alpha x 1 plus alpha 2 x 2 is equal to alpha 1 y 1 plus alpha 2 y 2 if these two conditions are satisfied then h is called a linear operator and a system represented by a linear operator is also called a linear system and if it is not then it's called a non-linear operator okay so supposedly h of x 1 plus x 2 is y 1 plus y 2 and h of alpha 1 x 1 plus alpha 2 x 2 but alpha and alpha 2 are just scaling is equal to alpha 1 y 1 and alpha 2 y 2 then h is called a linear operator or also a linear system if not then it is called a non-linear operator or non-linear system the next in the definition of systems are time invariant and time varying systems a system is called time invariant if a time shift in the input signal causes the same type shift in the output signal so if you have h of x t minus tau that is giving of y t minus tau this is called time invariant and if it doesn't satisfy this link relationship and then it's called time varying system linear time invariant systems or lti is a system which is linear and also time invariant then it is called lti or linear time invariant systems in addition to this we have we know about stable systems which system is bounded input bounded output stable if for any bounded input x defined by modulus of x is less than equal to k 1 the corresponding output y is also bounded defined by the same y less than equal to k 2 here k 1 and k 2 are finite real constants an unstable system is the one in which not all bounded inputs lead to a bounded output last one feedback systems a special class of systems of great importance consist of systems having feedback in a feedback system the output signal is fed back and added to the input system like this for radio astronomical applications we will restrict our discussions to linear and linear time invariant system as a system designed for reception of radio waves are linear systems so the next topic for today's lecture is Fourier series so we will try to introduce the Fourier series and then gradually move over to something called Fourier transform so Fourier series as you know or may know that it was developed by Jean Baptist Joseph Fourier and this is the continuous the continuous Fourier transform is important in mathematics engineering and physical sciences it is probably the most useful tool for data analysis in engineering uh the its counterpart for discretely sampled functions is discrete Fourier transform which is normally computed using so called fast Fourier transform the DFT has revolutionized modern society as it is ubiquitous in digital electronics and signal processing it is almost everywhere in signal processing sound processing image processing lots of imaging techniques are based on radio Fourier transforms like radio interferometric imaging spectral information in astronomical signals are developed by Fourier transform Fourier spectroscopy spatial features in signals from early universe power spectrum by spectrum etc are all developed based on this single piece of work called Fourier transform Fourier series leads to that will come in a bit radio astronomers are particularly avid users of Fourier transforms because Fourier transforms are key component in data processing bigger DCT searches and instruments and they are the cornerstones of interferometry and aperture synthesis so to summarize Fourier transform is a very important tool not only for the radio astronomy but also for electrical communication engineering so a lot of signal processing tools arises from Fourier transform are based on it before we jump to Fourier transform let's first start with the Fourier series as you said so we represent any periodic signal xt Joseph Fourier developed an expression called Fourier series this is in terms of an infinite sum of sines and cosines or exponentials Fourier series uses orthogonality condition Fourier series representation of continuous time periodic signals Fourier series representation of continuous time periodic signals so signal is said to be periodic if it satisfies the condition x of t is equal to x of t plus capital t where t is the time period capital t is a time period similarly in discrete signals x of n is the x of n plus capital n where n defines the period there are two basic periodic signals sines and cosine and also complex exponentials the signals periodic with period of t capital t is defined by 2 pi over omega naught where signal itself is xt is defined as cosine of omega naught t or xt is defined as e to the power j omega naught t where j squared is equal to minus 1 a set of harmonically related complex exponentials can be represented as phi of k t which involves e to the power j omega naught t or that omega naught can also be represented in terms of capital t by k times 2 pi over capital t so k can take any integral values positive or negative and zero according to the orthogonal signal space approximation of a function xt with n mutually orthogonal functions is given by x of t is is summation over k goes to minus infinity to plus infinity ak e to the power jk omega naught t where ak is the Fourier coefficient also known as the coefficient of approximation the signal x of t is also periodic with time period of capital t this equation represents Fourier series representation of periodic signal xt the term k is equal to 0 is a constant plus minus 1 having fundamental frequency of omega naught is called the first harmonic plus minus 2 having twice omega naught is second harmonic and so and so so the nth harmonic is k is equal to plus minus n so you can represent x of t a periodic signal in terms of infinite sums of periodic cosines and cosines so any periodic signal then can be decomposed into this summation over uh sines of cosines that's the Fourier series supposedly given a function then how will you derive this Fourier coefficient because that is the only thing you have to figure out

- A signal is said to be periodic if it satisfies the condition

$$x(t) = x(t+T) \text{ or } x(n) = x(n+N).$$

- Here T is the fundamental time period, ω_0 is fundamental frequency and is equal to $2\pi/T$.
- There are two basic periodic signals:

$$x(t) = \cos\omega_0 t \text{ (sinusoidal) \& } x(t) = e^{j\omega_0 t} \text{ (complex exponential)}$$

- These two signals are periodic with period $T = 2\pi/\omega_0$.
- A set of harmonically related complex exponentials can be represented as $\{\phi_k(t)\}$

$$\phi_k(t) = \{e^{j\omega_0 t}\} = \{e^{jk\left(\frac{2\pi}{T}\right)t}\} \qquad \text{where } k = 0, \pm 1, \pm 2 \ldots\ldots n \ldots\ldots$$

- All these signals are periodic with period T.

in order to do the representation so then what you do is if you simply take go from here you simply multiply by e to the power minus j the conjugate of this exponential term on both sides you get x of t e to the power minus j omega j n omega naught t and this side you basically end up having a of k because this cancels out okay hence you essentially then do a integral on both sides for sorry it doesn't cancel out it's k and this is n n so k minus n term survives sorry and so if you integrate over both sides the a of k is independent of the variable t that comes out of the integral and the int inside term is which remains is e to the power j k minus n omega naught t okay zero to t so you basically let's go back and start again so you have this function x of t given in some summation over k goes to minus infinity plus infinity a of k a k e to the power minus e to the power j k omega naught t you simply multiply both sides by e to the power minus j n omega naught t and so it is and then you do a integral on both sides from zero to capital t on the right hand side uh left hand side this thing remains as it is on the right hand side you can see that a k and the sum is a function of k and not t so they come out of the integral living inside the integral the integrand e to the power j k minus n omega naught t now you can um write e to the power j theta in terms of cosine theta plus j sine theta there is a Euler's formula if you do that then um this is the you come up with e to the power j k minus n omega naught t is cosine of k minus n omega naught t dt plus j the sine integral the we also examine that if k is equal to n then this integral goes to t if k is not equal to n then this goes to zero

- According to orthogonal signal space approximation of a function x (t) with n, mutually orthogonal functions is given by

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

- Where a_k is the Fourier coefficient also known as coefficient of approximation. This signal $x(t)$ is also periodic with period T.
- This equation represents Fourier series representation of periodic signal $x(t)$.
  - The term $k = 0$ is constant.
  - The term $k = \pm 1$ having fundamental frequency ω_0, is called as 1st harmonics.
  - The term $k = \pm 2$ having fundamental frequency $2\omega_0$, is called as 2nd harmonics.
  - The term $k = \pm n$ having fundamental frequency $n\omega_0$, is called as n^{th} harmonics.

Deriving Fourier Coefficient

- Multiply $e^{-jn\omega_0 t}$ on both sides to the equation above. Then we have

$$x(t)e^{-jn\omega_0 t} = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}\, e^{-jn\omega_0 t}$$

so uh we go back to that main equation and then we just put k is equal to n so a n t the the right hand side reduces to a n times t and the left hand side is still that integral so a n can be written in terms of the right left hand side integral and capital t that's how we get the value of um a k's uh which is done in the same way k and n are just arbitrary numbers so we do that and that's how we evaluate the uh coefficients of the fourier series coefficients of the fourier series now just take an example so how do you understand fourier series so you can take an arbitrary function f of x okay defined within the interval of point minus 0.5 to plus 0.5 and it can be represented by the series expansion of a 0 by 2 plus minus sorry um summation over n equal to 1 to infinity a n cosine of 2 pi n x plus b n sine of 2 pi n x okay in this particular case a n and b n are can be represented in terms of the function f x the similar way we are evaluating the the fourier components in the previous example and the set of basis functions cosine 2 pi n x and sine 2 pi n x are actually orthogonal so you have some orthogonality relationship by kronecker delta so delta m n equal to 1 when m is equal to n and 0 when m not equal to n if you generalize this interval from minus 0.5 to plus 0.5 to minus l by 2 to plus l by 2 the fourier series can be expressed in terms of f of x a 0 by 2 the first term of n equal to 0 then n equal to goes from 1 to infinity and a n cosine 2 pi n x by l and b n sine 2 pi n x by l where a n is given by this integral and b n by the same integral okay.

$$x(t)e^{-jn\omega_0 t} = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} e^{-jn\omega_0 t}$$

- By integrating both side we have

$$\int_0^T x(t)e^{-jn\omega_0 t}dt = \int_0^T \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} e^{-jn\omega_0 t}dt = \sum_{k=-\infty}^{\infty} a_k \int_0^T e^{j(k-n)\omega}$$

- And by using Euler's formula, $e^{j\theta} = cos\theta + jsin\theta$. So, we have,

$$\int_0^T e^{j(k-n)\omega_0 t}dt = \int_0^T \cos((k-n)\omega_0 t)\,dt + j\int_0^T sin((k-n)\omega_0 t)\,d$$

- Now we examine

So it can also be expressed in terms of the complex notation by consuming both the cosine and sine functions in this manner. So graphically you can see what is happening in this particular case there are this rectangular function which is become can be decomposed into several sine functions in this particular case. Next we come to the some properties of the fourier series any function f t can be represented by using fourier series only when the function satisfies Dirichlet conditions that is the function f t must be periodic and has finite number of maxima and minima there must be finite number of discontinuities in the signal f of t in the given time travel of time it must be integrable in the given period. So if it satisfies all these three Dirichlet conditions then that function can be represented by using fourier series.

- Now let again look at the following equation again

$$\int_0^T x(t)e^{-jn\omega_0 t}dt = \sum_{k=-\infty}^{\infty} a_k \int_0^T e^{j(k-n)\omega_0 t}dt$$

- Now, let put $k = n$, we can rewrite the equation as

$$\int_0^T x(t)e^{-jn\omega_0 t}dt = a_n T$$

Where, one can define a_n as $a_n = \frac{1}{T}\int_0^T e^{-jn\omega_0 t}dt$

Now, If we replace n by k we get $a_k = \frac{1}{T}\int_0^T e^{-jk\omega_0 t}dt$

Therefore, we can rewrite the equation $x(t)$ as

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{j(k-n)\omega_0 t}$$

Where, $a_k = \frac{1}{T}\int_0^T x(t)e^{-jk\omega_0 t}dt$

Consider an arbitrary function $f(x)$ defined in the interval $(-0.5,0.5)$. It can be represented by the series expansion

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{n=\infty} (a_n \cos 2\pi nx + b_n \sin 2\pi nx)$$

It can be represented by the series expansion

$$a_n = 2\int_{-1/2}^{1/2} f(x) \cos 2\pi nx \, dx,$$

$$b_n = 2\int_{-1/2}^{1/2} f(x) \sin 2\pi nx \, dx,$$

The set of **basis functions** $\{\cos 2\pi nx, \sin 2\pi nx\}$ are **orthogonal**:

$$\int_{-1/2}^{1/2} \cos 2\pi nx \ \sin 2\pi mx \, dx = 0,$$

$$\int_{-1/2}^{1/2} \cos 2\pi nx \ \cos 2\pi mx \, dx = \frac{1}{2}\delta_{nm},$$

$$\int_{-1/2}^{1/2} \sin 2\pi nx \ \sin 2\pi mx \, dx = \frac{1}{2}\delta_{nm}.$$

- **Basis functions** are building blocks for creating more complex functions. In other words, they are a set of k standard functions, combined to estimate another function—one which is difficult or impossible to model exactly.
- **Orthogonal basis functions** are basis function which are orthogonal to each other. That is their inner product result in zero $< x, y \geq \frac{1}{L}\int_{-L/2}^{L/2} x(k)y(k)dk$.

where, Kronecker delta function is defined as,

$$\delta_{mn} = 1; \ for \ m = n$$
$$= 0; \ for \ m \neq n.$$

So, if we generalize this interval of $\{-L/2, L/2\}$, Fourier series can be expressed as,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{n=\infty} \left(a_n \cos \frac{2\pi nx}{L} + b_n \sin \frac{2\pi nx}{L}\right)$$

with,

$$a_n = \frac{2}{L}\int_{-L/2}^{L/2} f(x) \cos \frac{2\pi nx}{L} \, dx,$$

$$b_n = \frac{2}{L}\int_{-L/2}^{L/2} f(x) \sin \frac{2\pi nx}{L} \, dx,$$

It can also be expressed in complex notation,

$$f(x) = \sum_{n=-\infty}^{n=\infty} \tilde{a}_n \, e^{\frac{i2\pi nx}{L}}$$

$$\tilde{a}_n = \frac{1}{L}\int_{-L/2}^{L/2} f(x) \, e^{\frac{-i2\pi nx}{L}} \, dx$$

$\tilde{a}_n$ is a complex quantity. Notice that there are negative frequencies as well.

- Any function f(t) can be represented by using Fourier series only when the function satisfies **Dirichlet's conditions.** i.e.
  - The function f(t) must be periodic and has finite number of maxima and minima.
  - There must be finite number of discontinuities in the signal f(t),in the given interval of time.
  - It must be integrable in the given period i.e., $\int_{-\infty}^{\infty} |x(t)| dt < \infty$ or $\int_a^b |x(t)| dt < \infty$

- **Linearity Property**: if $x(t) \leftrightarrow f_{xn}$ & $y(t) \leftrightarrow f_{yn}$ are the Fourier series pair, then the linear property states that $ax(t) + by(t) \leftrightarrow af_{xn} + bf_{yn}$
- **Time Shifting Property**: If $x(t) \leftrightarrow f_{xn}$ are the Fourier series pair, then the time shifting property states $x(t - t_0) \leftrightarrow e^{-jn\omega_0 t_0} f_{xn}$
- **Frequency Shifting Property**: If $x(t) \leftrightarrow f_{xn}$ are the Fourier series pair, then the frequency shifting property states $e^{-jn\omega_0 t_0} x(t) \leftrightarrow f_{x(n-n_0)}$
- **Time Reversal Property**: If $x(t) \leftrightarrow f_{xn}$ are the Fourier series pair, then the time reversal property states

$$x(-t) \leftrightarrow f_{-xn}$$

Other properties of fourier series linearity property x of t is f of x n and y of t is can be represented by the fourier series then the summation of the two multiplied by a a x and b y can be represented by the following fourier series. Time shifting property if x of t can be in the fourier series pair and then the time shifting property state the x of t minus t 0 can be represented e to the power minus j n omega naught t 0 times f of x n. Frequency shifting property if x of t and f of x n are fourier pair then frequency shifting essentially works in this manner so e to the power j n omega naught t 0 x of t is a fourier pair of f of x n minus n naught. Time reversal property x of minus t and f of minus x f f of minus x n are also the fourier pairs. There are other time scaling properties differentiation and integration properties multiplication and convolution properties and conjugate and conjugate symmetry properties they're just given for the completion.

- **Trigonometric Fourier Series (TFS)**
  - If $\sin(n\omega_0 t)$ and $\sin(m\omega_0 t)$ are orthogonal over the interval $(t_0, t_0 + 2\pi/\omega_0)$. So, $\sin(\omega_0 t)$ and $\sin(2\omega_0 t)$ forms an orthogonal set. This set is not complete without $\{\cos(n\omega_0 t)\}$ because this cosine set is also orthogonal to sine set. So, to complete this set we must include both cosine and sine terms. Now the complete orthogonal set contains all cosine and sine terms i.e. $\{\sin(n\omega_0 t), \cos(n\omega_0 t)\}$ where $n = 0, 1, 2 \ldots$
  - So, any function $x(t)$ in the interval $(t_0, t_0 + T,\ where\ T = 2\pi/\omega_0)$ can be represented as
    $$x(t) = a_0 \cos(0\omega_0 t) + a_1 cos(1\omega_1 t) + a_2 cos(2\omega_2 t) + \ldots + a_n cosn(n\omega_n t) + \ldots + b_0 sin(0\omega_0 t) + b_1 sin(1\omega_1 t) + \ldots + b_n sin(n\omega_n t) + \ldots$$
    $$= a_0 + a_1 cos(1\omega_1 t) + a_2 cos(2\omega_2 t) + \ldots + a_n cosn(n\omega_n t) + \ldots + b_1 sin(1\omega_1 t) + \ldots + b_n sin(n\omega_n t) + \ldots$$
  - Therefore, TFS is represented as
    $$x(t) = a_0 + \sum_{n=1}^{\infty} (a_n \cos(n\omega_n t) + b_n \sin(n\omega_n t)) \quad where\ (t_0 < t < t_0 + T)$$
  - where,

$$a_0 = \frac{\int_{t_0}^{t_0+T} x(t) \cdot 1 dt}{\int_{t_0}^{t_0+T} 1^2 dt} = \frac{1}{T} \cdot \int_{t_0}^{t_0+T} x(t) dt \qquad a_n = \frac{2}{T} \cdot \int_{t_0}^{t_0+T} x(t) \cdot \cos n\omega_0 t\, dt \qquad b_n = \frac{2}{T} \cdot \int_{t_0}^{t_0+T} x(t) \cdot \sin n\omega_0 t\, dt$$

Similar properties are also going to be valid we will see something similar coming up for the fourier transforms as well. Types of fourier series there is trigonometric fourier series if sine n omega naught t and sine m omega naught t are orthogonal over the interval t naught and t naught plus 2 pi by omega naught. So sine omega naught t and sine 2 omega naught t forms an orthogonal set this set is not complete without cosine of n omega naught t because this cosine set is also orthogonal to the sine set. So to complete this set we must include both cosine and sine terms. Now the complete orthogonal set contains all cosine and sine terms that is sine n omega naught t and cosine n omega naught t where n goes to 0 1 2 to infinity.

- **Exponential Fourier Series (EFS)**
  - Consider a set of complex exponential functions $\{e^{jn\omega_0 t}\}$ $(n = 0, \pm 1, \pm 2 \ldots)$ which is orthogonal over the interval $(t_0, t_0 + T)$. Where $T = 2\pi/\omega_0$. This is a complete set so it is possible to represent any function f(t) as shown below

$$f(t) = F_0 + F_1 e^{j1\omega_0 t} + F_2 e^{j2\omega_0 t} + \cdots + F_n e^{jn\omega_0 t} + \cdots F_{-1} e^{-j1\omega_0 t} + F_{-2} e^{-j2\omega_0 t} + \ldots + F_{-n} e^{-jn\omega_0 t} + \ldots$$

  - Therefore,

$$f(t) = \sum_{n=-\infty}^{\infty} F_n e^{jn\omega_0 t} \quad where\ (t_0 < t < t_0 + T)$$

  - The above equation represents exponential Fourier series representation of a signal $f(t)$ over the interval $(t_0, t_0 + T)$. The Fourier coefficient is given as

$$F_n = \frac{1}{T} \int_{t_0}^{t_0+T} f(t) e^{-jn\omega_0 t} dt$$

- **Relation Between Trigonometric and Exponential Fourier Series**
  - If we compare TFS and EFS we will arrive at the following relation

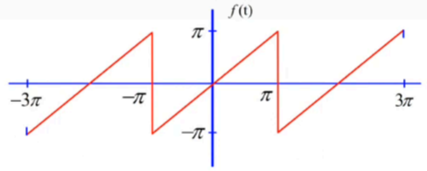
$$a_0 = F_0 \qquad F_n = \frac{1}{2}(a_n - jb_n)$$

$$a_n = F_n + F_{-n} \qquad F_{-n} = \frac{1}{2}(a_n + jb_n)$$

$$b_n = j(F_n - F_{-n})$$

So any function x of t in the interval of t naught and t naught plus capital t where capital t is given by 2 pi over omega naught can be represented by this expansion. Therefore x of t can be written as a naught plus summation over n equal to 1 to infinity a n cosine of n omega n t and b n sine n omega n t where t goes from t 0 to t 0 plus capital t where capital t is 2 pi over omega naught. In this case a can be can be represented can be evaluated by 1 over t t naught to t naught plus t x t dt a n can be in a similar way by represented by xt cosine of n omega naught t and b n as xt sine of n omega naught t dt integral of t naught to t naught plus t 2 pi over t in the front. There can be similarly Fourier's exponential Fourier series where you can express a arbitrary function in terms of exponential series and you can the Fourier coefficients can be given by 1 over t x into integral of t 0 plus t 0 t 0 2 t 0 plus t f of t to the power minus j n omega naught t dt. So let's look at an example consider f of t is the sawtooth wave as shown in the figure below and we have to find its Fourier series.

Consider f(t) is the saw-tooth wave as shown in figure below and we have to find its FS.



From the figure it is evident that $f(t) = t, \quad -\pi < t < \pi$ (6)

$$f(t + 2\pi) = f(t)$$

So, by using trigonometric FS we have

$$a_o = \frac{1}{\pi} \int_{-\pi}^{\pi} t \, dt = 0$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} t cos(nt)$$

$$= \frac{1}{n^2\pi} (cos(nt) + ntsin(nt))|_{-\pi}^{\pi} = 0$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} t \sin(nt) dt$$

$$= \frac{1}{n^2\pi} (sin(nt) - nt\cos(nt))|_{-\pi}^{\pi}$$

$$= -\frac{2\cos(n\pi)}{n} = -\frac{2(-1)^{n+1}}{n}$$

Therefore, $f(t) = 2(\frac{\sin t}{1} - \frac{\sin 2t}{2} + \frac{\sin 3t}{3} - \cdots)$

So the figure is evidently from the forms the function f of t is equal to is t small t where t goes from minus pi to plus pi and f of t plus 2 pi is equal to f of t so that 2 pi is the time period. So by using trigonometric Fourier series we have a naught 1 over pi integral of t dt minus pi to plus pi that is equal to 0 a n is 1 over pi t cosine n t dt that is given by the following and similarly the b n. So if we look into the a n and b n evaluation we can then go back and formulate the f of t as 2 times sine t by 1 minus sine 2 t by 2 so on. So in the previous example we just showed one example of a function and how it can be expressed in terms of the Fourier series where we have a function of sawtooth wave and we evaluated the the Fourier coefficients and then expressed the function in terms of the series. Before we move on we just recap what we have learned so far.

Radio Astronomy

FOURIER TRANSFORM

Deriving Fourier transform from Fourier series

- Consider a periodic signal $x(t)$ with period T. The complex Fourier series representation of $x(t)$ is given as

$$x(t) = \sum_{n=-\infty}^{\infty} a_k e^{jk\omega_0 t} = \sum_{n=-\infty}^{\infty} a_k e^{jk\frac{2\pi}{T_0}t}$$

- Now we let $\frac{1}{T_0} = \Delta f$, where Δf change in frequency. $a_k = \frac{1}{T_0}\int_{t_0}^{t_0+T} x(t)e^{-jk\omega_0 t}dt$

$$x(t) = \sum_{k=-\infty}^{\infty} \frac{1}{T_0} \int_{t_0}^{t_0+T} x(t)e^{-jk\omega_0 t}dt\, e^{j2\pi k\Delta f t}$$

We can also rewrite this as (for $t_0 = -T/2$)

$$x(t) = \sum_{k=-\infty}^{\infty} \left[\int_{-T/2}^{T/2} x(t)e^{-jk\omega_0 t}dt \right] e^{j2\pi k\Delta f t} \cdot \Delta f$$

CONTD...

Radio Astronomy

Now if we limit $T \to \infty$ then $\Delta f \to df$, using this we will arrive at

$$x(t) = \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} x(t)e^{-j2\pi f t}dt \right] e^{j2\pi f t} \cdot df$$

This can be written as

$$x(t) = \int_{-\infty}^{\infty} X[\omega]e^{j2\pi k\Delta f t} \cdot df$$

This $X[\omega] = \int_{-\infty}^{\infty} x(t)e^{-jk\omega_0 t}dt$ which is also known as Fourier transform and the above equation is known as Inverse Fourier transform:

Thus, we have,

$$\mathcal{F}\{x(t)\} = X[\omega] = \int_{-\infty}^{\infty} x(t)e^{-jk\omega_0 t}dt\text{, and}$$

$$\mathcal{IFT}\{X[\omega]\} = x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X[\omega]e^{jk\omega_0 t}d\omega$$

So, we can denote Fourier Pairs as $x(t) \leftrightarrow X[\omega]$ or $f(t) \leftrightarrow F[\omega]$.

So we learned in order to detect the faint signals of the sky from the sky it is critical to understand the nature and the characteristics of the signal. We learned about the signature of signals and the systems and on in our learning of the signal processing techniques the first one we learned is the Fourier series and we will be coming to the Fourier transform. Any periodic signal can be represented by the Fourier series. The main drawback of Fourier series is that it is only applicable to periodic signals.

It doesn't work on a periodic signals. So to overcome this shortcoming Fourier developed mathematical model to transform signals between time or spatial domain to frequency domain and vice versa which is called Fourier transform. Now Fourier transform has many applications in physics and engineering such as analysis of LTI system, radar, astronomy and signal processing etc. So how to derive Fourier transform from the Fourier series? Consider a periodic signal x of t with period of t. The complex Fourier series representation of x of t is given in terms of the following. x of t is summation over a k e to the power j k omega naught t.

Now let 1 over t naught is given as delta f, delta f is the change of frequency. Then we can define the a k the Fourier coefficients in terms of the integral as we did before. So a x of t then if you substitute the value of a k in the original definition of x of t you get the following expression. So now we take the limit of t tends to infinity or delta f then goes to df or infinitesimal frequency interval. In that case x of t goes to integral of minus infinity to plus infinity then within that another integral of x of t in e to the power minus j 2 pi f t dt and then there's a e to the power j 2 pi f dt outside as well.

Realize this the central one is actually a um or can be noted as x of omega because this is the Fourier transform of x of t. This x omega is nothing but a Fourier transform of x t e to the power minus j omega naught t dt which is also known as the Fourier transform and the above equation is known as inverse Fourier transform. So that's how we derive arrive at the definition of the Fourier transform itself and x of t and capital x of omega are Fourier pairs similar like any other arbitrary function of time and frequency. So discrete time Fourier transform

$$X[\omega] = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

$$IDTFT\{X[\omega]\} = x(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X[\omega]e^{j\omega n}d\omega$$

apart from the Fourier transform applicable to continuous signal it can also applicable to discrete time signal. So for a continuous time signal it's called continuous time Fourier transform discrete time signal it's called discrete time Fourier signal transform sorry.

So like it's defined like this if a sequence dt of the sequence x of n is given by x of omega integral of n equal to goes from minus infinity to plus infinity x n e to the power minus j omega n okay. So capital x can be broken up into real and imaginary part because it's a complex quantity. Their x real and x imaginary are the real and imaginary part. They can also be broken into cosine theta and sine theta. They follow the simple quadrature rule of x omega whole square.

$$DFT\{x(n)\} = X[k] = \sum_{n=0}^{N-1} x[n]e^{-jk\omega_N n}$$ where $k = 0,1,2,\ldots,N-1$ and $\omega_N = \frac{2\pi}{N}$

$$IDFT\{X[k]\} = x(n) = \frac{1}{N}\sum_{k=0}^{N-1} X[k]e^{-jk\omega_N n}$$

The norm of that is equal to amplitude of x real square plus amplitude of x imaginary square. They can also be represented as x of omega as x of omega mod e to the power minus j theta omega. The inverse dt of t is also defined as x omega as x of n as 1 over 2 pi integral of minus infinity to plus infinity capital x omega e to the power j omega n d omega. Discrete Fourier transform now if we observe the equations of c t f t and dt of t that is continuous time Fourier transform and discrete time Fourier transform. The continuous time signal has infinite number of data points for any delta t.

This makes impossible to store data in a digital system or some record. This makes c t f t impossible to implement on a machine. Discrete time signal has definite number of data points in a time delta t. This makes a suitable candidate from the perspective of storage. However, the t f t of a discrete time signal results in a continuous spectra in the frequency domain thereby making it impossible to be implemented on a machine for storage issues.

As a result, discrete Fourier transform was also defined where in the inverse space or the frequency space was also discretized for a discrete signal input. DFT is defined as discrete Fourier transform of x of n is x of k where n goes from 0 to n minus 1 x of n e to the power minus j k omega n n where cap k goes from 0 to n minus 1 and omega n is equal to pi by n capital n. Here the Fourier transform is calculated for a set of n equally spaced discrete frequencies or samples. Hence the sequence x of k is called discrete Fourier transform of the sequence small x of small n. That inverse Fourier transform is also defined as i d f t x of k that is given by x of n as 1 over n k 0 to n minus 1 x of k e to the power minus j k omega n small n.

So just take a look at the Dirichlet conditions. The function x t has infinite number of maxima and minima. There must be a finite number of discontinuities in the signal f of t in the given interval of time. It must be absolutely integrable in the given interval of time or sequence x t must be bounded. Now Parseval's relationship states that total energy may be determined by either computing energy per unit time x of t squared and integrating over total time or by computing energy per unit frequency and integrating over all the frequencies. So either you would compute this in one Fourier domain or the other and they're often related as energy density spectrum.

$$\int_{-\infty}^{\infty}|x(t)|^2dt < \frac{1}{2\pi}\int_{-\infty}^{\infty}|X[j\omega]|^2d\omega$$

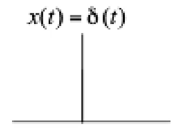
If $x(t) \leftrightarrow X[j\omega]$ & $y(t) \leftrightarrow Y[j\omega]$ are the Fourier transform pair, then

- **Linearity Property:** $ax(t) + by(t) \leftrightarrow aX[j\omega] + bY[j\omega]$
- **Time Shifting Property:** $x(t - t_0) \leftrightarrow e^{-j\omega_0}X[j\omega]$
- **Frequency Shifting Property:** $e^{-j\omega_0}x(t) \leftrightarrow X[j(\omega - \omega_0)]$
- **Time Reversal Property:** $x(-t) \leftrightarrow X[-j\omega]$
- **Time Scaling Property:** $x(at) \leftrightarrow \frac{1}{|a|}X[j\omega/a]$.
- **Differentiation Properties:** $\frac{dx(t)}{dt} \leftrightarrow j\omega X[j\omega]$
- **Integration Properties:** $\int_{-\infty}^{t} x(\tau)d\tau \leftrightarrow \frac{X[j\omega]}{j\omega} + \pi X[0]\delta(\omega)$.
- **Multiplication Properties:** $x(t) \cdot y(t) \leftrightarrow X[j\omega] * Y[j\omega]$
- **Convolution Properties:** $x(t) * y(t) \leftrightarrow X[j\omega] \cdot Y[j\omega]$.
- **Conjugate Properties**## **:** $x^*(t) \leftrightarrow X^*[j\omega]$
- **Conjugate Symmetry Properties**## **:** $X^*[j\omega] \leftrightarrow X[-j\omega]$

Few properties of Fourier transform linearity property a x t plus b y of t is equal to a the Fourier conjugate. Time shift property we have already seen in the Fourier series the same goes from here x t minus t 0 is just a phase multiplied by the Fourier transform of the x of t. Similarly there's a frequency shift property exactly similar time reversal property time scaling property differentiation integration properties multiplication convolution conjugate and conjugate symmetry properties are all tabulated over here. So let's take a few examples because we have been discussing about Fourier series and Fourier transform a lot.

Let's take a bit of examples so that it may be more clear. So first one what is the Fourier transform of x of t given by a delta function.

Question 1: Find FT of x(t)=δ(t).



Solution: $X(j\omega) = \int_{-\infty}^{\infty} \delta(t) e^{-j\omega t} dt = 1$

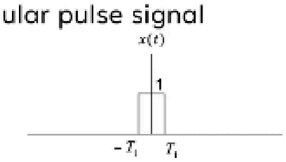
That is, the impulse has a Fourier transform consisting of equal contributions at all frequencies.

So it's just like an impulse. So if you do the math it will come out to be a constant in this case one. It's a very powerful transform. But what it says is a impulse in one domain is a constant broadband dc signal in the other domain.

We will come back this thing particularly in in we will definitely be using this in several derivations but also we're coming back to this particularly in terms of interference when we say when we come to the radio frequency interference signal we will come back to this particular property and refer to this property quite often. The other one is question number two we take a Fourier transform of a rectangular pulse signal where it is equal to one between minus t one and plus t one and goes to zero elsewhere.

Question 2: Calculate the Fourier transform of the rectangular pulse signal

$$x(t) = \begin{cases} 1, & |t| < T_1 \\ 0, & |t| > T_1 \end{cases}$$



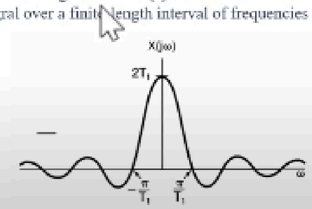
Solution: $X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt = \int_{-T_1}^{T_1} 1 e^{-j\omega t} dt = 2\frac{\sin \omega T_1}{\omega}$

Now, the Inverse Fourier transform is $\hat{x}(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} 2\frac{\sin \omega T_1}{\omega} e^{j\omega t} d\omega$

Since it is a IFT then we should get the rectangular pulse back so we can define e(t) such that $e(t) = \int_{-\infty}^{\infty} |x(t) - \hat{x}(t)|^2 dt = 0$

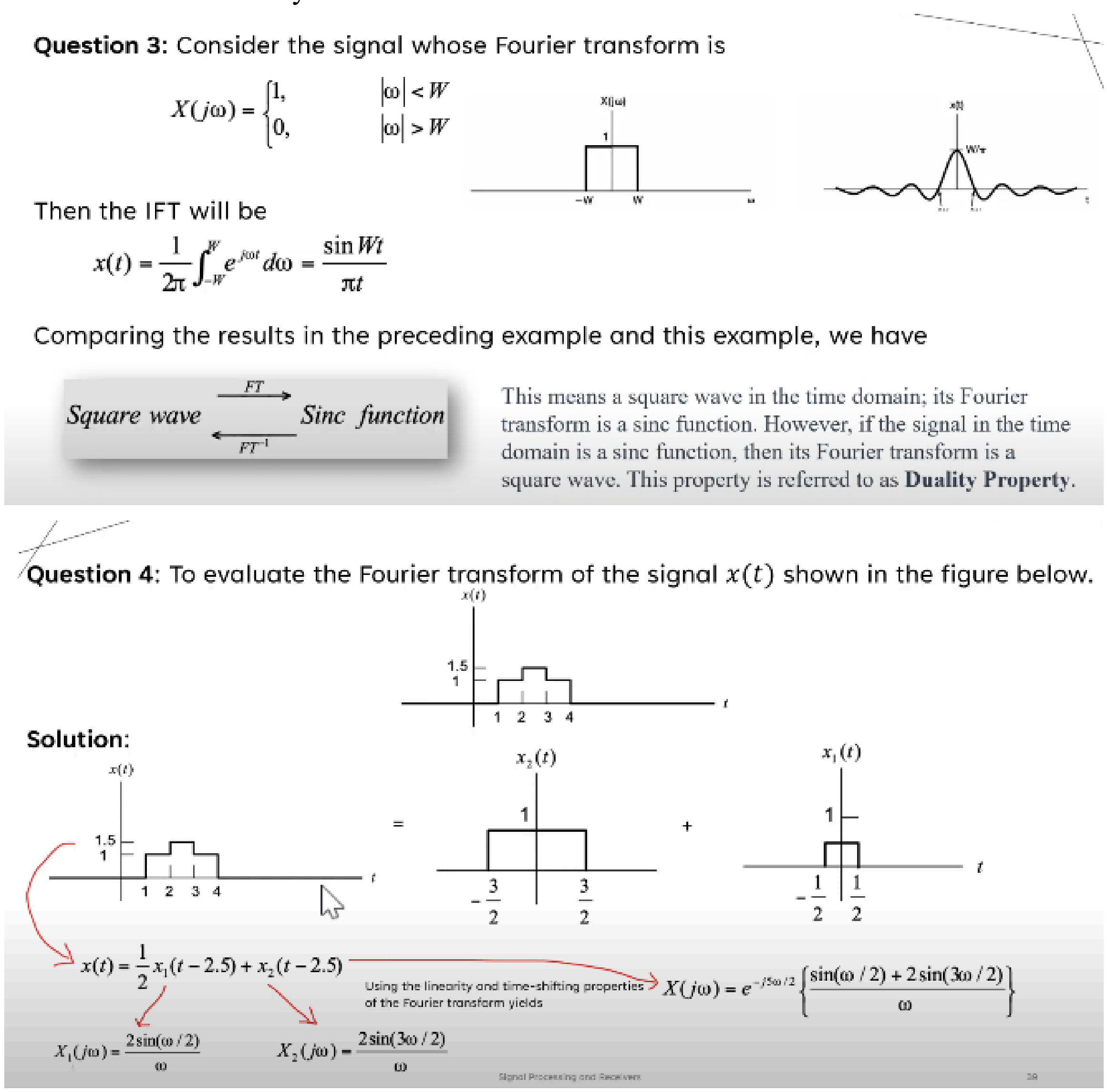
$\hat{x}(t)$ converges to $x(t)$ everywhere except at the discontinuity, $t = \pm T_1$, where $\hat{x}(t)$ converges to $1/2$, which is the average value of $x(t)$ on both sides of the discontinuity. In addition, the convergence of $\hat{x}(t)$ to $x(t)$ also exhibits **Gibbs phenomenon**. Specifically, the integral over a finite length interval of frequencies

$$\frac{1}{2\pi}\int_{-W}^{W} 2\frac{\sin \omega T_1}{\omega} e^{j\omega t} d\omega$$



As $W \rightarrow \infty$, this signal converges to $x(t)$ everywhere, except at the discontinuities. Moreover, the signal exhibits ripples near the discontinuities. The peak values of these ripples do not decrease as W increases, although the ripples do become compressed toward the discontinuity, and the energy in the ripples converges to zero.

If you do that you will find that it's kind of sine omega t one over omega kind of a function. However there is a little bit of a discrepancy in exactly the discontinuity near minus t one and plus t one. So if you go back and forth you see there's an abrupt change in the function caused by the this truncation.



So there's something called the Gibbs phenomena which gives rise to this alternating sinusoids which goes to extends to infinity. Hence it gives rise to the sinc kind of a function which you say. You can you can also derive the same thing so a square wave and the sinc function are kind of Fourier transform of each other and this is called a duality property. In this question we have a composite of composite of several square waves and so if you do the solution using linearity and time shifting properties you finally can derive the Fourier transform of the given function as x j omega e to the power minus j five omega by two times sine omega by two plus two sine of three omega by two by omega. So you have to use both the properties the linearity property that Fourier transform of a times f x plus b times g x is equal to a times capital f omega and b times capital g omega.

So if you follow that property like we discussed earlier just let's go back and show if you use this linear property and time shift property you can basically get the result of this particular question number four because they are composites of multiple rectangular waves. Find the discrete the next example is find the discrete time Fourier transform x n given by one three minus two and five.

Question 5: Find the discrete-time Fourier transform $x(n) = \{1, 3, -2, 5\}$.

Solution:

The DTFT of a sequence is defined as –

$$F[x(n)] = X(\omega) = \sum_{n=-\infty}^{\infty} x(n)e^{-j\omega n}$$

$$\Rightarrow X(\omega) = x(0) + x(1)e^{-j\omega} + x(2)e^{-j2\omega} + x(3)e^{-j3\omega} + x(4)e^{-j4\omega}$$

$$\therefore X(\omega) = 1 + 3e^{-j\omega} - 2e^{-j2\omega} + 5e^{-j3\omega} + 2e^{-j4\omega}$$

So the DTFT of a sequence is defined as given by this so if you do the do the series expansion you finally get you the solution given by x of omega. The question number six let us try to find the discrete Fourier transform of the following f of t is given by five plus two of cosine two pi t minus 90 degree plus three of cosine four pi t. Okay so if you do that DFT is defined by this expression f of n is summation of a small f of k e to the power minus j two pi over n is one n times k.

Question 6: let us try to find DFT of the following

$$f(t) = \underbrace{5}_{\text{dc}} + \underbrace{2\cos(2\pi t - 90^o)}_{\text{1Hz}} + \underbrace{3\cos 4\pi t}_{\text{2Hz}}$$

Solution: DFT is defined as

$$F[n] = \sum_{k=0}^{N-1} f[k]e^{-j\frac{2\pi}{N}nk} \quad (n = 0 : N-1)$$

We can rewrite this in matrix form as

$$\begin{pmatrix} F[0] \\ F[1] \\ F[2] \\ \vdots \\ F[N-1] \end{pmatrix} = \begin{bmatrix} \omega_N^{0\cdot 0} & \omega_N^{0\cdot 1} & \cdots & \omega_N^{0\cdot(N-1)} \\ \omega_N^{1\cdot 0} & \omega_N^{1\cdot 1} & \cdots & \omega_N^{1\cdot(N-1)} \\ \vdots & \vdots & \ddots & \vdots \\ \omega_N^{(N-1)\cdot 0} & \omega_N^{(N-1)\cdot 1} & \cdots & \omega_N^{(N-1)\cdot(N-1)} \end{bmatrix} \begin{pmatrix} f[0] \\ f[1] \\ f[2] \\ \vdots \\ f[N-1] \end{pmatrix}$$

where,

$$\omega_N^{nk} = -j\frac{2\pi}{N}nk$$

Now, let us sample at 4 times per second (ie. $f_s = 4Hz$) from $t = 0$ to $t = 3/4$. The values of the discrete samples are given by:

$$f[k] = 5 + 2cos(\tfrac{\pi}{2}k - 90^o) + 3cos\pi k \qquad \text{by putting } t = kT_s = \tfrac{k}{4}$$

$$\text{Therefore} \quad F[n] = \sum_{0}^{3} f[k]e^{-j\frac{\pi}{2}nk} = \sum_{k=0}^{3} f[k](-j)^{nk}$$

$$\begin{pmatrix} F[0] \\ F[1] \\ F[2] \\ F[3] \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{pmatrix} \begin{pmatrix} f[0] \\ f[1] \\ f[2] \\ f[3] \end{pmatrix} = \begin{pmatrix} 20 \\ -j4 \\ 12 \\ j4 \end{pmatrix}$$

Because,

$$f[0] = 8, f[1] = 4, f[2] = 8, f[3] = 0,$$

So if you do the the the discrete Fourier transform you finally be able to derive the values of capital f's given the value of the small f's and in f equal to zero is equal to eight f of one equal to four f of two equal to eight and f of three equal to zero if you substitute that you finally get your answer. Some other standard Fourier pairs the signum function is represented by sine of t is equal to one for t equal to greater than zero and minus one for t less than zero. So it's just a sine function and that you can express the x of t in terms of that. So uh where ut is also defined as one when t greater than equal to zero and zero for t less than zero. So if you apply the Fourier transform x of omega is given by xt e to the power minus j omega t dt which is replace the x of t by the sine function so it breaks up into two different functions because of the sign change for t greater than zero and t less than zero and if you do that math you finally end up having an expression with a limit.

SOME STANDARD FOURIER PAIRS

The signum function is represented by $sgn(t)$ and is defined as,

$$sgn(t) = \begin{cases} 1 \; for \; t > 0 \\ -1 \; for \; t < 0 \end{cases}$$

We can also represent $sgn(t)$ as

$$x(t) = e^{-a|t|}sgn(t); \;\; a \to 0$$

Or, $$x(t) = sgn(t) = \lim_{a \to 0} e^{-a|t|}sgn(t)$$

$$\Rightarrow x(t) = \lim_{a \to 0}[e^{-at}u(t) - e^{at}u(-t)]$$

Where,

$$u(t) = \begin{cases} 1 \; for \; t \geq 0 \\ 0 \; for \; t < 0 \end{cases}$$

Now applying FT,

$$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\,dt = \int_{-\infty}^{\infty} sgn(t)e^{-j\omega t}\,dt$$

$$\Rightarrow X(\omega) = \int_{-\infty}^{\infty} \left(\lim_{a \to 0}[e^{-at}u(t) - e^{at}u(-t)]\right) e^{-j\omega t}\,dt$$

$$\Rightarrow X(\omega) = \lim_{a \to 0}\left[\int_{-\infty}^{\infty} e^{-at}e^{-j\omega t}u(t)dt - \int_{-\infty}^{\infty} e^{at}e^{-j\omega t}u(-t)dt\right]$$

$$\Rightarrow X(\omega) = \lim_{a \to 0}\left[\int_{0}^{\infty} e^{-(a+j\omega)t}dt - \int_{-\infty}^{0} e^{(a-j\omega)t}dt\right]$$

$$\Rightarrow X(\omega) = \lim_{a \to 0}\left[\int_{0}^{\infty} e^{-(a+j\omega)t}dt - \int_{0}^{\infty} e^{-(a-j\omega)t}dt\right]$$

$$\Rightarrow X(\omega) = \lim_{a \to 0}\left\{\left[\frac{e^{-(a+j\omega)t}}{-(a+j\omega)}\right]_0^{\infty} - \left[\frac{e^{-(a-j\omega)t}}{-(a-j\omega)}\right]_0^{\infty}\right\}$$

$$\Rightarrow X(\omega) = \lim_{a \to 0}\left\{\left[\frac{e^{-\infty} - e^{0}}{-(a+j\omega)}\right] - \left[\frac{e^{-\infty} - e^{0}}{-(a-j\omega)}\right]\right\}$$

$$= \lim_{a \to 0}\left[\frac{1}{(a+j\omega)} - \frac{1}{(a-j\omega)}\right]$$

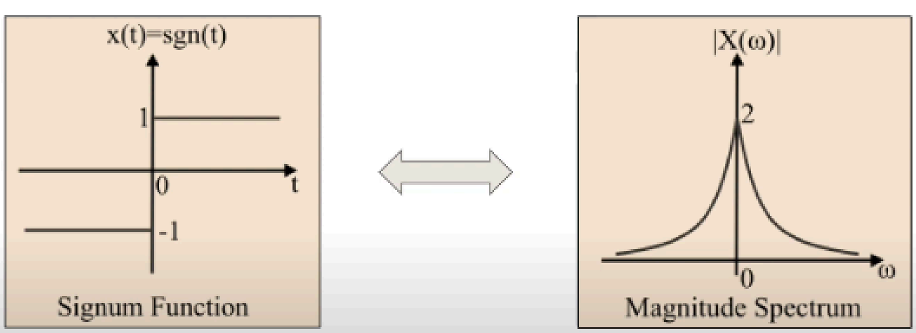
By solving this we have

$$\Rightarrow X(\omega) = \frac{1}{j\omega} - \frac{1}{(-j\omega)} = \frac{2}{j\omega}$$

SOME STANDARD FOURIER PAIRS

Therefore, the Fourier transform of the signum function is,

$$\boxed{X(\omega) = F[sgn(t)] = \frac{2}{j\omega}}$$



UNIT FUNCTION

The unit step function is defined as,

$$u(t) = \begin{cases} 1 \, for \, t \geq 0 \\ 0 \, for \, t < 0 \end{cases}$$

In order to find the Fourier transform of the unit step function, express the unit step function in terms of signum function as:

$$u(t) = \frac{1}{2} + \frac{1}{2}\mathrm{sgn}(t) = \frac{1}{2}[1 + \mathrm{sgn}(t)]$$

Now, from the definition of the Fourier transform, we have,

$$F[u(t)] = X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt = \int_{-\infty}^{\infty} u(t)e^{-j\omega t}dt$$

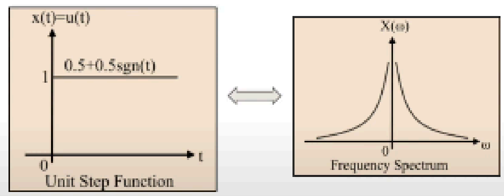
$$\Rightarrow X(\omega) = \int_{-\infty}^{\infty} \frac{1}{2}[1 + \mathrm{sgn}(t)]e^{-j\omega t}dt$$

$$\Rightarrow X(\omega) = \frac{1}{2}\left[\int_{-\infty}^{\infty} 1 \cdot e^{-j\omega t}dt + \int_{-\infty}^{\infty} \mathrm{sgn}(t) \cdot e^{-j\omega t}dt\right]$$

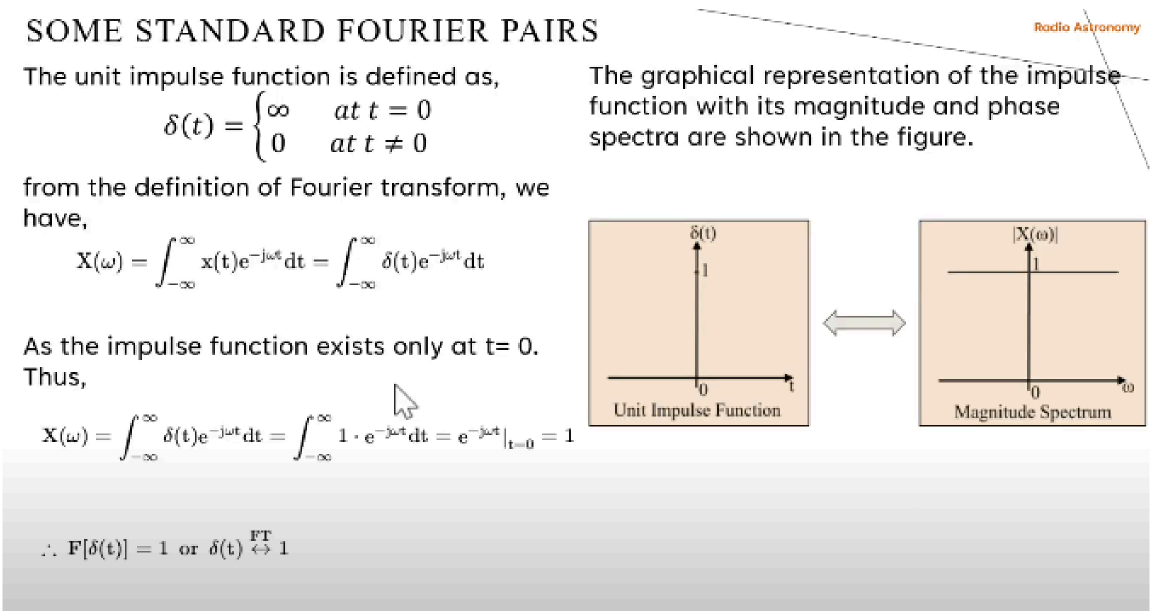
$$\Rightarrow X(\omega) = \frac{1}{2}\{F[1] + F[\mathrm{sgn}(t)]\}$$

The Fourier transform of the constant amplitude and the signum function is given by,

$$F[u(t)] = \left(\pi\delta(\omega) + \frac{1}{j\omega}\right)$$



So if you put the limit you finally get x of omega is two over j omega. So in if you draw the graph this is how the signum function looks like less than zero it is minus one and greater than zero it is plus one so the step function if you do the x of omega it looks like this. Another one is the unit function so it's one for t greater than equal to zero and zero for t less than zero so if the this is the plot of that function if you do the Fourier transform it goes to f of ut is equal to pi delta omega plus one over j omega so it looks like something like this. We have already shared this standard Fourier transform pair as a first example so a delta function goes to infinity for t equal to zero and it becomes zero for all others it's like a standard detect delta function and if you do the Fourier transform of that you finally go to one so that's a constant signal. A Gaussian function if you take a Gaussian function like this with a width e to the power minus a t square you finally get back a Gaussian function with the width equal to root over pi by a. Similarly you can get you can see exponential function e to the power minus a t times ut which is the unitary function in the step signal if you do that you finally get the magnitude of x of omega is one over square root of a square plus omega square for l values of omega so a one-sided exponential function it goes to this kind of a function.



There are some other list of Fourier pairs for your reference and use. We can also see a constant goes to delta function a box curve function goes to a sinc a triangle function goes to a sinc square function a Gaussian goes to a Gaussian but interesting feature is that a Gaussian with a wider width goes to a narrower width Gaussian in the Fourier domain and vice versa a wider Gaussian in the Fourier domain transforms into a narrower so this lighter curve in the center is a Fourier transform of this broader curve which is also in light and the darker curve in this in this plot is a Fourier transform of the wide Gaussian in the Fourier conjugate space.

GAUSSIAN FUNCTION

For a continuous-time function x(t) , the Fourier transform of x(t) can be defined as,

$$X(\omega) = \int_{-\infty}^{\infty} x(t)\ e^{-j\omega t}\ dt$$

Gaussian Function - The Gaussian function is defined as,

$$g_a(t) = e^{-at^2};\ \text{for all } t$$

Therefore, from the definition of Fourier transform, we have,

$$X(\omega) = F\left[e^{-at^2}\right] = \int_{-\infty}^{\infty} e^{-at^2}\ e^{-j\omega t}\ dt$$

$$\Rightarrow X(\omega) = \int_{-\infty}^{\infty} e^{-(at^2 + j\omega t)}\ dt = e^{-(\omega^2/4a)} \int_{-\infty}^{\infty} e^{[-t\sqrt{a} + (j\omega/2\sqrt{a})]^2} dt$$

Let,

$$[t\sqrt{a} + (j\omega/2\sqrt{a})] = u$$

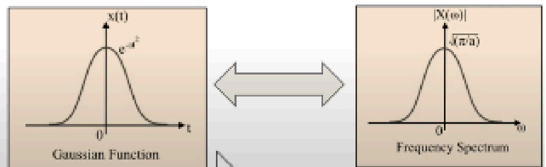
Then,

$$du = \sqrt{a}\ dt \text{ and } dt = \frac{du}{\sqrt{a}}$$

$$\therefore X(\omega) = e^{-(\omega^2/4a)} \int_{-\infty}^{\infty} \frac{e^{-u^2}}{\sqrt{a}}\ du = \frac{e^{-(\omega^2/4a)}}{\sqrt{a}} \int_{-\infty}^{\infty} e^{-u^2}\ du$$

$$\because \int_{-\infty}^{\infty} e^{-u^2}\ du = \sqrt{\pi}$$

$$\therefore X(\omega) = \frac{e^{-(\omega^2/4a)}}{\sqrt{a}} \cdot \sqrt{\pi} = \sqrt{\frac{\pi}{a}} \cdot e^{-(\omega^2/4a)}$$



SINGLE-SIDED REAL EXPONENTIAL FUNCTION

A single-sided real exponential function is defined as,

$$x(t) = e^{-at}u(t)$$

u(t)is the unit step signal and is defined as,

$$u(t) = \begin{cases} 1\ for\ t \geq 0 \\ 0\ for\ t < 0 \end{cases}$$

Then,

$$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt = \int_{-\infty}^{\infty} e^{-at}u(t)e^{-j\omega t}dt$$

$$\Rightarrow X(\omega) = \int_{0}^{\infty} e^{-at}e^{-j\omega t}dt$$

$$\Rightarrow X(\omega) = \int_{0}^{\infty} e^{-(a+j\omega)t}dt = \left[\frac{e^{-(a+j\omega)t}}{-(a+j\omega)}\right]_0^{\infty}$$

$$\Rightarrow X(\omega) = \frac{1}{-(a+j\omega)}[e^{-\infty} - e^{0}] = \frac{0-1}{-(a+j\omega)} = \frac{1}{a+j\omega}$$

Therefore, the Fourier transform of a single-sided real exponential function is,

$$F[e^{-at}u(t)] = \frac{1}{a+j\omega}$$

Magnitude and phase representation of the Fourier transform of a single-sided real exponential function. The Fourier transform of the one sided real exponential function is given by,

$$X(\omega) = \frac{1}{a+j\omega}$$

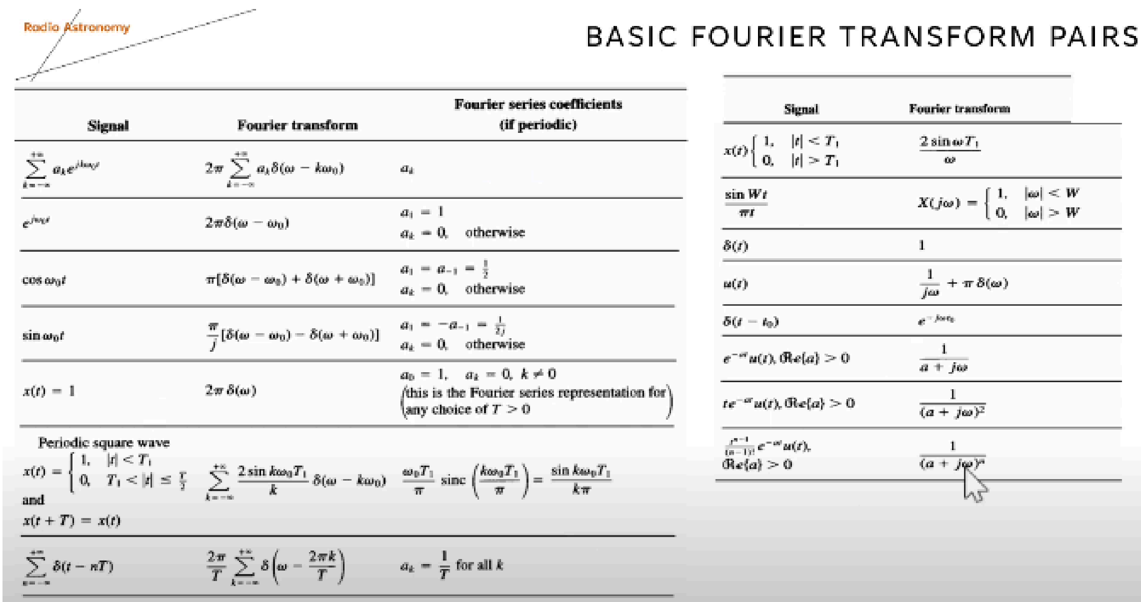
Multiplying it by the rationalizing factor, we get,

$$X(\omega) = \frac{a-j\omega}{(a+j\omega)(a-j\omega)} = \frac{a-j\omega}{a^2+\omega^2}$$

$$\Rightarrow X(\omega) = \frac{a}{a^2+\omega^2} - j\frac{\omega}{a^2+\omega^2} = \frac{1}{\sqrt{a^2+\omega^2}}\angle -\tan^{-1}\left(\frac{\omega}{a}\right)$$

Therefore, the magnitude and phase of Fourier series of single sided exponential function is given by,

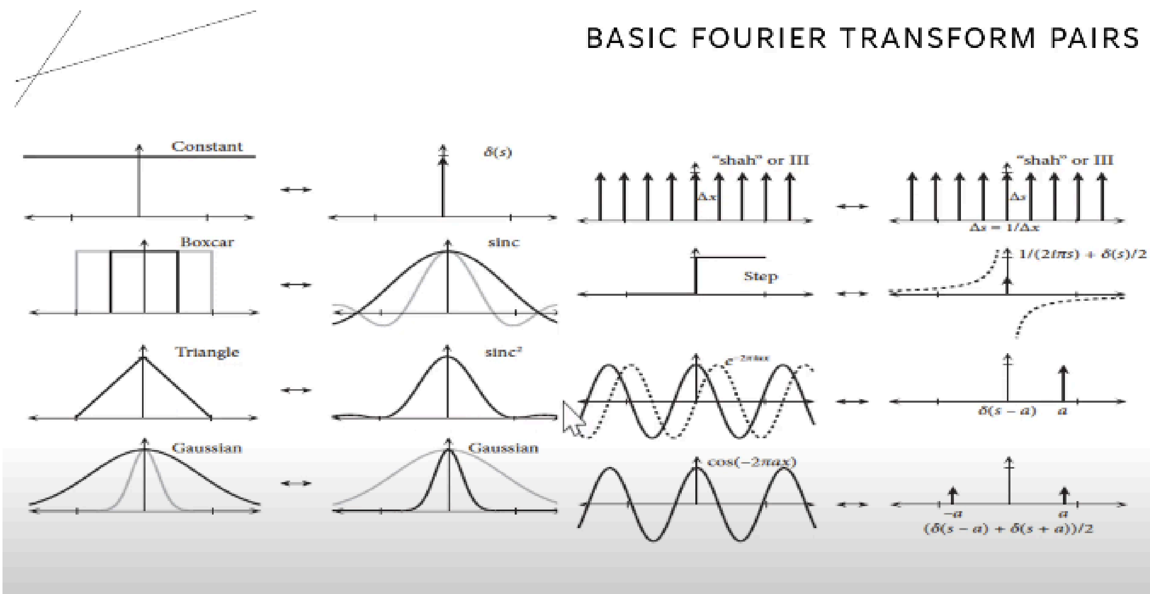
$$\text{Magnitude, } |X(\omega)| = \frac{1}{\sqrt{a^2+\omega^2}};\ \text{for all } \omega$$


Radio Astronomy

BASIC FOURIER TRANSFORM PAIRS

Signal	Fourier transform	Fourier series coefficients (if periodic)
$\sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}$	$2\pi \sum_{k=-\infty}^{+\infty} a_k \delta(\omega - k\omega_0)$	a_k
$e^{j\omega_0 t}$	$2\pi\delta(\omega - \omega_0)$	$a_1 = 1$ $a_k = 0$, otherwise
$\cos \omega_0 t$	$\pi[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)]$	$a_1 = a_{-1} = \frac{1}{2}$ $a_k = 0$, otherwise
$\sin \omega_0 t$	$\frac{\pi}{j}[\delta(\omega - \omega_0) - \delta(\omega + \omega_0)]$	$a_1 = -a_{-1} = \frac{1}{2j}$ $a_k = 0$, otherwise
$x(t) = 1$	$2\pi\,\delta(\omega)$	$a_0 = 1, \quad a_k = 0, \; k \neq 0$ (this is the Fourier series representation for any choice of $T > 0$)
Periodic square wave $x(t) = \begin{cases} 1, & \lvert t\rvert < T_1 \\ 0, & T_1 < \lvert t\rvert \leq \frac{T}{2} \end{cases}$ and $x(t + T) = x(t)$	$\sum_{k=-\infty}^{+\infty} \frac{2 \sin k\omega_0 T_1}{k} \delta(\omega - k\omega_0)$	$\frac{\omega_0 T_1}{\pi} \operatorname{sinc}\left(\frac{k\omega_0 T_1}{\pi}\right) = \frac{\sin k\omega_0 T_1}{k\pi}$
$\sum_{n=-\infty}^{+\infty} \delta(t - nT)$	$\frac{2\pi}{T} \sum_{k=-\infty}^{+\infty} \delta\left(\omega - \frac{2\pi k}{T}\right)$	$a_k = \frac{1}{T}$ for all k

Signal	Fourier transform
$x(t) \begin{cases} 1, & \lvert t\rvert < T_1 \\ 0, & \lvert t\rvert > T_1 \end{cases}$	$\frac{2 \sin \omega T_1}{\omega}$
$\frac{\sin Wt}{\pi t}$	$X(j\omega) = \begin{cases} 1, & \lvert\omega\rvert < W \\ 0, & \lvert\omega\rvert > W \end{cases}$
$\delta(t)$	1
$u(t)$	$\frac{1}{j\omega} + \pi\,\delta(\omega)$
$\delta(t - t_0)$	$e^{-j\omega t_0}$
$e^{-at}u(t), \Re e\{a\} > 0$	$\frac{1}{a + j\omega}$
$te^{-at}u(t), \Re e\{a\} > 0$	$\frac{1}{(a + j\omega)^2}$
$\frac{t^{n-1}}{(n-1)!} e^{-at}u(t)$, $\Re e\{a\} > 0$	$\frac{1}{(a + j\omega)^n}$



If you have a sinusoidal function with a single frequency then you get these two delta functions at those two frequency and minus one minus a and plus a and so and so. Fourier transform is very very important because this you can transform the time domain signal v of t into v of frequency you can also then transform the autocorrelation function in something called a power spectrum or spectral power density function if you do a Fourier transform so it's very much required in in terms of the signals and systems and we will also be using it more in the signal processing of radio astronomy as well. This brings us close to this particular lecture I hope you have enjoyed it as you know the disclaimer that we have not we have borrowed the materials from many many resources so a reference of them from where we have mainly borrowed the some some diagrams some segments expressions etc. So thanks for watching see you in the next lecture thank you very much.