Statistical Mechanics
Prof. Dipanjan Chakraborty
Department of Physical Sciences
Indian Institute of Science Education and Research, Mohali

Lecture - 54
Ideal Bose Gas

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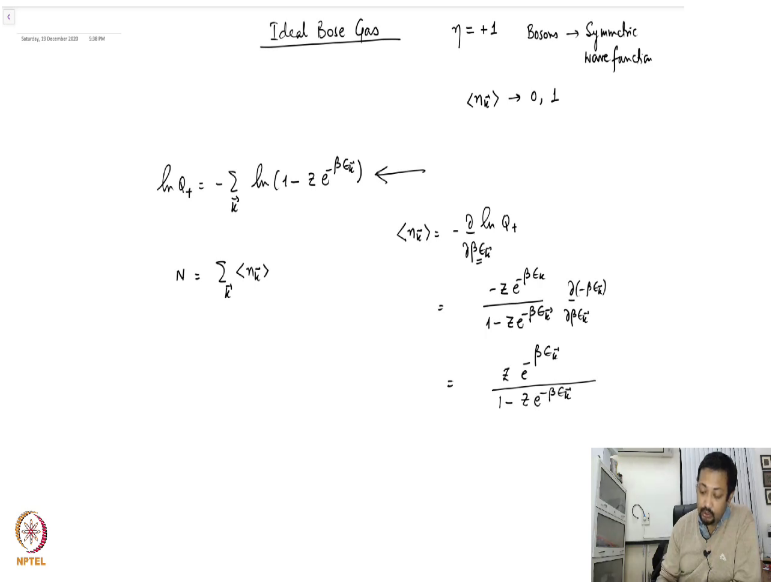

Welcome back. So, now, we want to look at an Ideal Bose Gas. We look at an ideal Fermi gas and now, we want to look at an ideal Bose gas. Recall that the variable eta takes a value plus 1 for a Bose gas and for a gas which is made up of bosons and for bosons, the wave function that we have are symmetric wave functions.

Not only that, in the Fermi gas, we saw that n k is restricted to be either 0 or 1 which is essentially the Pauli exclusion principle that states that not took more than one particle can have the same quantum number which means not more than one particle can occupy the same single particle level. Bose gas, bosons do not have these restrictions and therefore, these have some interesting implications as we will see later on when we go further into this.

Most importantly, you can see that since did not have these restrictions, more than one particle can come to a occupy the single particle level and therefore, it becomes interesting that as you keep on decreasing the temperature, the 0 energy state is very very special where you will see that all the particles go and sit over there, but this is what is called Bose Einstein condensation, but this is not really a condensation in a real space, it is a condensation in the momentum space. We will look at that later on.

Right now, our starting point is the grand canonical partition function which we had written down and which simply is minus k ln 1 minus Z e to the power minus beta epsa k. Now, the particle number if you recall was sum over n k and the average value of n k is del del beta epsa k minus of ln Q plus.

And if I do this, take this derivative, then essentially, I am taking the derivative with respect to one energy level and therefore, in the whole sum that you see in this ln Q plus, only one energy level will be picked up which means you are going to have 1 minus Z e to the power minus beta epsa k, then minus Z e to the power minus beta epsa k and then, del beta epsa k of minus beta epsa k.

So, that gives you Z e to the power minus beta epsa k divided by Z sorry 1 minus Z e to the power minus beta epsa k; 1 minus Z e to the power minus beta epsa k.

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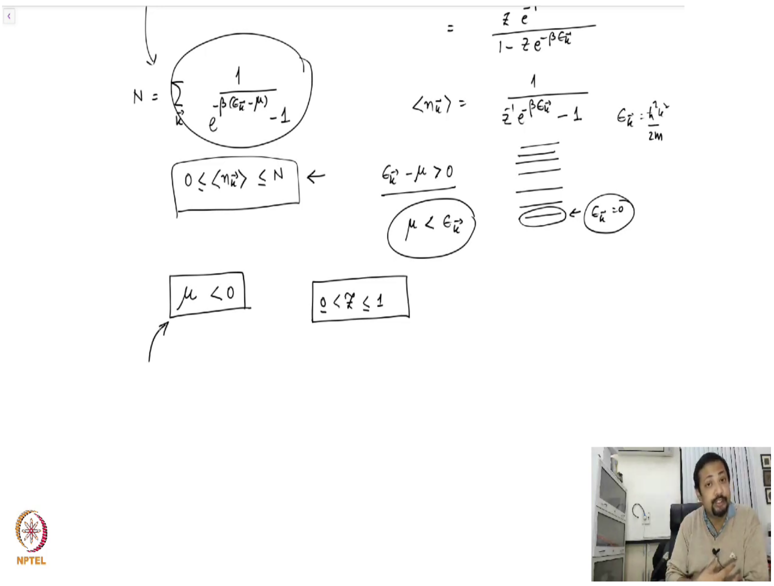

So, that you come up with the expectation value n k is equal to 1 over Z inverse e to the power minus beta epsa k minus 1 and this expression then takes the form N is sum over k 1 over e to the power minus beta epsa k minus mu minus 1 right. Now, clearly, you know that the restriction is n k is greater than 0 less than equal to the total particle number.

And it follows that if I look at this, now, since I have this restriction that occupation number in an energy level must be greater than e 0 less than equal to N right, it cannot be more than N of course, because you have a total of n number of particles. This effectively means if you look at this expression carefully, this effectively means that epsa k minus and this means that mu must be less than epsilon k, but this is true for any energy value. So, what should be the restriction on mu then?

Then one has to look at the energy levels and look at the lowest bound of this and the lowest bound of this corresponds to epsa k is equal to 0, the zero-momentum state because these are free particles right. So, I know that epsa k is equal to h square k square over twice m right.

So, therefore, it follows that since this restriction must be valid for any of the energy levels that I am considering, it must be also valid for the lowest energy level which is mean epsa k equal to 0. Therefore, it follows that for a Bose gas, mu is less than 0, the chemical potential of a Bose gas is less always less than 0 which means that the fugacity must be greater than equal to 0 less than equal to 1.

Now, this mu less than 0 is essentially, it is a reflection of the fact that the bosons feel an attractive potential. When we did the canonical formalism with bosons, we saw that the pair interaction was attractive right. So, it is simply statement of the fact that the chemical potential is less than 0 and adding more and more bosons to the ground state does not cost you any more energy right.

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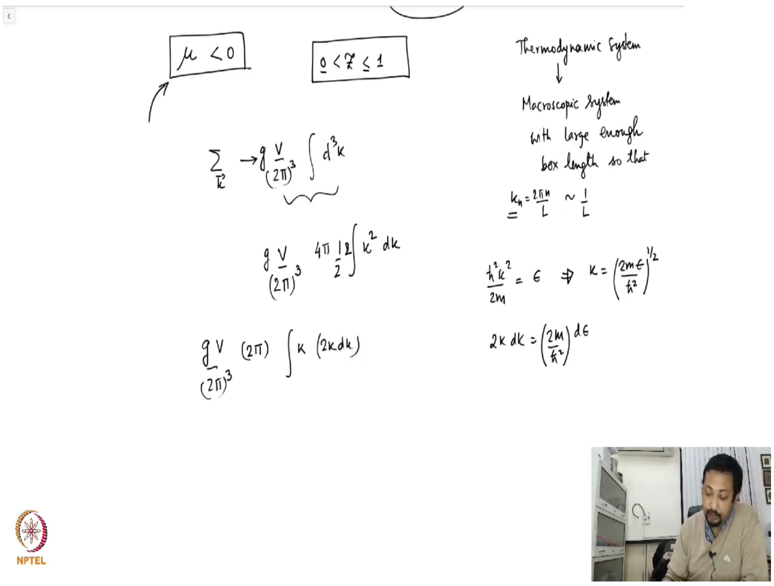

Now, how do we proceed? Well, the idea is simple, we have done it several times. So, what we write down is we replace the sum, we assume that we have a thermodynamic system which means it is a macroscopic system with a large enough box length so that my energy eigen values epsa goes as 1 over L.

Sorry the modes the discrete k modes that we saw that k n is 2 pi n over L, it goes as L, 1 over L and consequently, all the energy spectrum looks a continuous spectrum so that the discrete sum over k which is this can now be replaced by a continuous sum which we did for the fermionic system when we looked at ideal Fermi gas and that goes as d cube of k.

Now, here, I want to we did it in the Fermi for an ideal Fermi gas, we want to write down in a slightly different way, V there has to be a g factor over here that comes from the degeneracy, the internal degeneracy, internal degrees of freedom 2 pi whole cube times g V, this becomes 4 pi k square dk.

But h square k square is twice m sorry h square k square by twice m is epsilon. So, it follows that twice k dk is twice m over h bar square d epsa and the first equation also implies that k is twice m epsa over h bar square raised to the power half. So, I have g V over 2 pi whole cube 4 pi and then, I take a 2 divided by 2 and multiplied by 2 so that this becomes 2 pi and I have k 2k dk.

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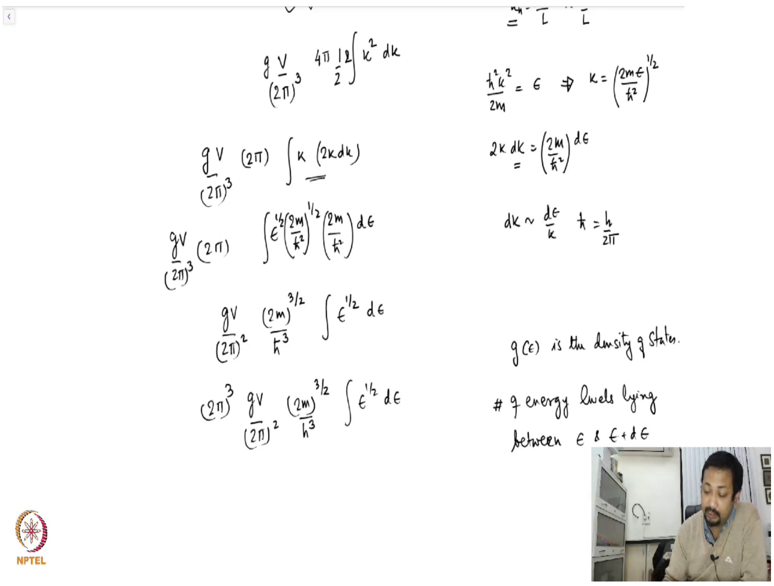

2k dk is straightforward, twice m over h bar square d epsa and this is straightforward twice m over h bar square half times epsa half and then, you have the integral gV over 2 pi whole cube times 2 pi. So, all the constants come outside the integral so that I have gV over 2 pi whole square and then, I have twice m raised to the power 3 by 2 and I have h bar whole cube right integral epsa half d epsa. This is further can be simplified by noting that h bar is h over 2 pi so that I have a 2 pi whole cube.

And this is 2 pi whole square times 2m raised to the power 3 by half divided by h cube integral epsa half d epsa so that this sum over k now goes to 2 pi gV over h cube times 2m 3 half integral epsa half d epsa. The reason I wanted to take this particular approach is that you will see that both of them are equivalent.

But because I wanted to highlight something over here, now this, I can write down g epsa d epsa where g epsa is the energy is the density of state which means the number of energy levels lying between epsilon and epsilon plus d epsilon is g epsilon d epsilon and for this particular case, this is going to be we will just write down this as epsa to the power half.

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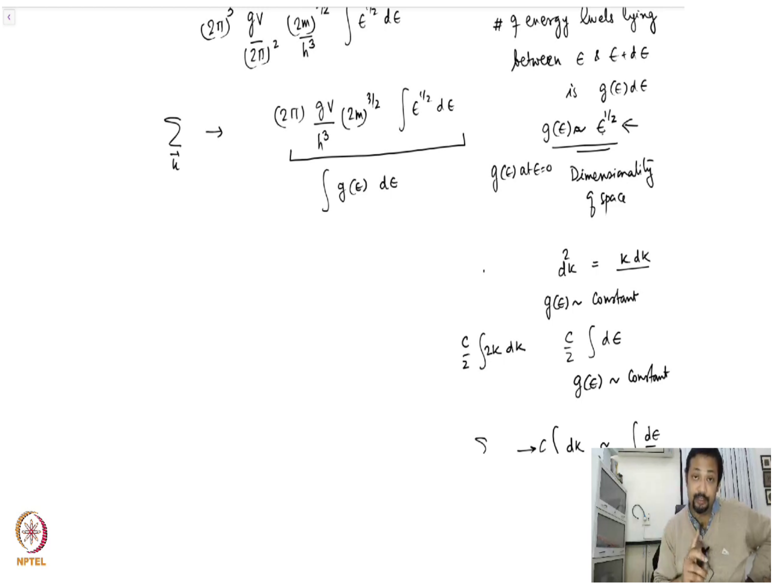

It is very interesting that this quantity depends on the dimension of dimensionality of space. Please note that where does it come in? It comes in over here, I have a three-dimensional system therefore, I have taken d cube k. If you had taken two-dimensional system, you would have d 2 k which would mean that this is k dk.

And if it is k dk, you immediately see that g epsa is constant because half k sorry not half k dk because just go back over here, this part when you had for a two-dimensional system, I am not worried about the constant factor, but I am just worried about the one which is under the integral, you would have k dk which you can write some constant times, let us call the constant as c k dk which is nothing but c by 2 d epsa 2k dk and this is d epsa.

And therefore, it follows that g epsa is going to be constant for when you are looking at a Bose gas or when you are looking at a free particle gas, whether it is a fermionic system or a bosonic system, the density of state is constant in 2D.

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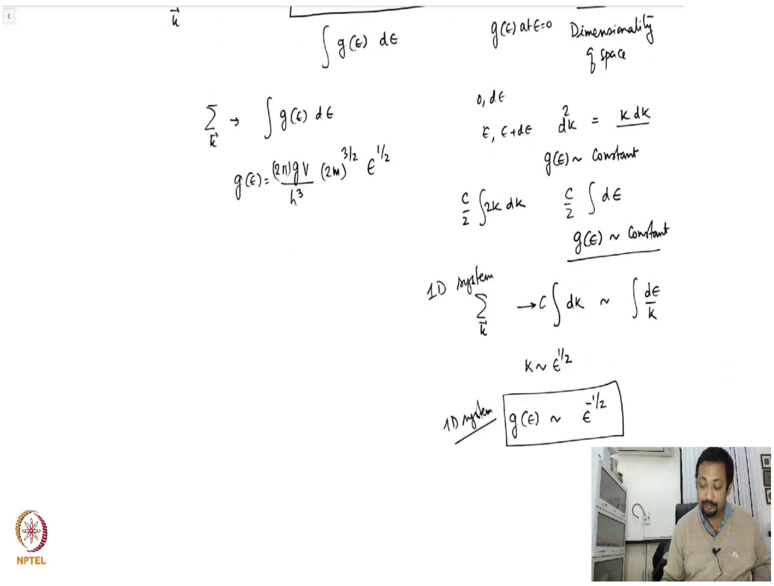

In 1D, this is just going to be dk right, the sum over k will just go over to some constant times dk and dk if you look at the relation between the energy and the momentum, I have dk going as d epsa by k so that this integral will go as integration d epsa over k some constant factor is always there outside, but that is not something which you have to worry about too much, but the idea is now, k goes as epsa to the power half so that your g epsa goes as epsa to the power minus half.

What is interesting to note that here, the density of states, the number of energy states g epsa at epsa equal to 0 is 0 in three-dimension, there are no energy levels at this particular energy that is what my density of states tell me. In contrast, if you go to two-dimensional case, then you see that all throughout the energy spectrum.

You have a constant number of energy, density of energies. So, wherever you consider your epsa to be, whether it is between 0 and d epsa or it is epsa and epsa plus d, the number of energy states are the same. In contrast, for a one-dimensional system, 1D system, the number of density, number of energy levels here epsa f equal to 0 diverges that is very interesting.

So, we will see what does it mean later on when we look at Bose Einstein condensation. Coming back to this now, I have sigma over k that goes to integration of g epsa d epsa with g epsa given by g V twice pi gV over h cube twice m raised to the power 3 half epsa to the power half right.

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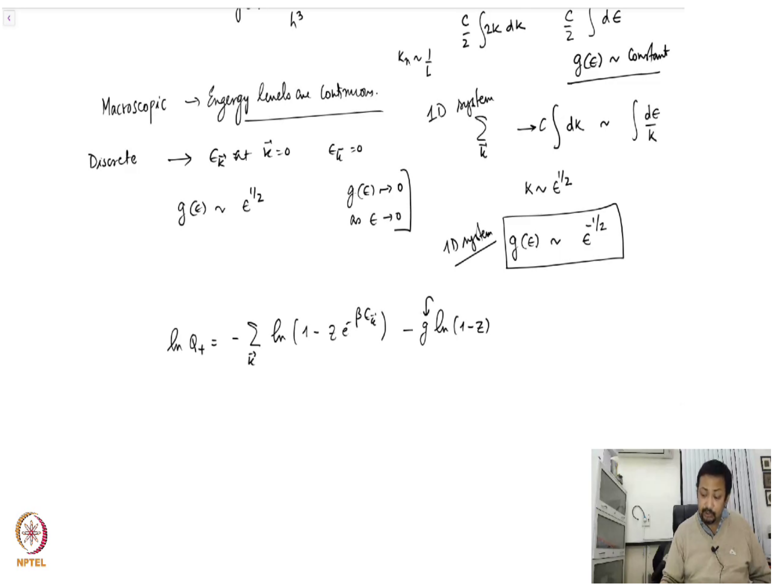

But note that even I am considering that my system is macroscopic so that energy levels are continuous in reality, this is not the case because you still have k n scaling with as 1 by L. So, you have a very tiny tiny difference between the 2 energy levels. But epsa k at k equal to 0 is

0. So, there is exactly one energy level which is there at the 0 at the zero-momentum state. But this clearly does not seem to be the case which we have already highlighted out.

While going from this to this, we clearly see that there is a problem because g epsa goes to epsa to the power half. Then discrete case, when we look clearly at the discrete case, we see that even at k equal to 0 the zero-momentum state, there is one energy level. But the moment we replace this sum by a continuous integral over the density of state, we see that at epsilon equal to 0, there is no energy level because g epsa is 0 as epsa tends to 0.

So, therefore, we have to be a little bit more careful in our analysis. So, what we have to do is we have to separate out the zero-energy level, the zero-momentum vector which we do it over here. We look at this expression over here and then, we see that I can write down this sum as ln Q plus as minus sum over k ln 1 minus Z e to the power minus beta epsa k minus g ln of 1 minus z.

There is no volume factor which comes in over here, you are just looking at one energy level and that has a degeneracy, internal degrees of freedom which is usually the spin which gives you the degeneracy level g right.

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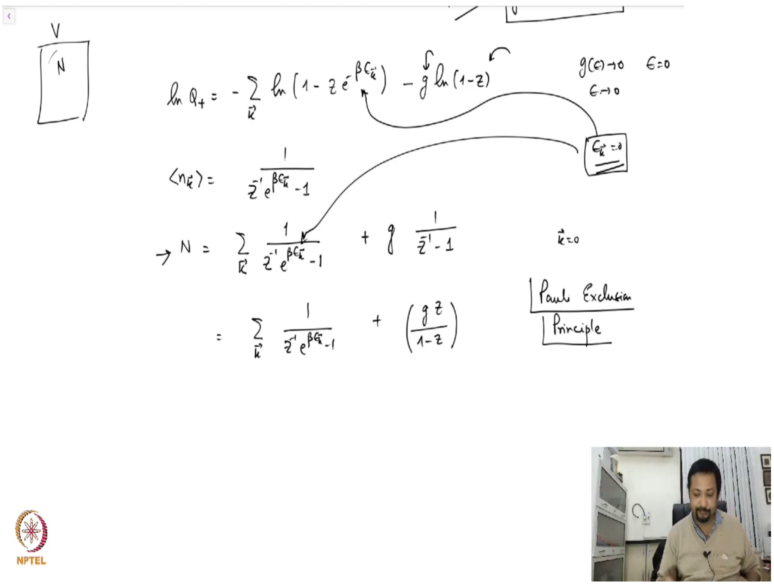

Similarly, I can take a look at the average, this quantity, this quantity over here, average occupation number which was n k as 1 over set inverse e to the power beta epsa k minus 1 and the total particle number was sum over k, this sum does not include the k equal to 0 value because that the value I have included over here and I can take care of this way also over here Z inverse e to the power beta epsa k minus 1 plus 1 by there is going to be a degeneracy factor of g 1 by Z inverse minus 1.

Where I have set epsa k is equal to 0, both here and here to get the last term. So, this gives me sum over k 1 over Z inverse e to the power beta epsa k minus 1 plus g Z 1 minus Z right. Now, the question is why do we want to take care of this k equal to the zero-momentum state special right, we did not do it for a fermionic system. Now, for a fermionic system, we are kind of saved by the Pauli exclusion principle. I will explain to you why we have to do it over here.

Now, the idea is in order to go forward, as we have seen that we convert this sum into an integral over the using the density of states, but that integral clearly ignores the zero-momentum state because there is no energy states lying at that particular value and the value epsa equal to 0 or in the neighborhood of epsa equal to 0. Therefore, that is a problem. In a fermionic system, even if you have a large system which had a large volume V and a cap particle number N.

Pauli exclusion principle would have tell you that you can only have 1 particle in that energy level. So, you are at most ignoring one out of n, but bosons do not have that. In bosons, for bosonics particles, more than one particle can go and sit at a given at any energy level. So, therefore, if you are; if you do not have sufficient thermal excitation to excite the particles to higher excited levels, all of the particles would like to go and sit at the lower and lower energy levels.

So, if you keep on decreasing temperature, they are going to sit or go and sit at this energy level. So, it can happen that a substantial number of particles are sitting at epsa k equal to 0, this the zero-energy level, but your partition function as well as this expression does not take into that take that into account right because your g epsa vanishes as epsa goes to 0 and it is therefore, precisely that you have to take care of this very very carefully.

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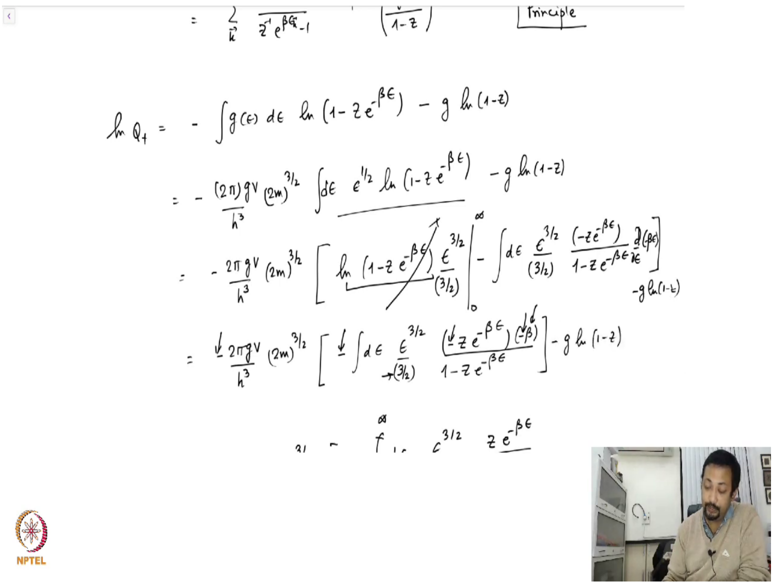

So, let us move forward ln Q plus is minus the sum of over k goes to integral g epsa d epsa and then, I have ln 1 minus Z e to the power minus beta epsa now, we do not talk about epsa subscript k g ln of 1 minus Z. So, let us see what was g epsa? g epsa was minus twice pi gV over h cube twice m raised to the power 3 half integral eps so, we write down d epsa epsa to the power half ln 1 minus Z e to the power minus beta epsilon minus g ln of 1 minus Z.

So, this means minus 2 pi gV, we are now very very familiar with how to handle such integrals we do an integration by parts where we take the log as the first function so that this becomes ln 1 minus Z e to the power minus beta epsilon and I have epsa to the power 3 half divided by 3 half minus integral d epsa.

Epsa to the power 3 half divided by 3 half and then, I have derivative of the log this function with respect to epsilon. So, I have 1 minus Z e to the power minus beta epsilon minus Z e to the power minus beta epsilon and then del d epsa of minus beta epsa close the bracket over here.

So that, this becomes twice pi gV over h cube twice m raised to the power 3 half, this vanishes between 0 and infinity, I do not have to worry about it, this is minus integral d epsa, epsa to the power 3 half divided by 3 half minus Z e to the power minus beta epsa 1 minus Z e to the power minus beta epsa and then, d d epsa of minus beta epsa is just minus beta.

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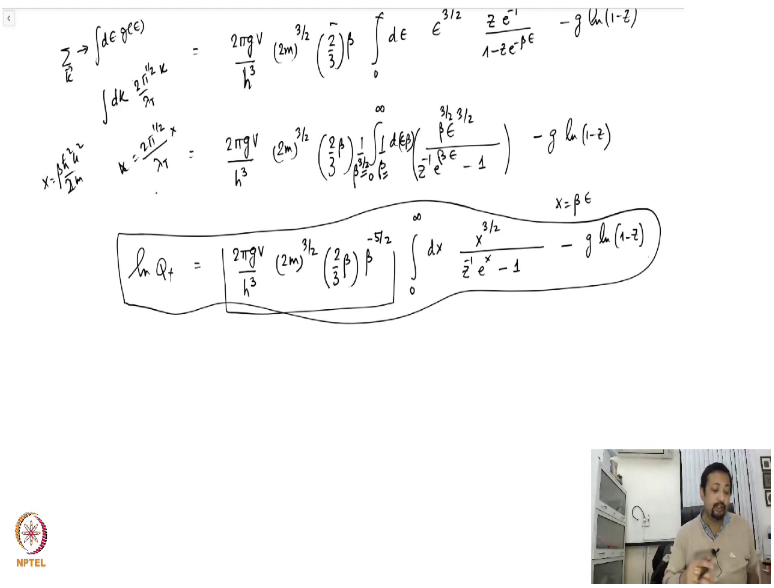

So that, this minus and this minus gives you a plus and this minus and this minus gives you a plus so that you have 2 pi gV over h cube twice m raised to the power 3 by 2. I have this as 2 by 3 and the beta I bring outside 2 by 3 beta and then, I have integral d epsa let us put the limits so that we do not miss anything later on. I have epsa to the power 3 half, I have Z e to the power minus beta epsa 1 minus Z e to the power minus beta epsa.

Our final simplification, 2 by 3 beta 0 to infinity d epsa is epsa to the power 3 half divided by Z inverse e to the power beta epsa minus 1. Now, do not forget that I also have this term. So, I have a minus g ln minus 1 minus Z which is being carried forward right.

I can also be a little bit more smart, I can substitute x equal to beta epsa and I can rewrite this expression as twice m raised to the power 3 half 2 by 3 beta, this is beta to the power 3 half and I have 1 over beta to the power 3 half 1 over beta beta times epsa so that I have beta to the power minus 5 half, a 3 half that comes from over here and the beta alone comes from over here and I have 0 to infinity dx x to the power 3 half divided by Z inverse e to the power x minus 1 minus g ln of 1 minus Z sorry let us write down.

This is my grand canonical partition function for a bosonic ideal Bose gas. Well, now, this is the part which looks extremely complicated. Let us see can we see whether this gives us something interesting because see the whole description or derivation of this the canonical partition function essentially has started from the fact that this integral, I can convert it into d epsa g epsa and the idea was that when we did the fermionic system, we did it in a slightly different way.

We essentially wrote this as in terms of dk itself times some function I think it was 2 pi to the power half lambda T times k right. So, yeah, so, x was beta h bar square k square over twice m and so that x k was 2 pi half raised to the power but divided by lambda T times x something like this was there and we had used this particular approach.

Now, I said that both of them are identical, we are going to get the same thing. I have just used different approaches for the two different cases so that you are familiar with both of them how you want to proceed right.

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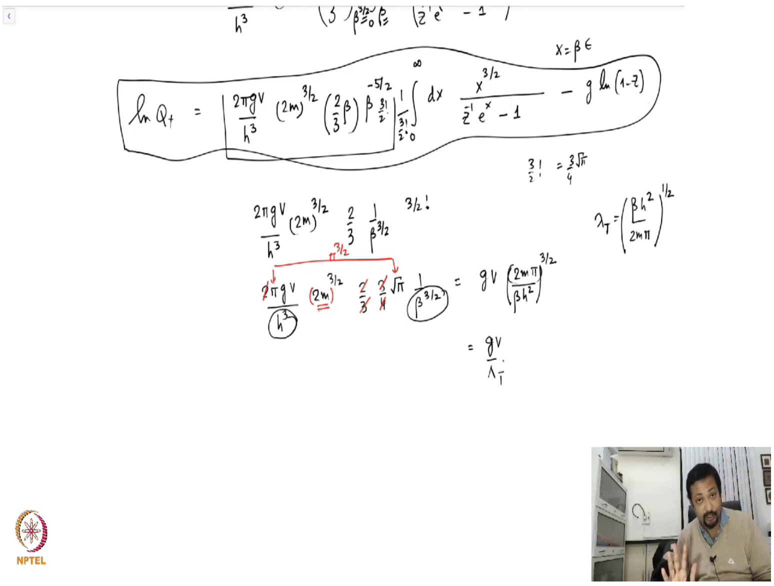

So, now, 2 pi gV over h cube twice m raised to the power 3 half 2 by 3, this gives you 1 over beta to the power 3 half. So, here, let us bring a 3 by 2 factorial divided by 3 by 2 factorial. So, we will put it over here itself, I have a 3 by 2 factorial and I have a 3 by 2 factorial.

And the 3 by 2 factorial in the numerator I include over here this times 3 by 2 factorial. So, the expression that I am looking at is simply this. Now, this quantity 3 by 2 factorial is 3 by 4 square root pi so that I have 2 pi gV over h cube twice m 3 by 2 2 by 3 3 by 4 square root pi 1 over beta to the power 3 half.

So, you should already get a feeling where we are going. So, this should be now, let us cancel. The cancellation is 3, 3 gets cancelled, this gives you 2 and this gives you 2. Remember this is 2m raise to the power 3 half ok. So, this 2 and this 2 cancels with the 4 and this pi and this square root pi gives you a pi to the power 3 half so that which couples with this 2m raised to the power 3 half to give you gV 2m pi raised to the power 3 half.

And then, I have this beta to the power 3 half and I have h cube which I can easily include as beta h square and if you remember, your de Broglie wavelength which was beta h square over 2m pi half, then this quantity is gV over lambda T which was exactly the pre factor we had when we looked at Fermi integrals or fermionic systems.

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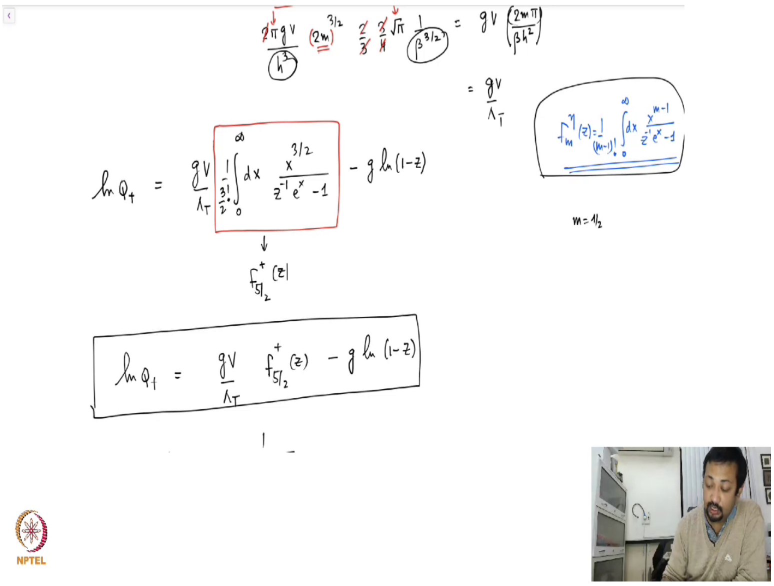

So, my partition function becomes ln Q plus is gV over lambda T integral dx 0 to infinity 1 over 3 half factorial x to the power 3 half Z inverse e to the power x minus 1 minus g ln of 1 minus Z.

Now, clearly this is the part; this is the part which looks very very familiar and if you recall f m eta of Z was 1 over m minus 1 factorial 0 to infinity dx x to the power m minus 1 Z inverse e to the power x minus 1, this is was how we had defined the function and therefore, this is f of 5 half plus Z. So, your grand canonical function takes the form gV over lambda T f 5 half plus Z minus g of 1 minus Z right.

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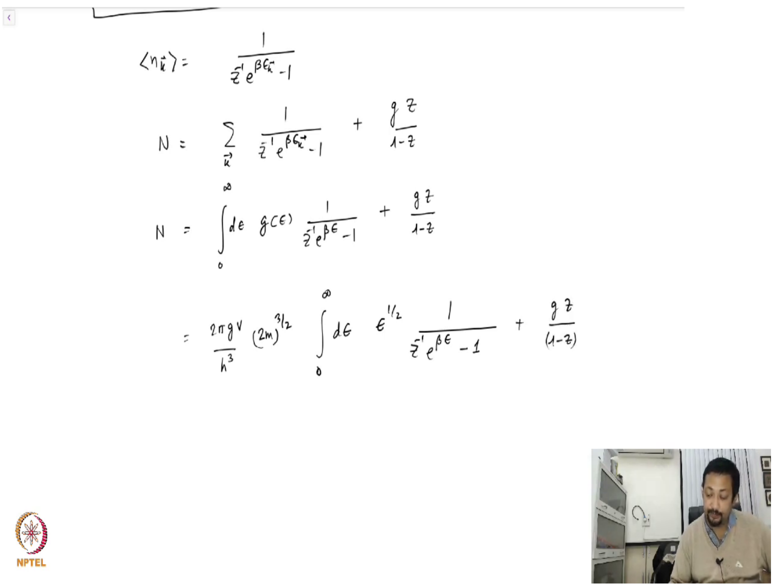

Average expectation number was 1 over Z inverse e to the power beta epsa k minus 1 and we separated the zero-momentum state out so that we have z inverse e to the power beta epsa k minus 1 plus g z 1 minus z. This means that if I go to the continuum limit, I have 0 to infinity g epsa d.

So, one has to first write down the measure d epsa g epsa 1 over Z inverse e to the power beta epsa minus 1 plus g Z 1 minus Z. This again is 2 pi gV over h cube twice m raised to the power 3 half 0 to infinity d epsa epsa to the power half 1 over Z inverse e to the power beta epsa minus 1 plus g Z 1 minus Z right.

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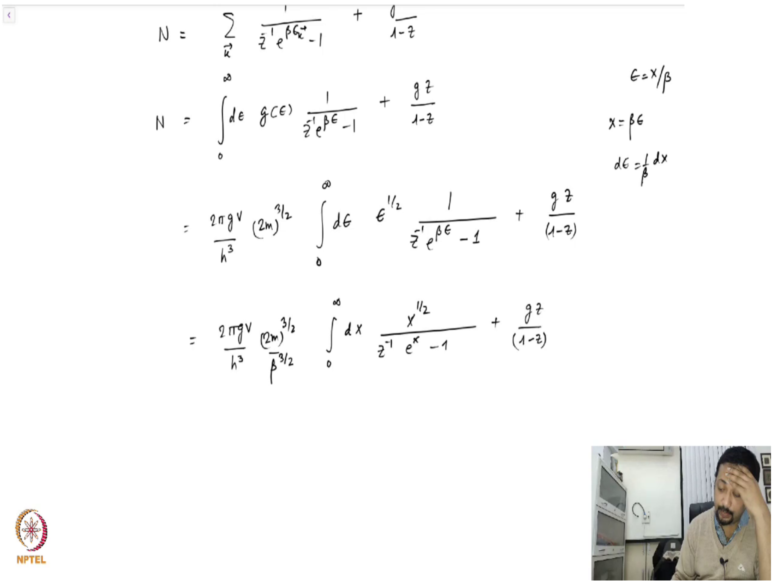

And I have 2 pi gV over h cube times twice m raised to the power 3 half. I can again I will again substitute x equal to beta epsa here so that d epsa is 1 by beta dx and epsa is x over beta so that I have beta to the power 3 half and then, I have 0 to infinity dx x to the power half Z inverse e to the power x minus 1 plus g Z 1 minus Z.

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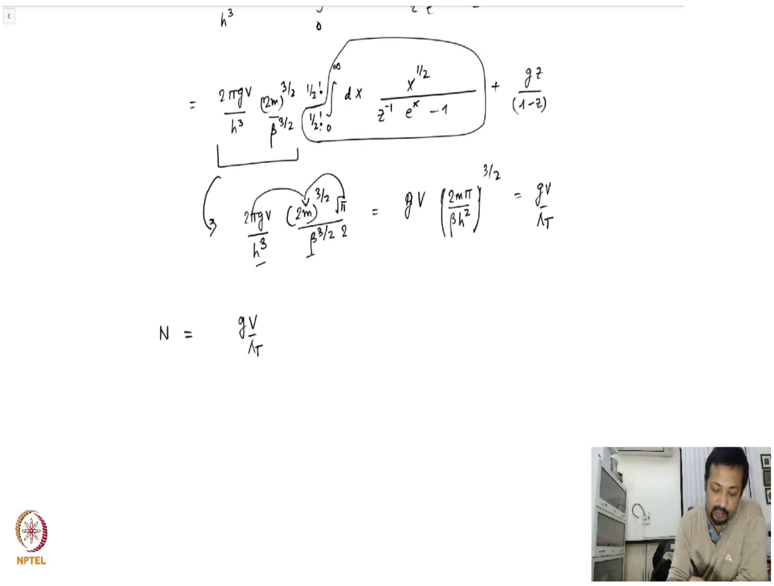

One needs to look at this factor over here again and I mean this is just a repetition of what we did earlier but let us just do it very quickly multiply and divide throughout by half factorial so that you will have this particular quantity will give you 2 pi gV over h cube twice m raised to the power 3 half square root pi over beta to the power 3 half.

So, this has to be square root pi by 2 which is gV the pi I mean square root pi will go inside over here to give you 2m pi raised to the power 3 half h cube. And beta to the power 3 half can be combined to give you beta h square so that you have gV over lambda T.

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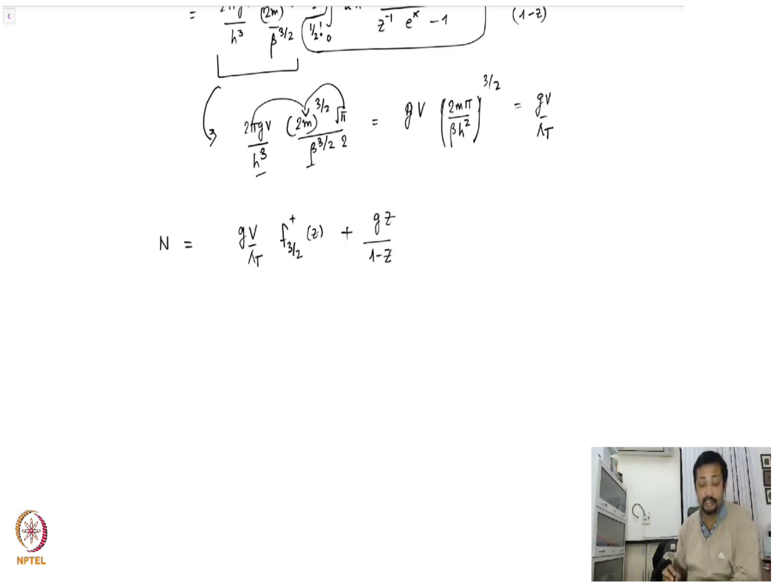

So, this total particle number then becomes gV over lambda T and this integral is very very familiar to me in the sense I know that we use the this thing for m equal to half sorry m equal to 3 half and therefore, I have f of 3 half plus of Z minus sorry plus g of Z 1 minus Z.