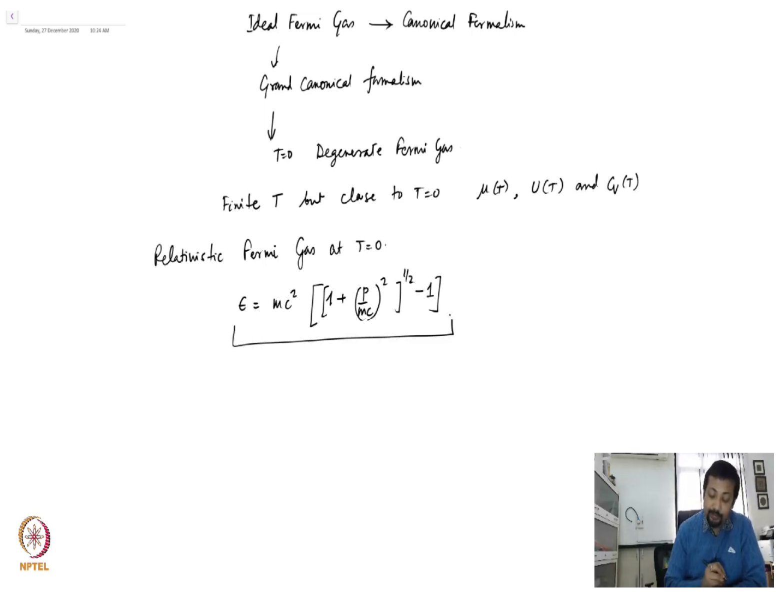
Statistical Mechanics
Prof. Dipanjan Chakraborty
Department of Physical Sciences
Indian Institute of Science Education and Research, Mohali

Lecture - 53
Relativistic Fermi Gas at T=0

(Refer Slide Time: 00:23)

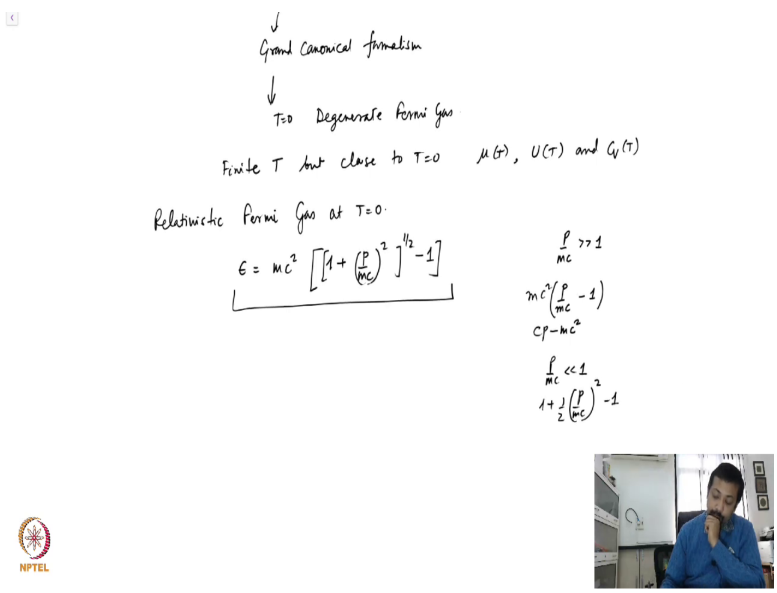


So, welcome back. Today having done all everything about ideal Fermi gas, we have looked at this ideal Fermi gas in the canonical formalism as well as in the grand canonical formalism.

So, we have looked at the T equal to 0 which is the degenerate Fermi gas and we have also looked at a finite temperature; finite temperature, but close to T equal to 0, we have seen how mu, the chemical potential depends on this temperature, how the internal energy depends on the temperature and the specific heat depends on the temperature.

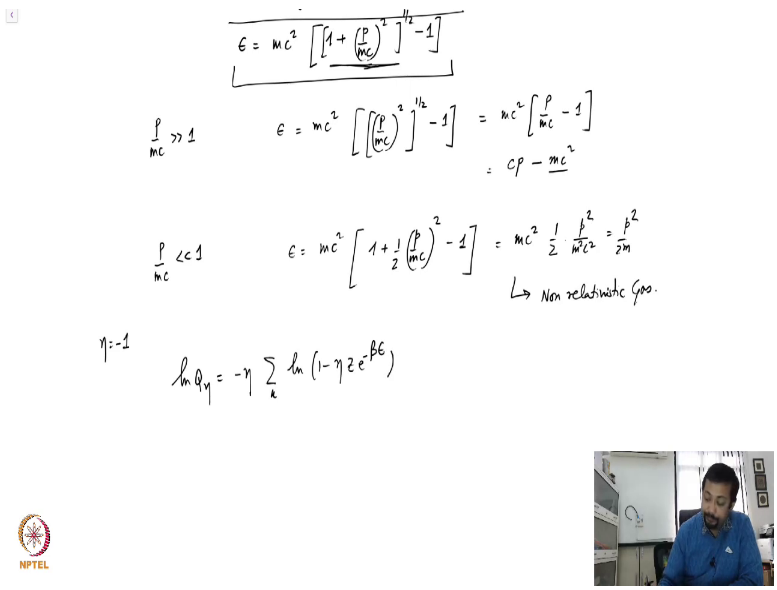
In the current lecture, what we are going to do is we are going to study a relativistic Fermi gas at T is equal to 0 so that we can write down this as mc square 1 plus P over mc whole square half minus 1 and here, I have taken care of the rest mass, I have subtracted it out from my in this relation. So, I want to study how this Fermi gas is going to behave at T equal to 0.

(Refer Slide Time: 02:16)



So, before we go ahead with the calculation, let us just take two look at two limits of this.

(Refer Slide Time: 03:07)

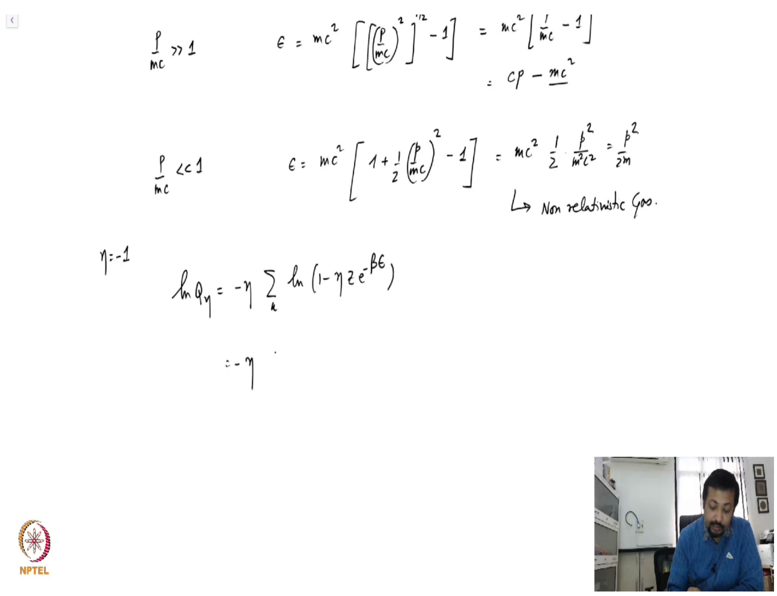


First of all, one if you look at P over mc much much larger than 1, you will see that epsilon is mc square, this is the term which terminates so, P over mc whole square and then, I have raised to the power half minus 1 which becomes mc square P over mc minus 1 which is just c times p minus mc square, this is your rest mass energy as you already know and this is the case of an ultra relativistic gas .

In contrast, when you have P over mc as much much less than 1, then if you have mc square, I can expand this quantity in the brackets and term in a binomial series so that I am going to have P over mc square and keep up to this order minus 1 which gives you mc square times half p square over m square c square which gives you p square over twice m.

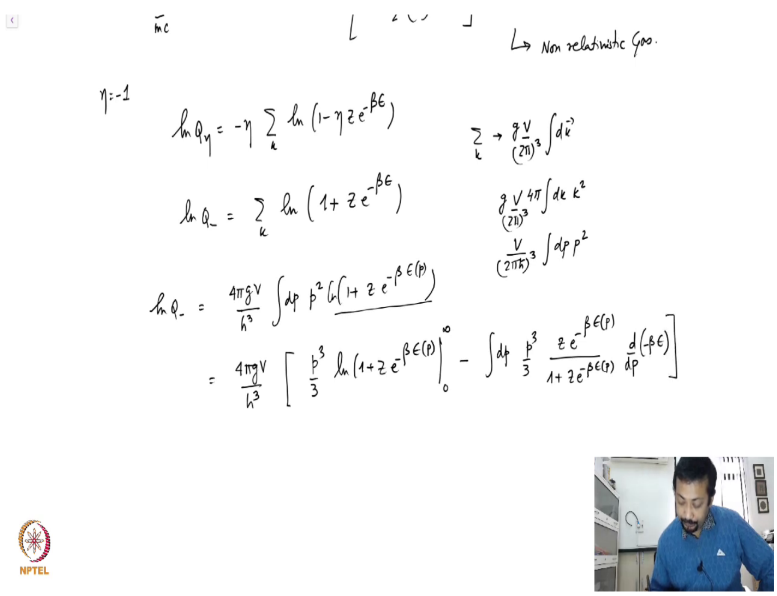
So, therefore, this gives you the non-relativistic gas and when you look at this energy expression, you are kind of doing both just have to figure out the limits when you finally, want to know the results.

(Refer Slide Time: 05:07)



So, let us start with this, since it is a fermionic system, I have eta is equal to minus 1 and therefore, ln of Q eta, just recall was minus eta sum over k ln of 1 minus eta Z e to the power minus beta epsa which is minus eta.

(Refer Slide Time: 05:13)

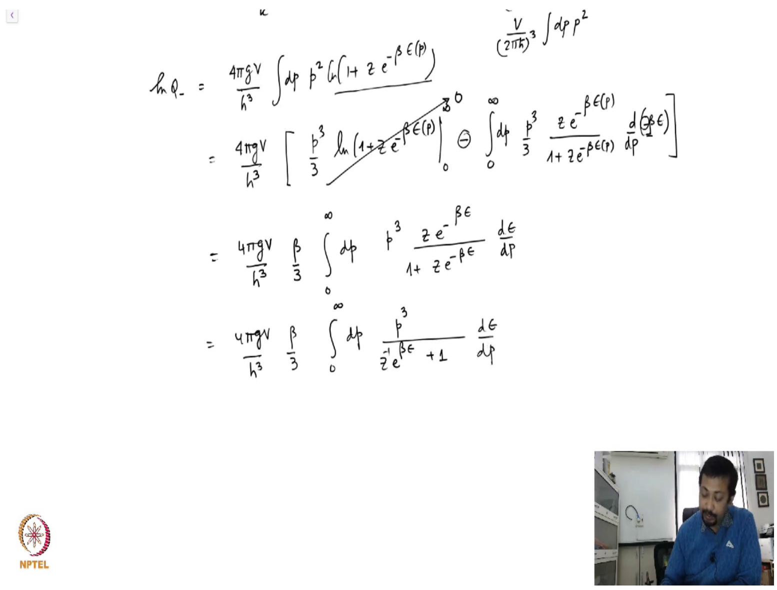


In this case, for a fermionic system, I know that this is going to be ln of Q minus which is going to be sum over k ln 1 plus Z e to the power minus beta epsilon. We want to convert this sum into an integral over the energies so that we know that sum over k will go as v over 2 pi cube integral of dk and this integral again we can write down this as in three-dimension, this is going to be k square. We will look at 3D system only.

Once you have this as 3D, I can bring, I can rewrite this equation as v over 2 pi whole cube. Now, you see that here the momentum is in terms of p so, I want to write it down in terms of p which will give me dp p square v over 2 pi h bar whole cube and I am going to have a factor g v and a factor 4 pi that is not over here, but a factor 4 pi that will come over here. So, that ln of Q minus becomes v over 4 pi gv so, let us over h cube integral dp p square ln of 1 plus z e to the power minus epsilon of p.

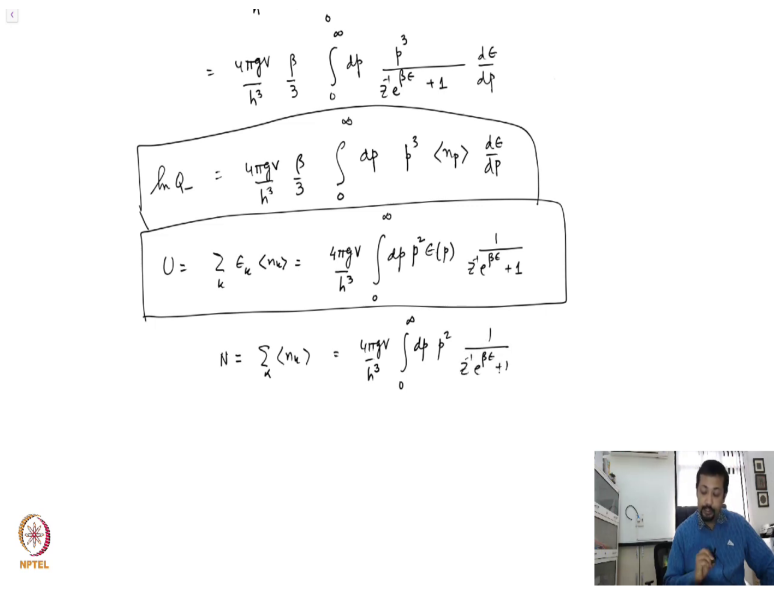
I can integrate by parts that is what we have been doing so far so that this becomes gv over h cube p cube over 3 ln 1 plus Z e to the power minus beta of epsilon p, this goes from 0 to infinity minus integral dp p cube over 3 divided by the derivative of this term the log term with respect to p so, that is going to be Z e to the power minus beta epsa of p and then, I have Z e to the power minus beta epsilon right and then, I have d dp of minus beta epsilon. It is understood that epsilon is a function of p.

(Refer Slide Time: 08:07)



This term as you well know by now, it is going to be 0. So, I am left out with 4 pi gv over h cube, the beta factor comes outside and this minus and this minus gives you a plus so, you have a beta over 3 and then, I have 0 to infinity dp p cube Z e to the power minus beta epsa 1 plus Z e to the power minus beta epsa. It is understood that beta, the epsilon depends on p, the momentum d epsa dp. Let us write down this in the form that I know 0 to p dp p cube over Z inverse e to the power beta epsa plus 1 d epsa dp.

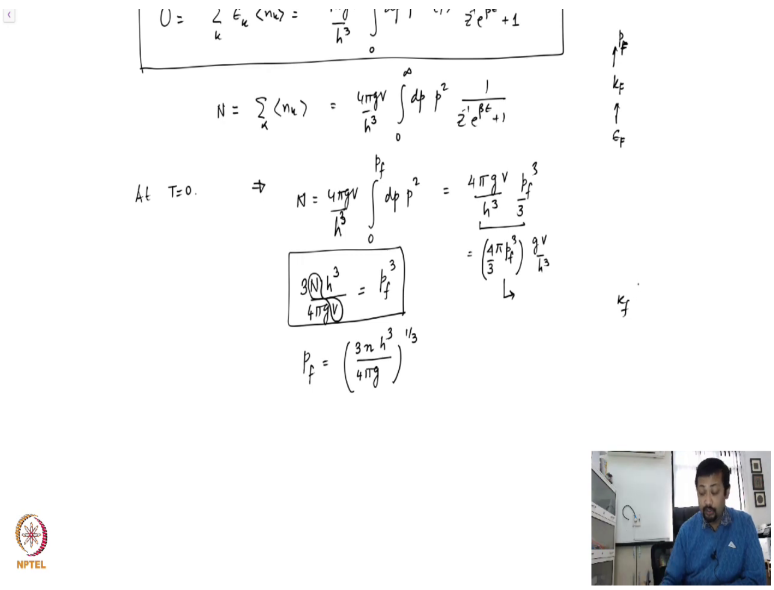
(Refer Slide Time: 09:14)



And this I know that I can write down this as 4 pi gv over h cube beta over 0 to infinity dp p cube np d epsa dp. We have seen this in the standard results of a Fermi gas. So, this is going to be your ln of Q minus. The energy is going to be sum over k epsa k average of n k which is going to be 4 pi gv over h cube 0 to infinity dp, I will have epsa of p, there is going to be a p square and then, I am going to have 1 over Z inverse e to the power beta epsa plus 1. So, these are the two results that we are primarily interested in.

If you notice, then you see I have not yet worked out the result for average of N so, where the total N is going to be sum over k average of n k and this is going, this will give you the same result that we obtained when we did the degenerate Fermi gas. So, this is going to be 4 pi gv over h cube 0 to infinity dp p square 1 over z inverse e to the power beta epsa plus 1 right.

(Refer Slide Time: 11:06)

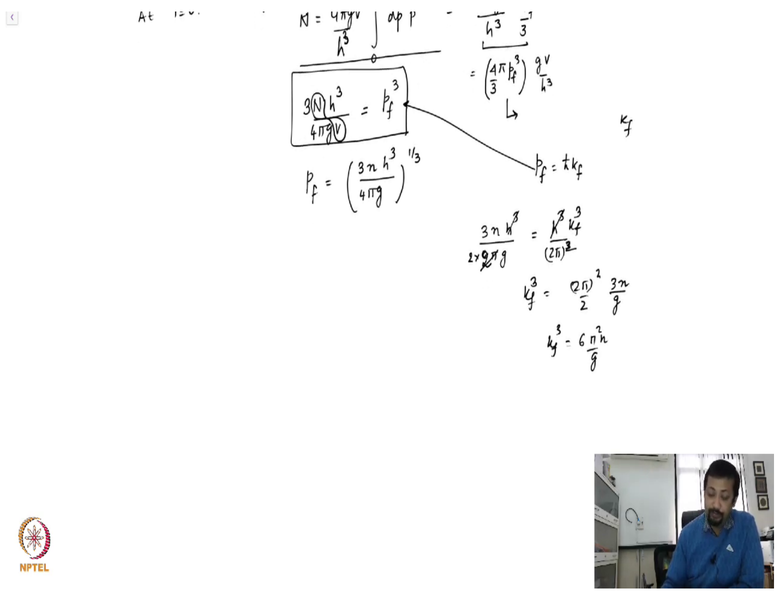


Now, we are looking at T equal to 0 so that this implies that I have a fermicy so, all energy level up to the Fermi energy is filled up and corresponding to this Fermi energy, I know that i have a k F and therefore, a momentum p F. So, the idea is to figure out this momentum p F and which I can do from this relation right so that would mean that N is going to be 4 pi gv over h cube 0 to p f dp over p square that gives me 4 pi gv over h cube p f cube divided by 3.

So, I have N h cube divided by 3 N h cube 4 pi gv is going to be p f cube. If you look up this, look at this expression, then you see I can take this some things out 4 by 3 pi p f cube gv over h cube right. So, this is the volume of a sphere of radius p f and this essentially gives you the Fermi momentum p f. So, that p f is going to be N by v I can combine it into small n so, this

N and this v I can combine into a small n, I will be having 3 N h cube divided by 4 pi g raised to the power one-third. This result is identical to the value of k f which we got when we worked out the degenerate Fermi gas.

(Refer Slide Time: 13:28)

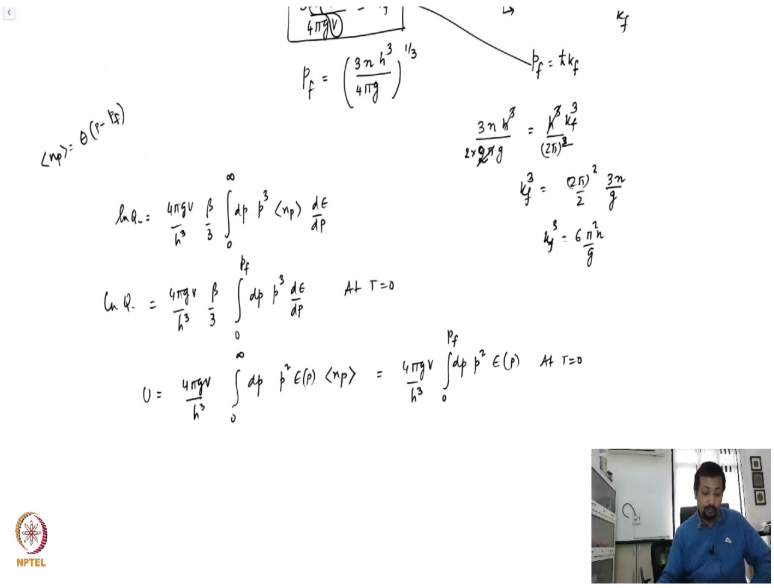


In fact, you can clearly see that if i write p f as h bar k f and substitute this relation over here, then I am going to have 3 n h cube 4 pi g is going to be h cube k f over 2 pi whole cube and you see h cube, h cube is going to cancelled away, I am going to have 2 into 2 pi instead of 4 pi so, one 2 pi factor cancels to give me 2 pi whole square so that k f is 2 pi whole square by 2 3 n g so, this is cube 3n sorry 3n divided by g.

This is 4 pi square so, this gives you 6 pi square n over g as k f whole cube, it is the same result that we have obtained before; that we have obtained before.

The reason they are same is because the energy epsa p does not enter this expression anywhere ok. What enters is only the dispersion relation and as long as the since the dispersion relation is same therefore, one has the same expression.

(Refer Slide Time: 14:50)

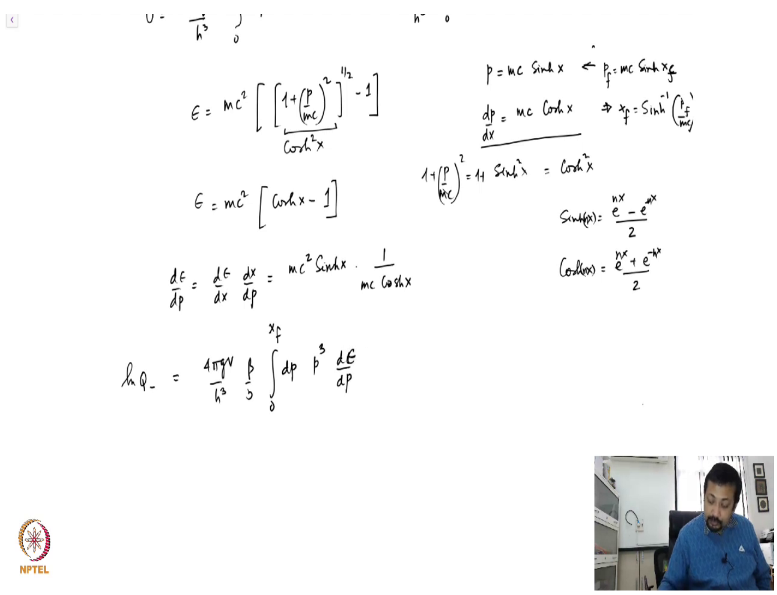


So, now, I have ln of Q minus which was 4 pi gv over h cube and then, I had beta over 3 dp p cube n p d epsilon dp. Strictly at T is equal to 0, I know that I have n p is going to be theta of p minus p f which essentially means that all momentum levels from 0 to p f are filled just as we have seen for Fermi gas at degenerate Fermi gas at T is equal to 0 so that essentially, this limit which goes from 0 to infinity is now replaced by beta over 3 0 to p f dp p cube d epsa dp, this is strictly at T is equal to 0 and we will use this later on.

The energy is going to be 4 pi gv over h cube 0 to infinity dp p square epsa as the function of p and then, you have average of n p which again translates to 4 pi gv over h cube 0 to p f dp p square epsilon p at T is equal to 0. One has to please let us go back and just check whether we have done it correctly otherwise, we have to do it again ok. This looks right to me.

Anyway, if we have anything, if we have made some mistake, we will come back and correct it. So, now, let us before we go ahead, I have to take a look at this derivative, and I have to use this.

(Refer Slide Time: 16:47)



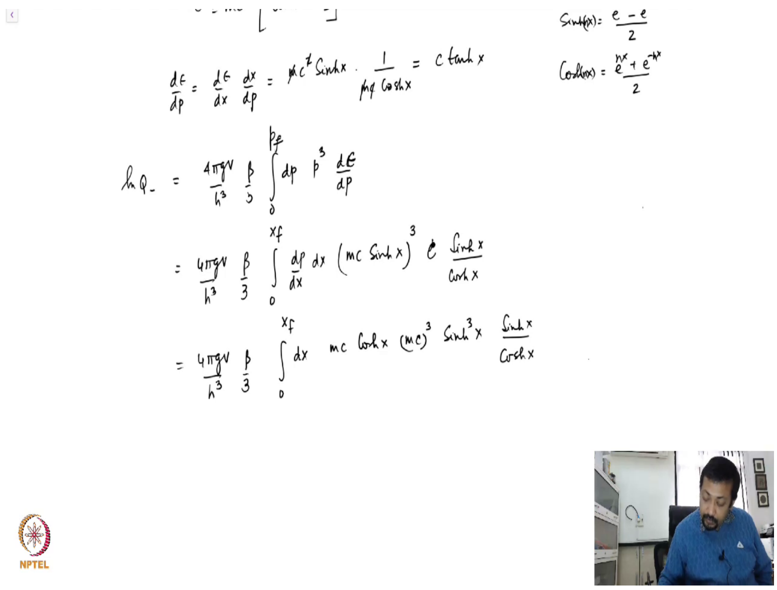
And energy, the relation in terms of momenta was not very simple we have 1 plus p over mc whole square this raised to the power half minus 1. So, we will make some substitutions to make our lives a little bit easier.

So, we go from a variable p to x such that p is mc sin hyperbolic x and therefore, one should note that dp dx which you are going to use frequently is mc cosh hyperbolic x and 1 plus p over mc whole square is 1 plus sinh square x which becomes cosh square x, this becomes cosh square x so that you have mc square cosh x minus 1, this is the relation.

D epsa dp which we are also going to need is going to be d epsa dx times dx dp, d epsa dp is simply mc square sin hyperbolic x times we have calculated this, this becomes 1 over mc cosh x. Please remember that sin hyperbolic x is e to the power x minus e to the power x minus 2 cosh hyperbolic x is e to the power x e to the power minus x by 2.

If you have n, if you have n sitting over here so, this becomes n x, n x, n x, n x. So, this you can use to calculate derivatives if you are unfamiliar with this hyperbolic functions. So, with this in our hand, let us just first write down ln of Q minus which is 4 pi gv over h cube beta over 3 0 to x f where x f is defined from this relation so, we will have p f is going to be mc sin hyperbolic x f which implies x f is going to be sin hyperbolic inverse p f over mc.

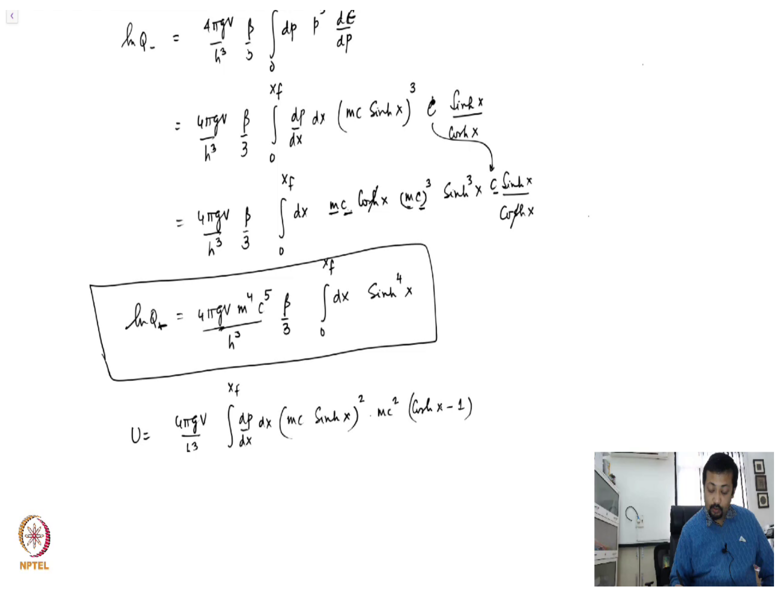
(Refer Slide Time: 19:52)



So, this becomes dp p cube d epsa dp which we write down as 4 pi gv over h cube beta. Note that this is strictly at T is equal to 0, we are looking at the degenerate Fermi gas. So, I am sorry this in this case, we have the limit of p f, we will write down the limit of x f only after we have made the change of variables this is dp dx times dx and this is p over sorry, this is going to be mc sinh x whole cube, d epsa dp is going to be let us look at this relation m, m cancels out c, c cancels out one factor so that you have c of tan hyperbolic of x.

So, we will write d epsa d p as sin hyperbolic of x divided by cosh hyperbolic of x. So, that this becomes 4 pi gv over h cube beta over 3 0 to x f dx and dp dx is mc cosh x mc whole cube sinh hyperbolic cube of x and then, you have sinh x over cosh x.

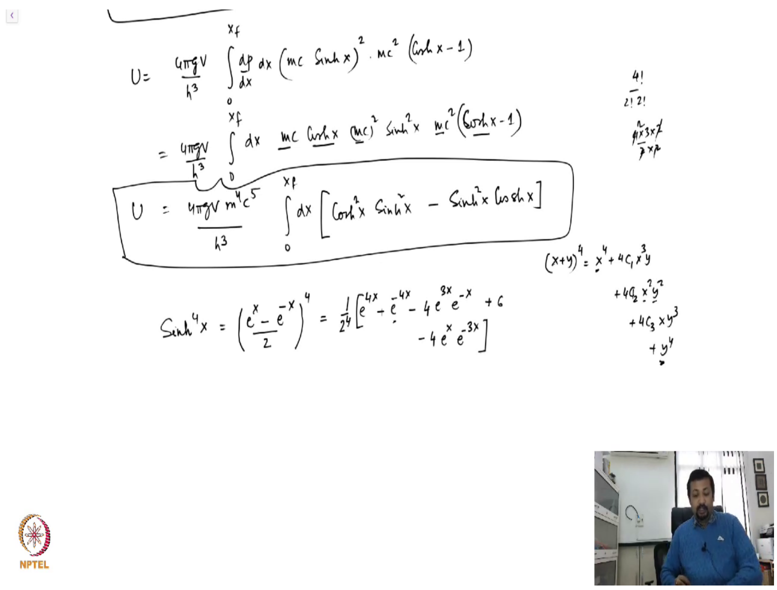
(Refer Slide Time: 21:32)



This, this cancels out, I have m 4 and there is a c factor here that one should not miss so, m 4, m times m cube is m 4 so, I am going to have 4 pi gv m 4 and I have 3 plus 1, 1 plus 3 is 4 plus 1 more is 5 so, I am going to have c 5 h cube beta over 3 0 to x f dx sinh 4 of x. This is going to be ln of Q minus.

The energy U is going to be 4 pi gv over h cube 0 to x f, now we are going to straight forward do it p square is mc sinh x whole square and then, I have epsa of p which in terms of this is going to be, in terms of x is going to be mc square cosh x minus 1.

(Refer Slide Time: 23:00)



Now, so, let us quickly write down dx dp dx is going to be cosh x mc whole square sinh square x mc square cosh x minus 1. One factor of m here, two factor of m here so, cube 4. So, you are going to have 4 pi gv m 4 c 5, the same factor h cube 0 to x f dx now, cosh x times cosh x gives you cosh square x sinh square x and then, you are going to have minus sinh square x cosh x. So, this is the internal energy of this gas.

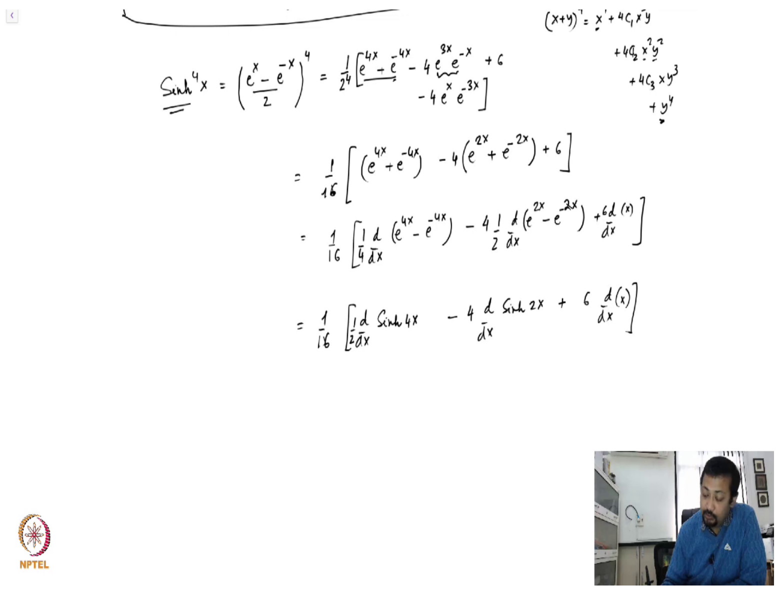
Now, the point is how do you integrate? It is a little bit tiresome, but one should be able to do it and we will do this exercises just try to see if you can follow the calculations, I mean you can try to repeat this calculations at you know at your own time.

Sinh 4 x is e to the power x minus minus x by 2 raised to the power 4 which is if you recall that your binomial expansions x to the power 4 plus 4c 1 x cube y plus 4c 2 x square y square plus 4c 3 x y cube plus y to the power 4, we are just going to use this one. So, e to the power 4 x which is this term plus e to the power minus 4x, 4c 1 is 4 factorial divided by 3 factorial which gives you a 4 here, x cube is e to the power 3x, y is minus e to the power minus x so, you have minus x.

Here, 4c 2 is 4 factorial divided by 2 factorial divided by 2 factorials. So, 4 factorial 2 factorial 2 factorial so, 4 into 3 by 2.

If you want to calculate, this is the exact formula and you can see that this is actually 6, but x square is e to the power 2x, y square is e to the power minus 2x so, we are just left out with plus 6. 4c 3 is again going to be minus 4 here of course, you are going to have e to the power x e to the power minus 3x and this already you have taken over here so, our bracket closes here and then you have 2 to the power 4.

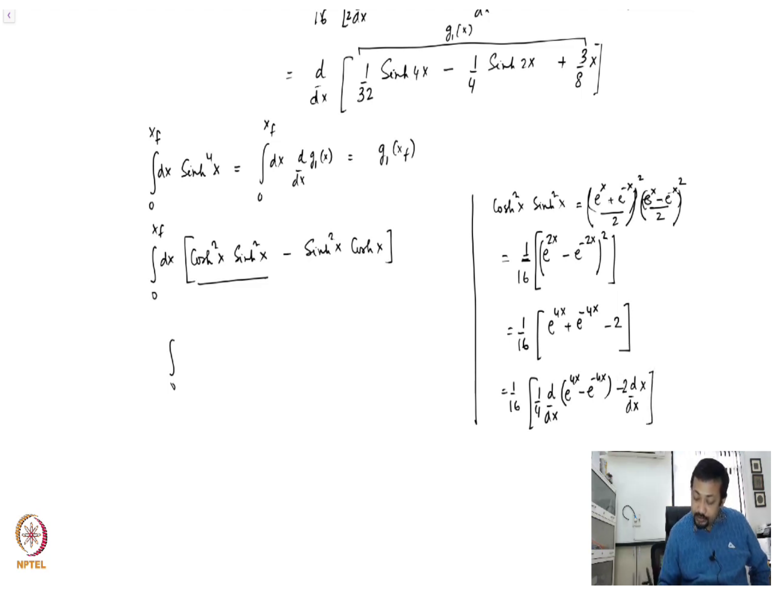
(Refer Slide Time: 26:23)



2 to the power 4 is 1 over 16. Now, e to the power 4x. So, [FL] let us just first write down this and then we will minus 4 this is e to the power 2x plus e to the power minus 2x plus 6. Now, these are exponentials. So, I can equally write them as d dx of e to the power 4x minus 4x 1 by 4 right minus 4 d dx of e to the power 2x minus 2x divided by half plus d dx of 6 times d dx of x it is that simple.

So, 1 over 16, this is where the trick lie. Since, I am going to integrate this. So, if I write down as d derivative of the sub function, then it is much much easier for me. The one-fourth I can take 2 inside to write down this as sin hyperbolic of 4x, again I can take 2 inside to write down this as 4 d dx of sin hyperbolic of 2x plus 6 d dx of x nothing much you can do over here.

(Refer Slide Time: 27:56)



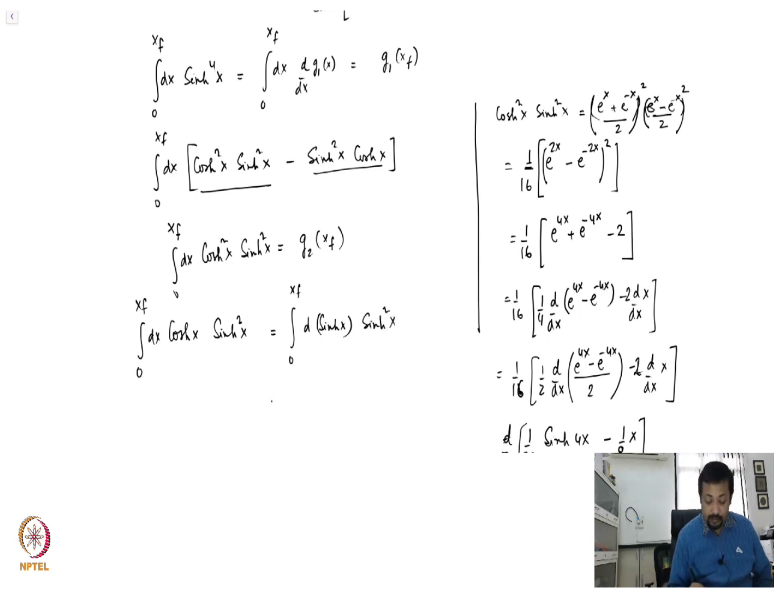
So, this becomes 1 over 32, let us take the derivative out, the operator out and 1 by 16 inside so, I have 1 by 32 sin hyperbolic of 4x minus one-fourth sin hyperbolic of 2x plus 6 over 16, if I am going to do it divide cancel out by 2, then I will have 3 over 8 times x quite an elegant result. Similarly, so, therefore, let us just first write down 0 to x f dx, this is going to be 1 by 32 so, let me call this function let us say g 1 of x is integral dx 0 to x f d dx of g 1 of x which gives me g 1 of x f.

Let us look at the energy. Now, I have two terms over here 0 to x f dx cosh square x sinh square x minus I have sinh square x cosh x yeah. So, let us look at this.

So, I have 0 to. So, here let us do the calculation cosh square x sinh square x, this is going to be e to the power x plus e to the power minus x whole square times e to the power x minus x whole square so, which is going to be e to the power 2x, I was tempted to just write it down as e to the power 2x minus e to the power minus sorry, this has to be 2 whole square this divided by 2 whole square.

So, overall, you have 1 by 16 times e to the power minus 2x whole square because this you can write down a plus b times a minus b whole square. So, you have a square minus b square whole square, and this is going to be 1 over 16 e to the power 4x plus e to the power minus 4x minus of 2. So, again one write down; one writes this down as d dx of e to the power 4x minus e to the power of minus 4x 1 over 4 minus d dx of so, 2 times x.

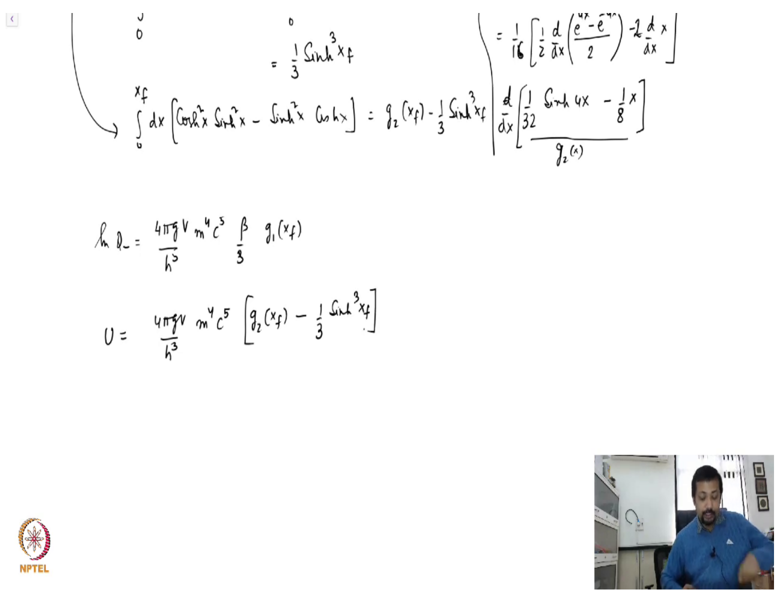
(Refer Slide Time: 31:25)



So, I have 1 by 16 half d dx of e to the power 4x minus e to the power minus 4x divided by 2 minus twice d dx of x. So, I will have 1 over 32; let us take the d dx outside, sin hyperbolic of 4x minus 1 by 8 times x. So, this is the integral.

So, therefore, 0 to x f dx cosh square x sinh square x, let us call this function as g 2 of x, then I am going to have as simply g 2 of x f. This is easy 0 to x f dx sinh square x cosh x. Let us just change the order over here to write it down as cosh x sinh square x which means 0 to x f d of sin hyperbolic x sin square x.

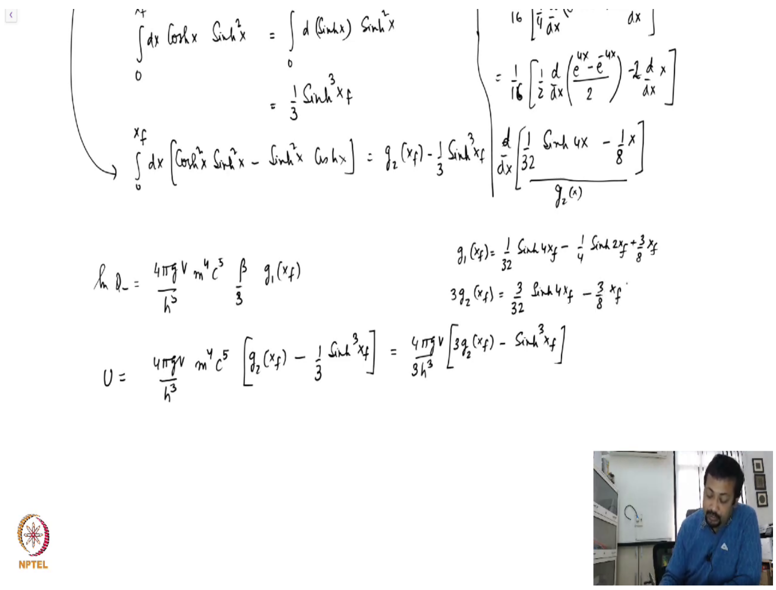
(Refer Slide Time: 33:00)



And this is going to be sinh hyperbolic whole cube x f 1 by 3. So, this integral then evaluates to 0 to x f dx cosh square x sinh square x minus sinh square x cosh x is going to be g 2 of x f minus one-third sin hyperbolic cube of x f that is the final answer good.

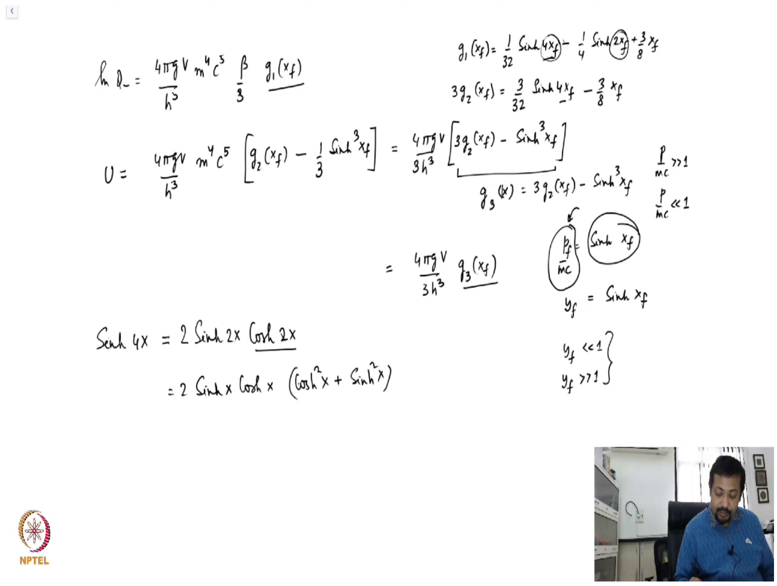
So, it is a very tedious algebra, I agree with you, but let us go ahead. Since, we have come so far let us push it forward a little bit. ln of Q minus is going to be 4 pi gv over h cube m 4 c 5 beta over 3 and the integral was integration of dx sin hyperbolic 4 of x raised to the power 4 of x which we had was g 1 of x f and U, the energy was 4 pi gv over h cube m 4 c 5 g 2 of x f minus one-third, I am going to have sin cube of x f.

(Refer Slide Time: 34:57)



Well, I can write down this as 4 pi gv over h cube; 3h cube since this also has a factor 3 and then, I am going to write down this as 3 times g 2 of x f minus x f where g 1 of x f is going to be 1 by 32 4x f and then, I am going to have minus 1 by 4 sin hyperbolic of 2x and I will going to have plus 3 by 8 of x f, g 2, we will write down 3g 2 of x f is going to be; this is going to be multiplied by this so, it is going to be 3 over 32 sin hyperbolic of 4x f minus 3 by 8 of x f.

(Refer Slide Time: 36:13)

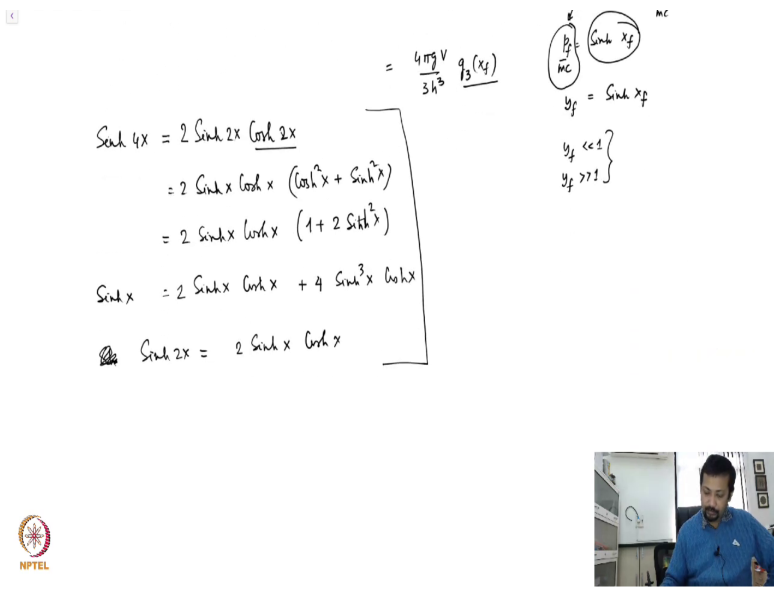


And let us call this together as g 3 of x which is equal to thrice g 2 of x f minus sinh cube hyperbolic cube of x f so that I have the result 4 pi gv over 3 h cube g 3 of x f that is it. So, so far so good. A little bit more algebra is left out because what I want to do now is I know that p f is going to be sin hyperbolic this over x f.

So, I know this so, let us call this y f. So, I want to express all this 4x f in terms of x f because this is the primary variable of concern to me which is y f and sin hyperbolic x f which essentially because in the very beginning of the class, I had seen that p mc much much less than 1 corresponds to a non-relativistic gas and p mc much much larger than 1 corresponds to an ultra-relativistic gas.

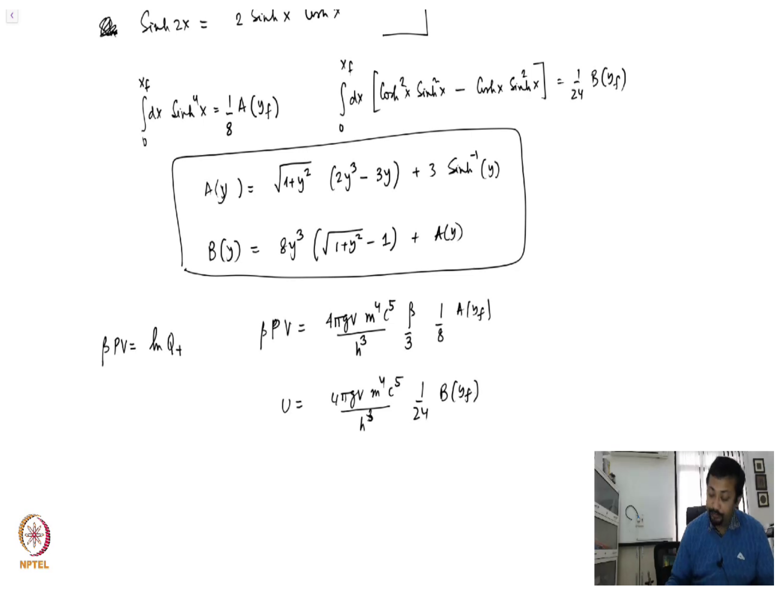
So, I will be looking for the limit of y f much much less than 1 and y f much much greater than 1, what is going to be the behaviour of this function and this function, but before even we do that essentially, I have to express this 4x f and 2x f in terms of sinh hyperbolic of x f and that is not very easy difficult to do. So, sinh hyperbolic of 4x is going to be twice sin hyperbolic 2x cosh hyperbolic 2x. This you can further break down as x cosh x cosh hyperbolic of 2x is going to be cosh square x plus sinh square x.

(Refer Slide Time: 38:29)



If you want to write down predominantly in terms of sinh square x, you are going to just as cosine double for; double angle formulas to sin except that one the sin changes sinh square of x cosh x sin hyperbolic of x which is 2x sin hyperbolic x cosh hyperbolic x plus 4 sinh hyper cube and then, you are going to have cosh x. So, this becomes your sinh x and sinh 2x is going to be 2 sin hyperbolic of x cosh hyperbolic of x right. So, you can do this.

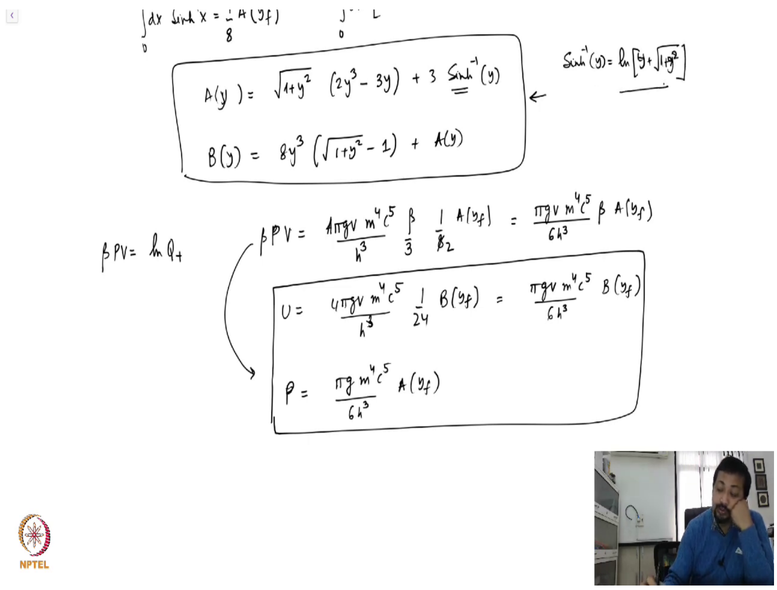
(Refer Slide Time: 40:20)



And in terms of this formulas, if you just do it, then sinh 4 x becomes A of y f and 0 to x f dx I had cosh square x sinh square x minus cosh x sinh square x which was the expression which entered for the energy, I will call this so, this is going to be 1 by 8 A y and this is going to be 1 by 24 B of y f, where your A of y f is going to be square root 1 plus y square twice y cube minus 3y plus 3 sin hyperbolic inverse of y. So, we will write down this as A of y. B of y is going to be B of y; B of y is going to be 8y cube 1 plus y square minus 1 plus A of y.

So, little bit complicated expression, but we have come to the fag end of this exercise and just we are done with this and we are left out with little bit more steps. So, one should note that beta pv is going to be ln of Q plus so that I have the expression beta times pressure times volume is 4 pi gv m 4 c 5 h cube beta over 3 and then, I have 1 by 8 of A of y f and the energy was 4 pi gv m 4 c 5 over h cube 1 over 24 B of y f.

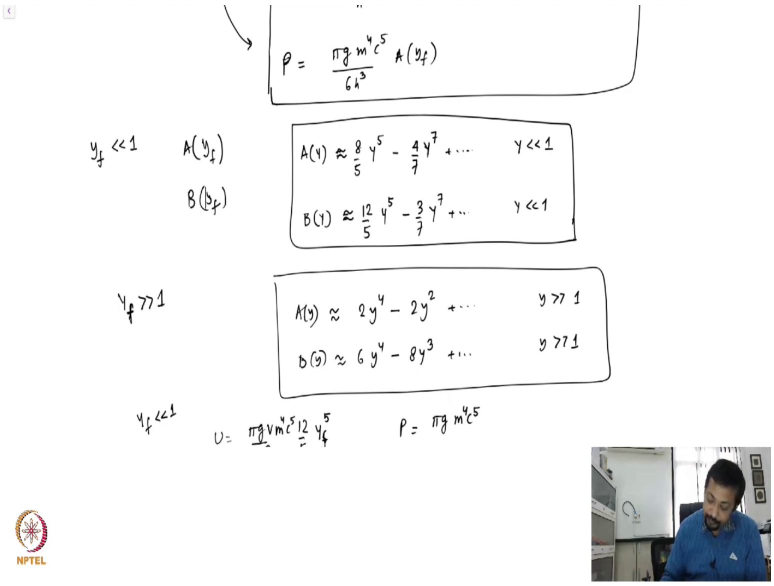
(Refer Slide Time: 43:14)



So, this clearly simplifies to the 4, 4 gets cancelled over here to give you pi gv m 4 c 5 divided by 6 h cube beta A of y f and this is also going to give you pi gv m 4 c 5 6 h cube B of y f. So, that the thermodynamic pressure, therefore, takes care of the volume factors. So, let us just use this quickly to write down thermodynamic pressure of this gas is going to be pi gv m 4 c 5 over 6 h cube, the beta, beta cancels out, I will have A of y f.

So, either v also cancels out over here so, you have only pi g and this is the final expression that we have been trying to do where A y and B y are complicated functions which are given over here, but let us just check whether we have come at the right result.

(Refer Slide Time: 44:36)



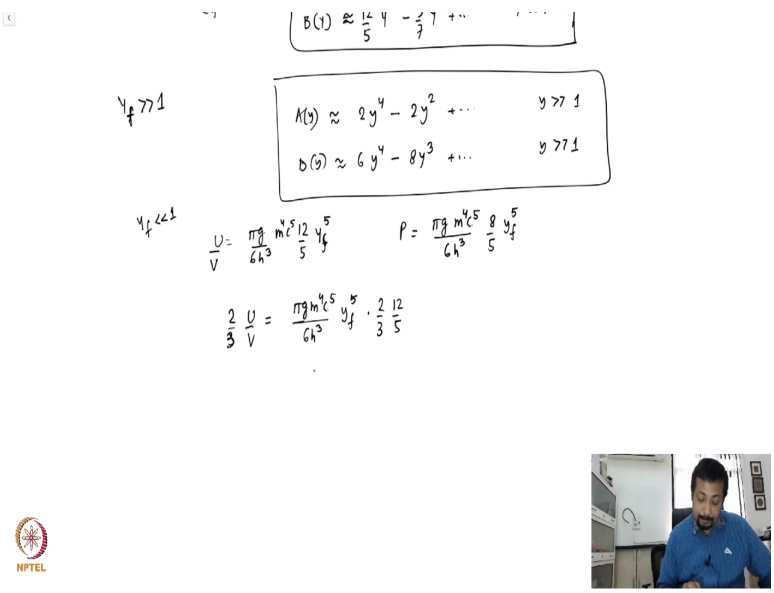
So, I am interested in y f much much less than 1, what happens to A of y f and B of y f. Here, I have this sin hyperbolic inverse of x so, I am going to use the expression that sin hyperbolic y is going to be log of y plus square root 1 plus y square. This I am going to use in the two limits, I will not work out the limits, but I will just write it down. It is very easy.

So, for y very very less than 1, this is 8 by 5 y to the power 5 and then, you have 4 by 7 y to the power 7 plus higher order terms when you have y much much less than 1 and you have B of y is going to be 12 by 5 y to the power 5 minus 3 by 7 y to the power 7 plus higher order terms. This is the first expansion that you are looking at.

For y much much larger than 1, A y, this quantity is going to be we will use an approximate sign over here since this is an asymptotic expansion twice y 4 minus twice y square for y much much less than 1 and B y is going to be 6 y 4 minus 8 sorry this is cube no, this is y square and this is going to be cube for y larger than 1.

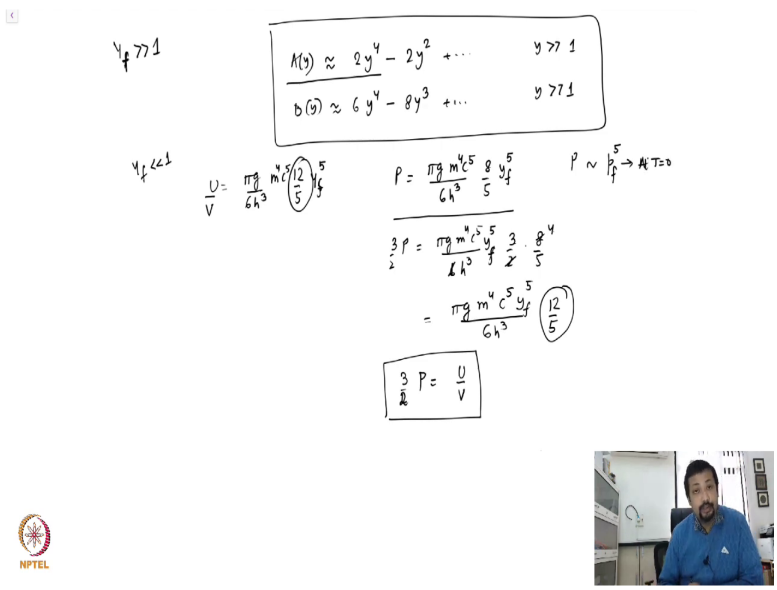
And this means that if I look over this expression for these two, for the first limit, I have U as pi gv over 6 h cube b of y f right, I have this expression for U, b for y f much much less than 1 is going to be 12 over 5 y f to the power 5 and the pressure is going to be pi g sorry, I have missed out the m 4 c 5, it is going to be pi g m 4 c 5.

(Refer Slide Time: 48:01)



Let us in this expression, let us take the volume to the other side so that it is become clearer more easier to handle, I will have 6 h cube and it is going to be 8 over 5 y f to the power 5 right. So, two-third times U over V is going to be pi g over m 4 c 5 divided by 6 h cube y f raised to the power 5, 2 by 3 into twelve by 5 and does it help?

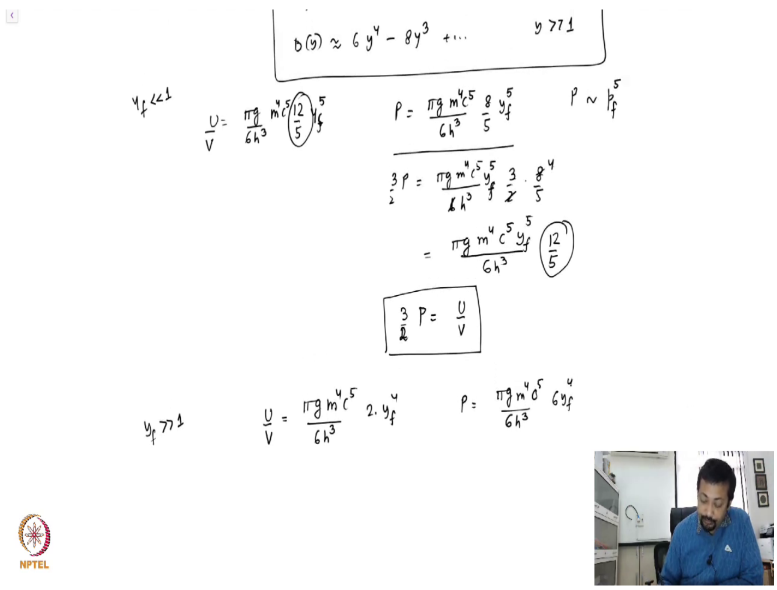
(Refer Slide Time: 48:58)



Now, from this expression, if you see if I multiply this by 3 by 2 p, this is going to be pi g m 4 c 5 over 6 h cube times there is a y f raised to the power 5 and then, I have 3 by 5 times 8 by; 3 by 2 times 8 by 5, this gives me a 4, 3 times of 4 is 12 and you have the result pi g m 4 c 5 y f raised to the power 5 6 h cube you have 12 over 5.

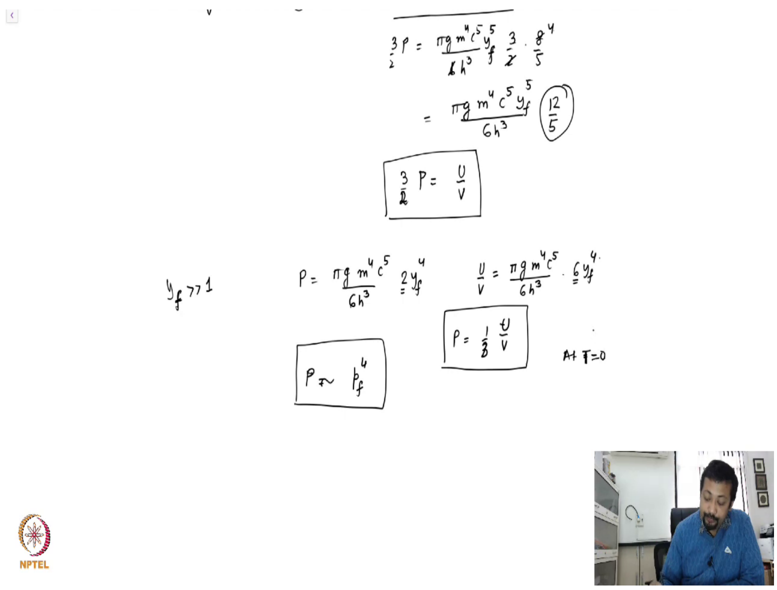
Now, you clearly see that this factor is same as the factor over here so that you know that 3 by 2 pressure is going to be U by V which is the case for a non-relativistic gas as we have seen. So, once again we have done, everything is consistent, we should also know that the pressure is proportional to p f to the power 5 just as we found out when we did the degenerate Fermi gas.

(Refer Slide Time: 50:18)



Now, comes the other limit y f much much larger than 1 and then, you have U over V which is going to be pi g over m 4 c 5 6 h cube and then, I have 2 times y f raised to the power 4, the pressure was pi g m 4 c 5 6 times h cube and this was 6 times y f.

(Refer Slide Time: 50:57)



Now, in the opposite limit of y f much much larger than 1, let us look at A y, A y is leading term is 2 y to the power 4 and therefore, pressure is going to be pi g m 4 c 5 over 6 h cube 2 times y f raised to the power 4 and U by V is going to be pi g m 4 c raised to the power 5 6 h cube times 6 y f raised to the power 4.

From this, it is immediately clear these two factors that the pressure is going to be one-third U over V for as we expect for an ultra-relativistic gas. Also, we also note that the thermodynamic pressure is going to be proportional to the p f raised to the power 4 at T is equal to 0. So, here also one notes that this is at T is equal to 0.

Now, you might want to wonder that why did we go through all this trouble of doing this, but interestingly end of this finds application in astrophysics, particularly in dwarf stars, white dwarfs, but that is for a more special field. So, we leave our discussion here at this point, you know how to handle such systems.