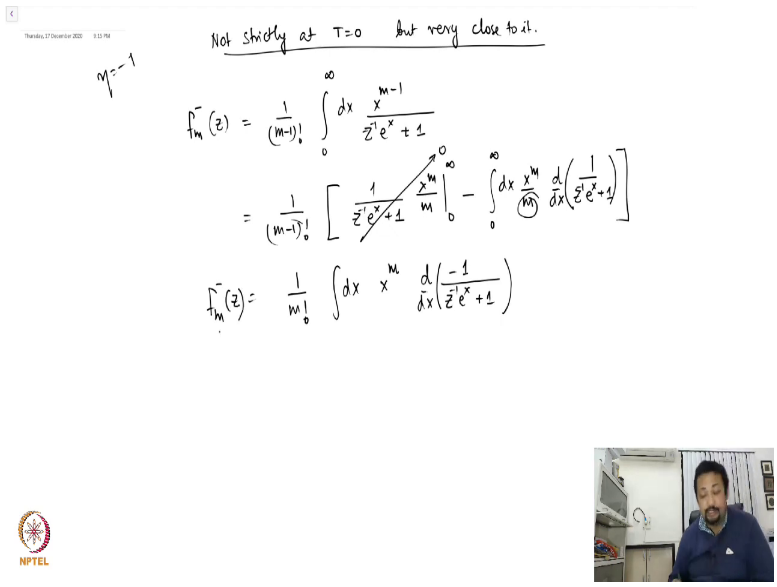
Statistical Mechanics
Prof. Dipanjan Chakraborty
Department of Physical Sciences
Indian Institute of Science Education and Research, Mohali

Lecture - 52
Ideal Fermi Gas close to T=0, Chemical Potential and Specific Heat

(Refer Slide Time: 00:24)

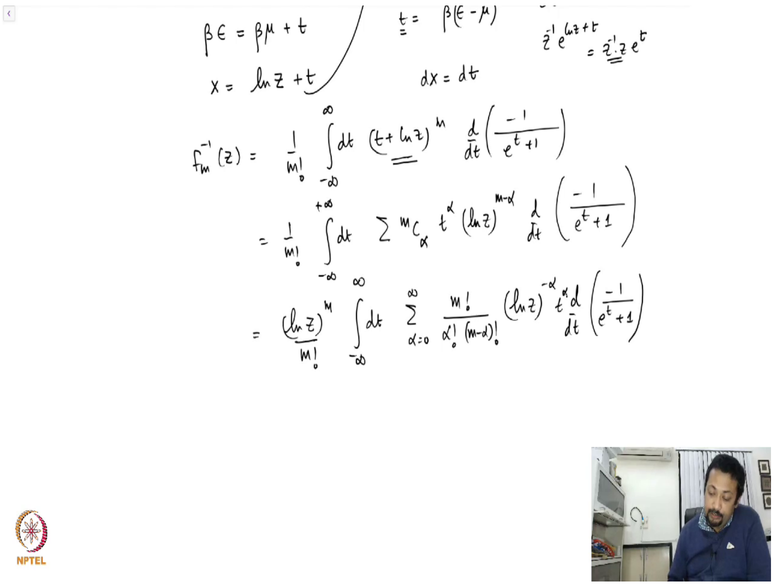


So now, we want to understand that what happens, if we are not strictly at T equal to 0, but very close to it right. So, we start off with the integral that we have defined now it is a fermionic system therefore; we put a minus that says that this is for a fermionic system where eta is equal to minus 1. And this we have defined as m minus 1 factorial, 0 to infinity dx x to the power m minus 1, Z inverse e to the power x plus 1.

The idea is first to integrate by parts, and if I want to integrate by parts so, 1 over Z inverse e to the power x plus 1 and integral of x to the power m minus 1 dx is going to be x to the power m 0 to infinity and then, I am going to have 0 to infinity dx x to the power m over m. And I am going to have d dx of 1 over Z inverse e to the power x plus 1 right.

So, that this is going to vanish at x equal to 0 this vanishes at x equal to infinity e to the power x diverges so, this 0 in both the limits and my m times m minus 1 factorial is 1 over m factorial integration dx x to the power m d dx of minus 1 over Z inverse e to the power x plus 1. So, I have f m minus of Z is equal to this.

(Refer Slide Time: 02:38)



Now, I am very very close to t equal to 0. So, that we write the beta epsa as beta mu plus t and t is beta epsa minus mu. Since, I am really very close to t equal to 0 you see the values of t will run from minus infinity to plus infinity right. So, that X is going to be ln Z plus t right and dx is going to be dt.

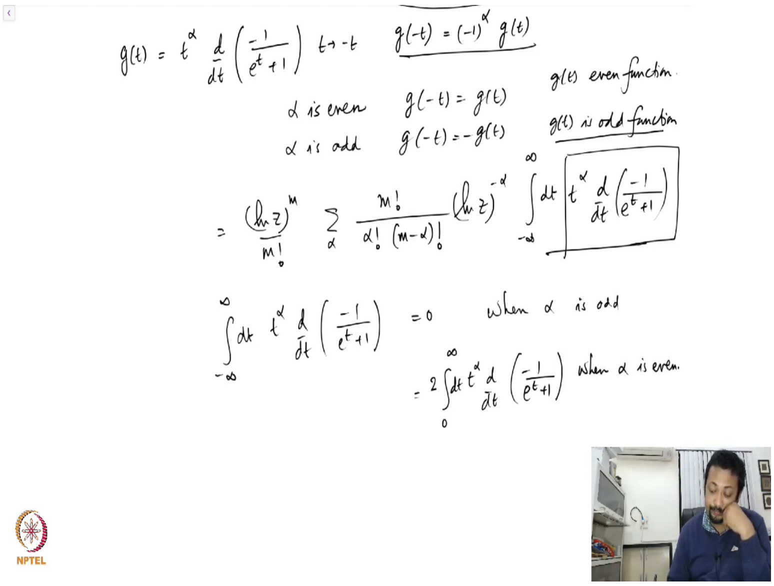
So, that f inverse of z is going to be this is the low temperature expansion we are looking at very close to t equal to 0 minus infinity to plus infinity dt t plus ln Z raised to the power m, substitute over here and then you have d dt of minus 1 e to the power t plus 1 right.

Z inverse e to the power x is going to be Z is e to the power beta mu. So, that this is going to x is going to be ln z plus t so, that you are going to have e to the power ln Z plus t which is going to be Z inverse times Z plus t, e to the my mistake Z inverse times Z times e to the power t and this is 1 therefore, you have e to the power t plus 1.

Now, this is a binomial expansion, I can do a binomial expansion over here and this expansion is going to be m C alpha t to the power alpha ln Z raised to the power m minus alpha d dt of minus 1 e to the power t plus 1.

So, that I can take ln Z raised to the power m divided by m factorial outside and minus infinity to plus infinity dt sum over alpha equal to 0 to infinity, I will have m factorial, alpha factorial, m minus alpha factorial ln Z raised to the power minus alpha and then I am going to have d dt of minus 1 e to the power t plus 1. There is a t to the there is a t to the power alpha that is missing over here so, we include that.

(Refer Slide Time: 05:56)



Now, note that g of t lets call this as t to the power alpha d dt of minus 1 e to the power t plus 1. If you note this then you will realize that this is going to be g of minus t is going to be minus 1 raised to the power alpha so, if you substitute t form, if you substitute t for minus t then you are going to get g of minus t as g of t.

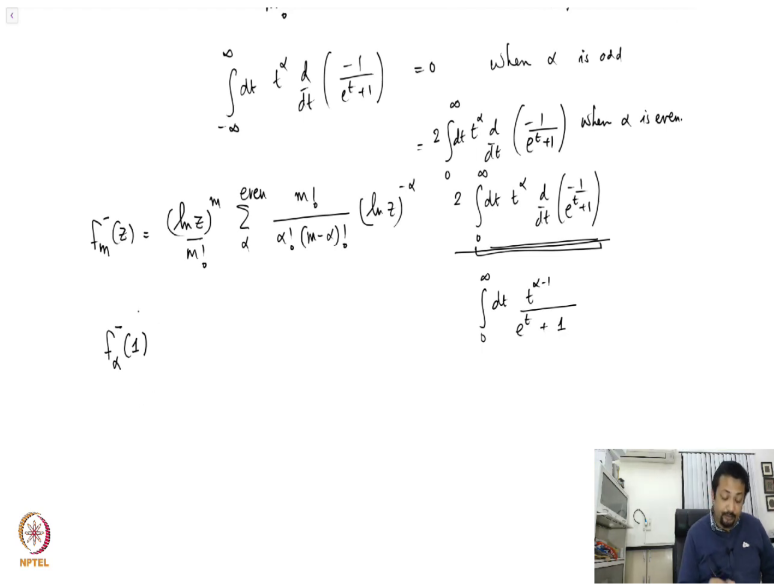
So if you look at it, then you see that if alpha is even if alpha is even then g of minus t is exactly g of t that means, g of t is an even function. If alpha is odd g of minus t is minus g of t so, that g of t is odd function. Now, this we are going to utilize, let us see how we are going to utilize. I can take this integral over here.

So, that I have ln Z raised to the power m m factorial sum over alpha m factorial, alpha factorial, m minus alpha factorial, and then I have ln Z raised to the power minus alpha, I have integral minus infinity to plus infinity dt t to the power alpha d dt of minus 1 e to the power t plus 1.

Now, this I have just now said that this is going to be an even function, if alpha is even and it is going to be an odd function is alpha is odd. So, that this integral minus infinity to plus infinity dt is going to be t to the power alpha d dt of minus 1 e to the power t plus 1 is going to be 0, when alpha is odd.

And this follows from that the fact that when alpha is odd g t is an odd function, and is going to be integration there is going 2 factor dt t to the power alpha d dt of minus 1 e to the power t plus 1, when alpha is even; good.

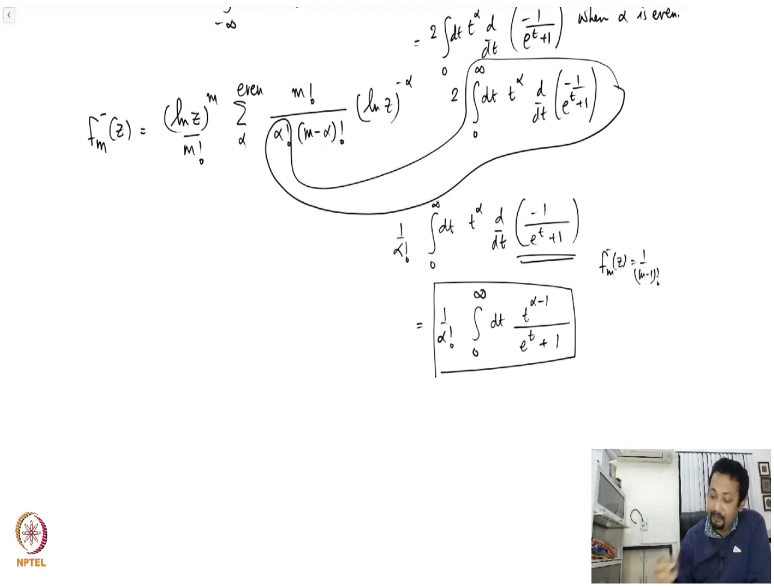
(Refer Slide Time: 08:36)



So, therefore, I have f m minus of Z is going to be ln of Z raised to the power m; m factorial, I then have the sum over alpha but now, alpha only even values are allowed so, that I have m factorial, alpha factorial, m minus alpha factorial. I have ln Z raised to the power minus alpha and then, I have a 2 0 to infinity dt t to the power alpha d dt of minus 1 e to the power t plus 1 right.

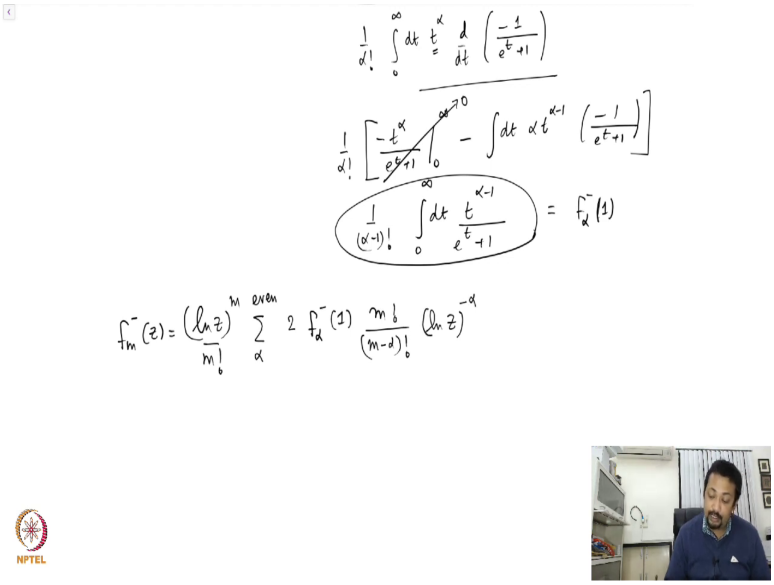
If you are clever enough then you will see that I can in integrate again by parts and I will come up that this integral is going to be 0 to infinity dt t to the power alpha minus 1 e to the power t plus 1, and this is exactly f alpha minus so, that I have so if you want to recall [FL].

(Refer Slide Time: 10:10)



So, now I can combine this integral with 1 by alpha factorial and t to the power alpha d dt of minus 1 e to the power t plus 1. So, that this is 1 over alpha factorial 0 to infinity, I can integrate by parts again here and it is going to be t to the power alpha minus 1 e to the power t plus 1.

(Refer Slide Time: 10:56)

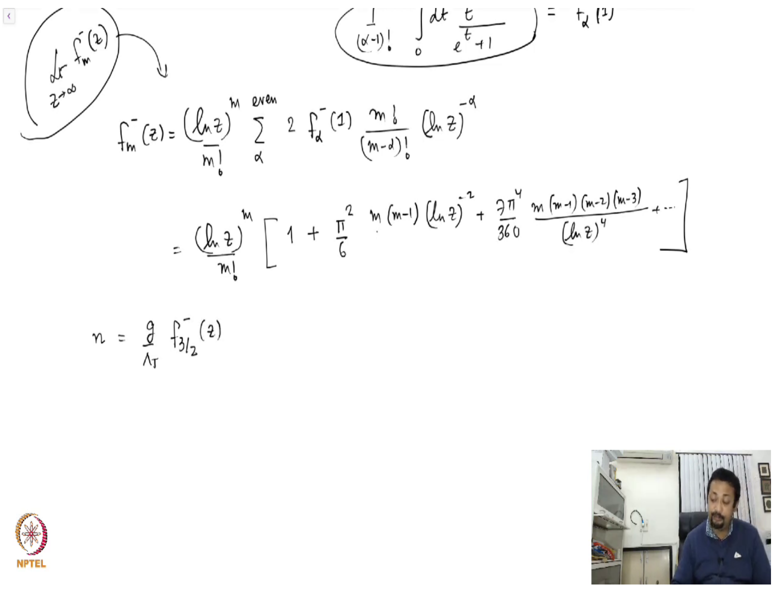


So, that you immediately realize that this function if you recall I had this as m minus 1 factorial, to give me 1 over alpha factorial 0 to infinity dt t to the power alpha d dt of minus 1 e to the power t plus 1. I can integrate by parts this one again. So, what happens if I integrate by parts? I can take this as the first function so, that you have t to the power alpha then integral of dt is gives you minus e to the power t plus 1 0 to infinity.

Then, you have minus dt derivative of this is t to the power alpha minus 1 and then you have essentially minus 1 of e to the power t plus 1, this vanishes and you have alpha minus 1 factorial 0 to t infinity dt t to the power alpha minus 1 divided by e to the power t plus 1, strikingly familiar right because, this is f of alpha minus 1.

So, that my low temperature expansion becomes ln Z raised to the power m divided by m factorial, I have sum over alpha which are e take even values of alpha l out twice f alpha minus 1 times m factorial divided by m minus alpha factorial and then, I have ln Z raised to the power minus alpha right.

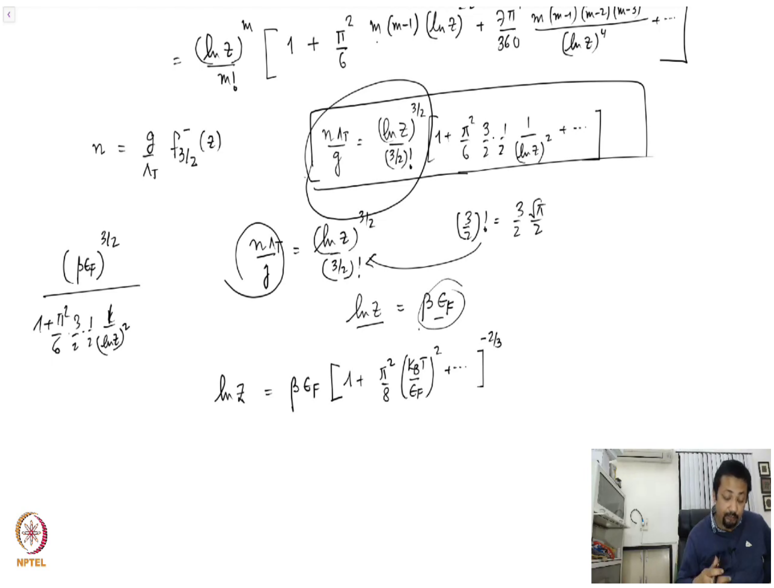
(Refer Slide Time: 12:39)



So, this is the limiting value limit of Z to infinity f m minus of Z, if you take this is the approximation that you get. So, I can now write it down m m factorial using the values of f this functions is going to be pi square over 6 m into m minus 1, ln Z minus 2 plus 7 pi 4 over 360 m into m minus 1 into m minus 2 into m minus 3 divided by ln Z raise to the power 4 plus higher order terms; good.

What do I do with it? Well our starting point again is to look at the number density. The number density we know that for a quantum ideal gas is g over lambda t f 3 half minus Z for a fermionic system right, and I am looking at the limit of z to infinity.

(Refer Slide Time: 13:58)



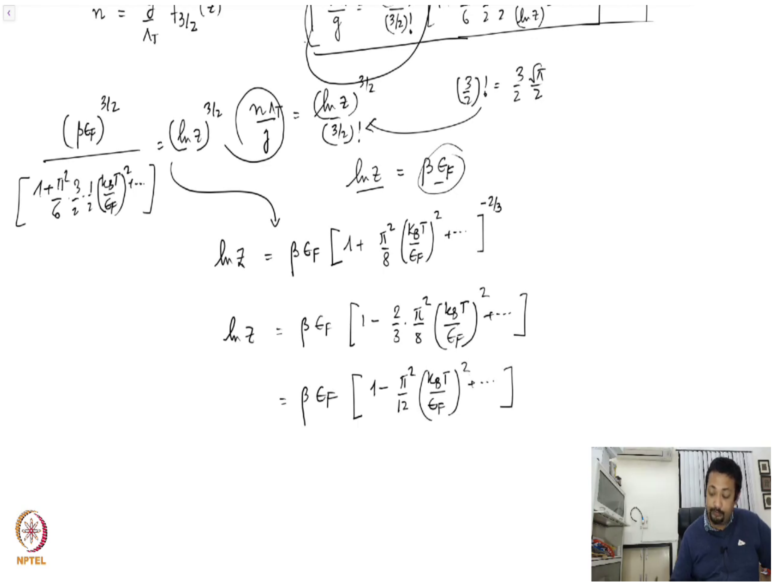
So that means, n lambda t over g is going to be ln Z m is 3 half. So, I am going to have 3 half factorial 1 plus pi square over 6 m into sorry m is 3 half and therefore, I have 3 half into half and I have 1 over ln Z whole square we will keep it as higher order terms. The lowest correction is when this is equal to this.

So, that I have n lambda t over g is going to be ln Z raised to the power 3 half by 3 by 2 factorial, and 3 by 2 factorial you should note is 3 by 2 square root pi over 2. So, once you put in over here and if you compute ln Z from this term you will see that you are going to get ln Z is equal to beta epsa F, which is the Fermi energy at 0 temperature.

Then, if I have this I can write down ln Z invert this over here which is going to be beta epsa F 1 plus pi square over 8 ln Z is beta epsa F to the leading order. So, we will write this as K B T over epsa F whole square plus higher order terms raised to the power minus 2 by 3 right. Why is that? Because you see to the leading order I have determined that ln Z is going to be beta epsa so exactly at 0 temperature right, which I also get it by ignoring all the other terms in the series.

Once I have that then essentially I have beta epsa F, raised to the power 3 half if I look at this expression this whole equation I take this whole equation divided by 1 plus pi square over 6 times 3 by 2 times half 1 over ln Z whole square, but ln Z is equal to beta epsa F I am very very close to the 0 temperature.

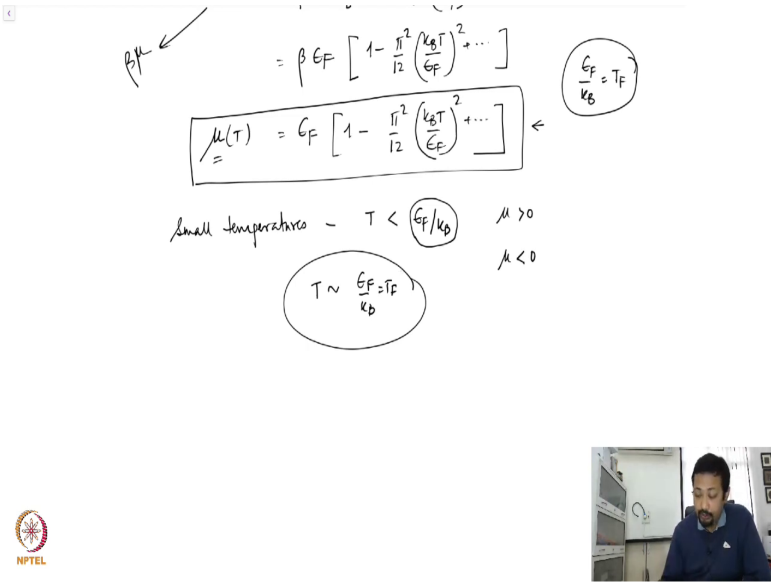
(Refer Slide Time: 16:37)



So, here I am going to write down beta epsa F 1 by beta epsa F whole square which is nothing but K B T over epsa F whole square plus higher order terms, and this is ln Z raised to the power 3 half right.

So, this then leads me to this equation and you see that ln Z is going to be beta epsa F 1 minus 2 by 3 times pi square over 8 K B T for epsa F whole square plus higher order terms which is going to be beta epsa F 1 minus pi square over 12 K B T over epsa F whole square plus higher order term.

(Refer Slide Time: 17:35)



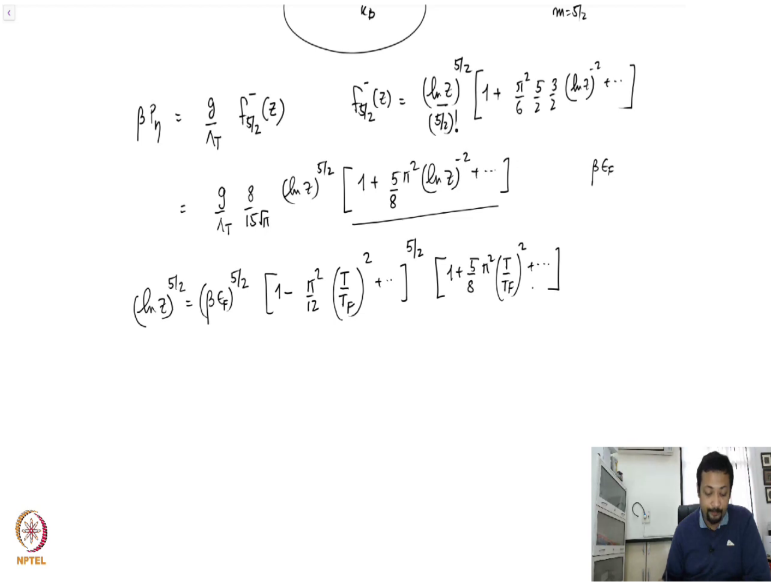
So, that the chemical potential very close to 0 not exactly at 0, since this is going to be beta mu is going to be epsa F 1 minus pi square over 12 K B T over epsilon F whole square plus this. So, that as you go away from zero temperature you see the chemical potential decreases with as a quadratic in temperature.

So, we looked at the chemical potential as a function of temperature and you realize from looking at this equation that for small temperatures, which are less than epsilon F over K B can the chemical potential is positive. So, mu is greater than 0. This particular quantity epsilon F over K B is called the Fermi temperature T F right.

In contrast for high temperatures mu is less than 0 mu becomes the chemical potential is negative, and the zero crossing happens at temperature which is of the order of epsilon over K

B that is equal to your T F of the order it does not exactly happen that T F, but it happens of the order of T F.

(Refer Slide Time: 19:10)



Now, our next interest lies in looking at the pressure. So, I know that beta P eta is g over lambda T f 5 by 2 minus of Z, and I am looking at the low temperature expansion and therefore, I am interested in the expansion that we did a little while ago right, this is the one.

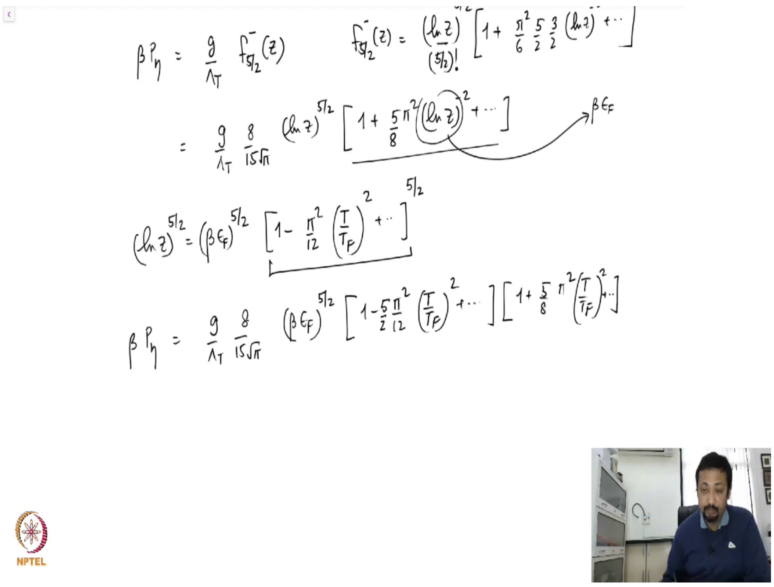
So, we will only keep up to ln Z up to this order and therefore, f of 5 by 2 minus of Z is ln Z raised to the power 5 by 2 divided by 5 by 2 factorial times 1 plus pi square over 6 m m minus 1 is 3 by 2 m is 5 by 2 here, and I have ln Z raised to the power minus 2 plus higher order terms.

So, that this expression is g over lambda T 8 by 15 the 5 by 2 factorial square root pi and then I have ln Z raised to the power 5 by 2 I have 1 plus 5 pi square over 8 ln Z raised to the power minus 2 plus higher order. All I have to do now is to substitute ln Z from the here.

So, let us do that. So, ln Z raised to the power 5 by 2 is beta epsa F raised to the power 5 by 2 and then I have this term, which is raised to the power 5 by 2 that is 1 minus pi square over 12. I have K B T over epsa F whole square which we will write down as T over T F whole square plus higher order terms raised to the power 5 by 2.

And in this term I am going to replace since I am looking at the leading order correction I am just going to replace this ln Z as beta epsa F which is T over T F raised to the power square plus higher order terms sorry ok, this discussion comes little later.

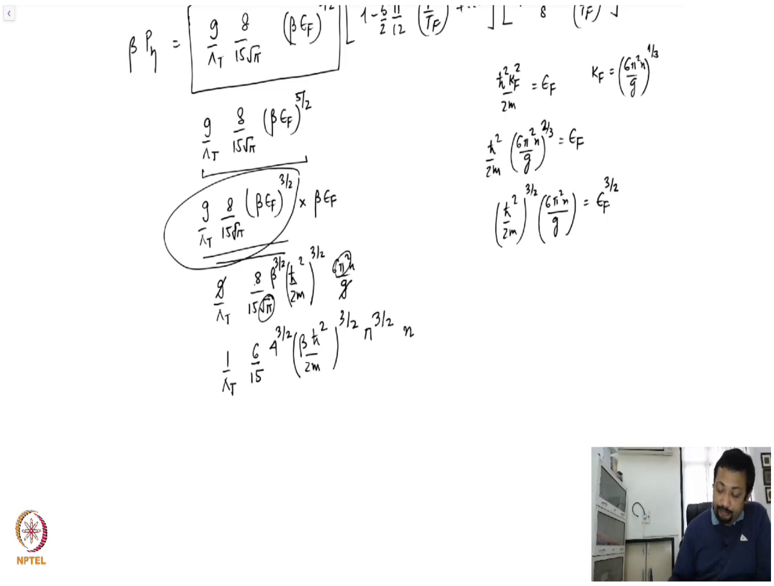
(Refer Slide Time: 21:55)



So, here just I am going to replace this by beta epsa F. So, that beta times P eta the pressure becomes g over lambda T 8 over 15 square root pi ln Z 5 by 2 raised to the power 5 by 2 is beta epsa F raised to the power 5 by 2, then I have 1 minus pi square over 12.

Since, this bracketed quantity is raised to the power 5 by 12, I will use a binomial expansion to write it down like this way and then I have T over T F whole square plus higher order terms times 1 plus 5 by 8 pi square T over T F whole square plus higher order term.

(Refer Slide Time: 22:54)



Now, let us look at this expression, I have g over lambda T 8 over 15 square root pi beta epsa F raised to the power of 5 by 2 and I want to express this in terms of the number density n. So, even if you have forgotten how epsa F depends on n there is always a way out.

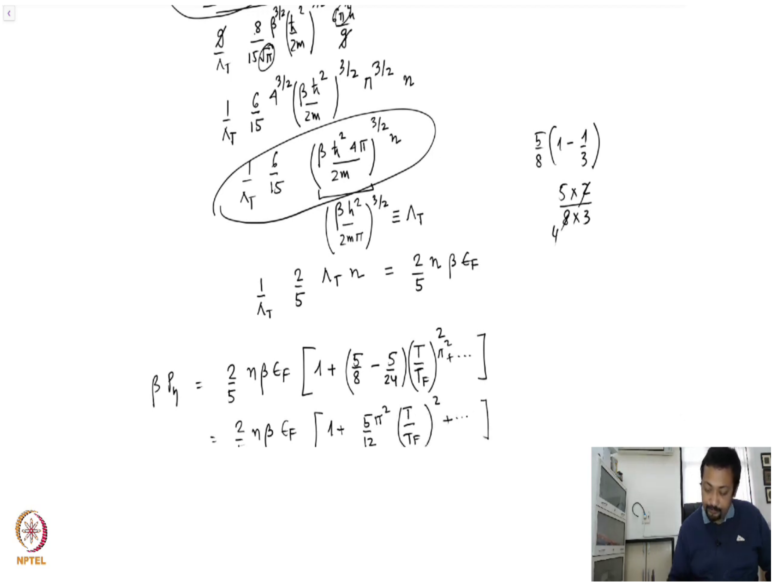
Remember you can always start from this expression which is your epsa F and your Fermi wave vector K F is defined as 6 pi square n over g raised to the power 1 by 3. So, that I have h square over twice m 6 pi square over n over g raised to the power of 1 third sorry I raised to the power 2 third is going to be your epsa F.

This part now I split, I write down g lambda T over 8 over 15 square root pi times beta epsa F raised to the power 3 half into beta epsa F. I mean this part I am interested in simplifying. Then, I have to look at epsa f to the power 3 half and epsa f to the power 3 half is h square over twice m raised to the power 3 half and 6 pi square over n over g is going to be your epsa F raised to the power 3 half.

So, let us substitute; I have g over lambda T 8 over 15 square root pi h bar square over twice m beta to the power 3 half, this raised to the power 3 half. Now you should know where I am going. Things should be clear to you now and then I have 6 pi square n over g; this part is just this thing. I have a beta epsa F that multiplies this; g g gets cancelled out right.

And then, I have 1 over lambda T, I have 6 over 15, I have beta h bar square over twice m raised to the power 3 by 2. Now I have an additional factor 8, which I am going to write down as 4 to the power 3 by 2 and I have a pi square; pi to the power 3 by 2. There is a pi square in the numerator there is a square root pi in the denominator that gives you pi to the power 3 half and then finally, I have n.

(Refer Slide Time: 25:50)

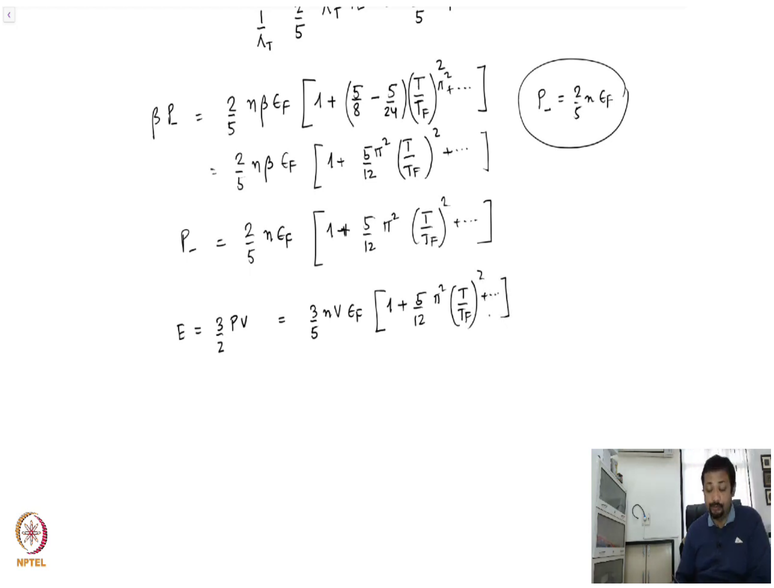


So, 1 over lambda T 6 over 15 beta h bar square 4 pi over twice m raised to the power 3 by 2 times n, and if you simplify this just this part this is beta h square over twice m pi raised to the power 3 half and which you immediately identify as lambda T. So, your this whole expression then boils down to 1 over lambda T this is going to be 2 by 5 and then you have a lambda T and then n so, which is equal to 2 by 5 n.

Do not forget that you have an additional beta epsa F lying outside so, you have this. So, therefore, your beta P eta is 2 by 5 n beta epsa F. Now, I have to take care of this, this is to the leading order it is 1 plus 5 by 8 minus I have 5 by 24 T over T F whole square plus higher order term.

So, 5 by 8 1 minus one-third that is 5 into 2 by 8 into 3 one gets it, 5 by 12; sorry. So, this is 2 by 5 n beta epsilon F 1 plus 5 by 12, there is a pi square somewhere; is it? Yes there is a pi square which we have missed, so we will T over T F whole square plus higher order term.

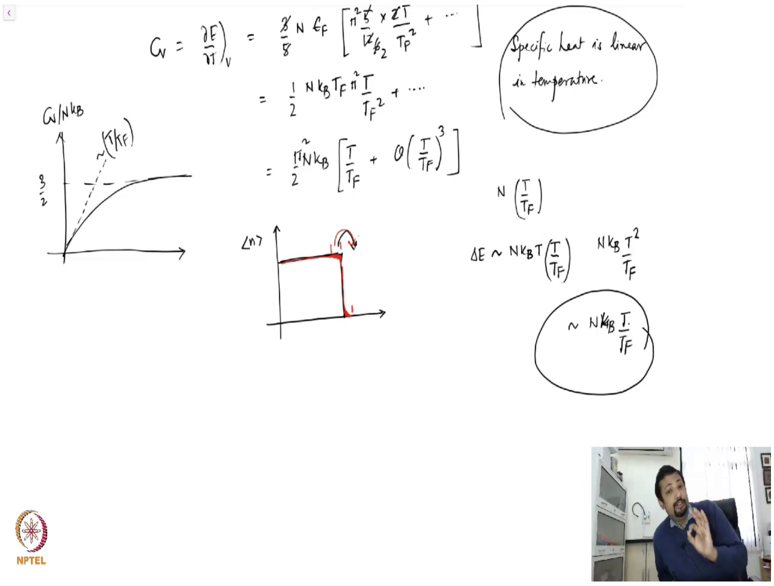
(Refer Slide Time: 28:01)



P eta the pressure of the Fermi gas is n beta epsa F sorry n is 2 5 n epsa F 1 plus 5 by 12 pi square T by TF whole square plus. Now, this behavior is completely unlike an ideal classical gas, because in an ideal classical gas at zero temperature it has a zero pressure. But I know that at zero temperature the pressure of a Fermi gas, the eta is minus here so, we will put a p minus, we will put a P minus which is going to be 2 by 5 n times epsa F.

So, even at zero temperature the Fermi gas do exert certain amount of pressure, the pressure is not 0. Now, the energy is 3 by 2 P times V right, which means, I have three-fifth n times the volume times epsa F 1 plus 5 by 12 pi square T over T F whole square plus higher order terms.

(Refer Slide Time: 29:38)



Now, the specific heat is del E del T at constant volume, which is your C v that is going to be 3 by 5 N epsa F times 5 by 12 into 2 T over T F square plus higher order terms. So, a little more simplification gives me N epsa F is K B. So, that this is going to be 3 by 2 so, a little more simplification gives me this cancels to give you 6, this cancels to give you 2 and the 5 5 cancels out, you have half NK B and the epsa F is K B T F. So, I will write down this as K B T F time's T over T F square plus higher order terms.

So, that you have NK B there is a pi square which we have missed there has to be a pi square here. There has to be a pi square half NK B pi square by 2NK B and then I have T of over T F plus order T over T F whole cube, these are my higher order corrections. So, clearly you see that very close to the zero temperature the specific heat is linear in temperature and this is true for any arbitrary dimension you consider, we consider a three dimensional case, but in any arbitrary dimension you consider this is going to be true.

So, one can plot the specific heat with respect to temperature and you will get a result which will look like this and this is your. So, this is C v over NK B and this is the classical dolong petite limit and here this part is going to be proportional to T over T F a linear scaling.

Now, as we said that this is going to be linear in any arbitrary dimension that you consider, it does not depend on the dimension and we will see that this is in stark contrast to a bose gas. But the reasoning is very very simple. Recall that at T equal to 0 the Fermi Dirac the occupation number looks like a step function. So, when you increase the temperature what you at most do is you basically have something which is like this.

So, you are exciting this many number of p particles and taking into a higher excited states over here. So, this fraction goes over here from here to here, the higher excited state. Now, that number of fraction is T over T F times N and each of this particle carry an energy which is K B T. So, that the total this is NK B T over times T over T F, which is NK B.

So, we will not write equal to but we will just say the change in the energy that you that happens from going from the here to here, the exciting because of thermal fluctuations that you have is T square over T F. And therefore, your specific heat which is proportion which is basically delta E delta T. And therefore, the specific heat which is the derivative of this is will go as NK B T over T F. And this result is surprisingly independent of the dimension of the system.