Fluid Dynamics for Astrophysics
Prof. Prasad Subramanian
Department of Physics
Indian Institute of Science Education and Research, Pune

Lecture - 07
Conservation laws: Momentum conservation and Euler equation

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So, we are back and we are now going to be talking about the second of the Conservation laws that we promised we would cover that of Momentum conservation. And as I said momentum conservation is essentially just Newton's second law of motion F equals m a except with some notable differences when it comes to fluid elements, we are not talking about F equals m a for point particles or for rigid bodies, we are talking about F equals m a for a fluid element right.

So, the right hand side here m a is simply the rate of change of momentum. The left hand side well is simply the force acting on the fluid element right. So, there are two reasons the momentum can change it can be. So, what we have written down here is, when we say momentum in this case we are really talking about rho u this is what we really momentum per unit volume ok. So, rho u is momentum per unit volume right.

So, this momentum per unit volume when you integrate over a volume element it is it becomes simply momentum. So, the momentum can change because there is an intrinsic there is a d over d t kind of there is an actual change of momentum within the volume.

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Momentum conservation

- Newton's second law of motion: $\mathbf{F} = m\mathbf{a}$
- The RHS ($m\mathbf{a}$) is simply the rate of change of momentum
- The momentum can change because of "intrinsic" ($\partial/\partial t$) change inside the volume:

$$\int \frac{\partial(\rho \mathbf{u})}{\partial t} \, dV$$

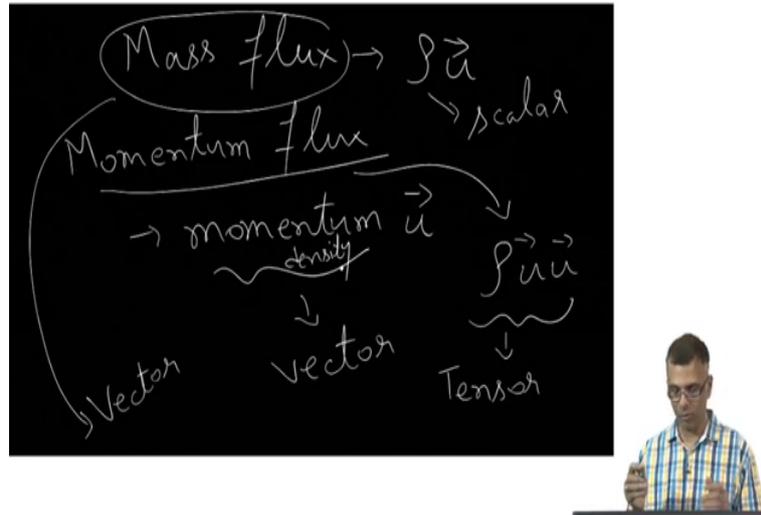- ..and also due to the flux of momentum through the bounding surface

$$\int (\rho \mathbf{u})\mathbf{u} \cdot d\mathbf{A}$$

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It can also change because there is a flux of momentum through the bounding surface right.

You know you in order to get the mass flux you simply mass flux was just you recall for instance right and momentum flux well momentum times u in a way not quite. What is the main difference here? This is a scalar whereas, this is a vector. So, therefore, the momentum flux while mass flux is a vector mass flux is a vector, but because momentum itself is a vector, momentum flux is essentially rho u u, where it is not a u dot u it is an outer product not an inner product and so, this is a tensor.

We will discuss this. We will sort of specify what exactly rho u u is and then we will come to it. But just wanted to you know draw the analogy in order to get the mass flux you simply multiply with the mass density with the velocity, in order to get the momentum flux you simply multiply the momentum density with u momentum density sorry and that gives you the momentum flux.

So, the momentum can change because of intrinsic change of momentum inside the volume and also momentum flux through the bounding surface just like we discussed for the mass continuity equation. There is a bounding surface and there is a flux of momentum through the bounding surface either outside or inside and then you what you do is you do an integration over the surface element right.

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And you add both of this up before doing that just like we. So, the surface integral I beg your pardon here, I am calling a surface integral element d A and here I am calling it d S it is really the same thing ok. So, you can just like we use Gauss's divergence theorem for you know we use Gauss's divergence theorem in this situation.

Just like we use Gauss's divergence theorem in this situation to go from here to here similarly, we are going to use Gauss's divergence theorem here in order to convert the surface integral into a volume integral except the divergence theorem that we will be using is for tensors and simply because you know the momentum flux is now a tensor.

So, we can use this, we can transform the surface integral into a volume integral like this and so, go from here to here using Gauss's divergence theorem.

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So, the rate of change of momentum is this. Now notice how similar it looks to what we wrote down for the continuity equation for the mass conservation equation except instead of just rho you have rho u right. And the flux is essentially rho u u x where this is not an inner product it is an outer product.

So, this quantity is a tensor ok. The reason we went from the surface integral to the volume integral was because this term was in terms of a volume integral and you want to put both on a same footing and so, that is what we do and as before this statement is valid for arbitrarily small volumes ok.
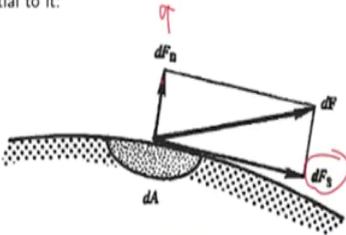
And so, if the dV is taken to be arbitrary arbitrarily small the integration essentially becomes invalid I mean since we are integrating only over one volume element and that means, that

you know the integrand itself is equal to 0 just like the notice the analogy with you know what we did for mass conservation is exactly the same.

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So, before saying that we said F equals m a. So, far we have only done m a here and what is F? F can the rate of change of momentum is equal to force of course, this is F equals m a this is Newton's law and the force on the fluid element can have a volume component due to body forces say due to gravity and that can be written as you know rho g dV.

The g is really a proxy for any kind of body force it can be gravity, it can be some other kind of body force also some by body force I mean a force that is acting all throughout the fluid not only on the surfaces its acting throughout the fluid throughout the body of the fluid like gravity.

So, in that case you would have to integrate all through the volume and not merely on the surface forces are a little trickier. So, forces can be acting throughout the body of the fluid for instance gravity, but they can also be acting on a surface one example of a surface force pressure.

Pressure is force per unit area of course, but the pressure acts only on the surface and as we have discussed earlier in the course pressure is only one element of the stress tensor that is a normal element the other components of the a stress tensor are shear stresses which are essentially on the same footing as that of pressure. Force per unit area except the very fact that you are talking about area means you are talking about the boundary.

So, this is what we just said recall we can have forces that are on a surface area element that are normal to it as well as tangential to it. Like here you see you have a normal force as well as a tangential force. These are tangential forces these are forces that are you know the normal force acts in the same direction as the vector that characterizes the surface element right.

So, this is the outward outwardly directed normal that characterizes the surface element and the normal force is acting along the same direction whereas, the tangential forces are along the surface. This can be an example of a tangential force another example for tangential force can be one which is acting into the plane of this screen.

So, taken together forces that these kinds of forces are called shear forces and so, these are the two different kinds of forces the main difference between a body force and a surface force well is surface force acts only on the surface whereas, body forces act throughout the body of the fluid not only on the surface right.
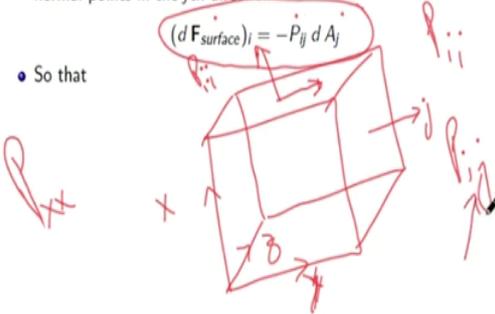
So, the ijth component of the pressure tensor represents the force in the ith direction felt by an area element whose outward normal points in the jth direction. Let us pause for a minute and try to understand what this means, let us draw a little box like this and so, let say X Y and X cross Y would be I believe this would be Z. So, you can have a force that is acting like this can also have a force that is acting like this right.

So, the force in the ith direction suppose this is the i direction and the area element is characterized also by the i direction in which case the force would be P i that is i and anything else any other force like this would be something like a P i j and the j would be something like this. So, if that is the ith direction and this is the jth direction. So, the j in this case would be along the Y axis and the i would be along the X axis right.

So, something like this would be P x x whereas, something like this would be a P x y or P y x right. So, that is all this is. This is simply accounting for the fact that you know you can have there is something called a pressure which is ultimately at the end of the day it is a d F over d A except you know the direction of the F and the direction of the A need not be the same one can be along i and one can be along j and i need not be the same as j.

In which case you need to have something in between this is just a matrix multiplication right. So, this is a vector and that is a vector. So, by necessity what is in between needs to be a matrix and the elements of that matrix are the elements of the pressure tensor. The diagonal elements are simply the scalar pressure the off diagonal elements like this are the shear stresses ok.

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So, when you integrate this, the F surface the ith direction of the F surface is you know when you integrate over the surface element that points whose outward normal is along the jth direction. So, you integrate that and you apply in order to; in order to go from here to here again, you apply Gauss's divergence theorem. It is the same thing you went from an area integral to a volume integral and this thing is nothing, but a divergence just written in a different kind of form this is just a divergence.

So, it is the same thing you went from an area integral to a volume integral and in doing so, you would encounter a divergence and this is just that. So, this is just the an application of the Gauss's divergence theorem right. So, the total force on a fluid element would be a body force and surface forces and in use by using Gauss's divergence theorem you convert the surface integral into a volume integral like that.

The advantage being you can put both of those on the same footing and you can have a volume integral for the whole thing exactly like we did for the mass conservation equation no difference ok. So, this is the total force. So, this is the F and we did the m a here. So, we equate this to F. So, this is the; this is the finally, the you know when I write divergence of P, the P is really a tensor. So, the divergence of a tensor is defined by this ok.

Momentum conservation - the pressure tensor

- The ijth component of the pressure tensor represents the force in the ith direction felt by an area element whose outward normal points in the jth direction:

$$(d\, F_{surface})_i = -P_{ij}\, d\, A_j$$

- So that

$$(F_{surface})_i = -\int P_{ij}\, d\, A_j = -\int \frac{\partial P_{ij}}{\partial x_j}\, dV$$

- So the total force on a fluid element is

$$\int \rho g\, dV - \int p\, d\mathbf{A} = \int \left[\rho g - \nabla . \mathbf{P} \right] dV$$

- where

$$(\nabla . \mathbf{P})_i = \frac{\partial P_{ij}}{\partial x_j}$$

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The ith element of the divergence is simply d P i j d x j that is how divergence of a tensor is defined. You really do not need to know much more about tensors this is good enough. We will move to slightly less complicated I mean this is the most general way of writing down the momentum conservation equation and do not despair, if this seems a little too involved for you I just wanted to write down the most general way and then reduce it to slightly more familiar forms such as the Euler equation we will get to that in a minute.

So, taken together. So, this was F and we have already written down m a taken together the momentum conservation equation is this. So, this would be the F and this would be the ok.

Momentum conservation - putting it together

- Taken together, the momentum conservation equation is

$$\frac{\partial}{\partial t}(\rho \mathbf{u}) + \nabla \cdot (\rho \mathbf{u}\mathbf{u}) = -\nabla \cdot \mathbf{P} + \rho \mathbf{g}$$

- Note, $\mathbf{u}\mathbf{u}$ is a second order tensor, in the sense of

$$\mathbf{a}\mathbf{b} = \begin{bmatrix} a_1 b_1 & a_1 b_2 & a_1 b_3 \\ a_2 b_1 & a_2 b_2 & a_2 b_3 \\ a_3 b_1 & a_3 b_2 & a_3 b_3 \end{bmatrix}.$$

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And here is what I just said I said a few slides ago that the quantity u u this is an outer product ok. It is a second order tensor in the sense that this is how the outer product of a two vectors a b is defined and this is the matrix a 1 b 1 a 1 b 2 so on so forth. So, that is what u u is. If you are wondering what an outer product is well this is an outer product.

Now, does this look familiar if you did not have this pesky u u here and if you only had the rho here and the rho u here this would be and of course, the right hand side would be 0 this would just be the mass continuity equation this is the momentum continuity equation right.

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Momentum conservation: "conservative" form

- Neglect body forces (**g**)
- The "conservative" form of the momentum conservation equation is

$$\frac{\partial}{\partial t}(\rho \mathbf{u}) + \nabla \cdot (\rho \mathbf{u}\mathbf{u} + \mathbf{P}) = 0$$

- which is of the form: *partial time derivative of quantity + divergence of flux of that quantity = 0*

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So, now if only for some reason the right hand side was 0, if this was 0 then writing you know the left hand side equal to 0 would be the conservative form of the momentum equation.

In other words, if there was no pressure gradient there were no body forces then either that or if there are no body forces and you are able to somehow cleverly you know take this pressure gradient into this term, absorb all the you know elements of the pressure gradient of the divergence of the pressure tensor into here, then we can write down the momentum conservation equation in a conservative form.

And that is what we will try to do here you neglect body forces that is one thing first and doing exactly what I just said you take this in here which is again of the form partial time

derivative of the quantity plus the divergence of the flux of that quantity is equal to 0 right. So, this is the conservative form of the momentum conservation equation.

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Momentum conservation for inviscid fluids: The Euler equation

- For an *inviscid* fluid, its enough to consider the scalar pressure

$$P_{ij} = p\,\delta_{ij}$$

; i.e., only the diagonal elements of the pressure tensor are non-zero

- Furthermore, we expand

$$\frac{\partial}{\partial t}(\rho \mathbf{u}) + \nabla \cdot (\rho \mathbf{u}\mathbf{u} + \mathbf{P}) = 0$$
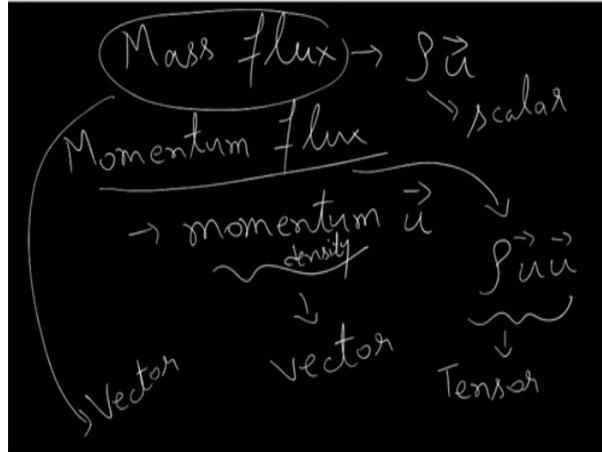
and use the mass continuity equation

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0$$

to get the Euler equation (*work it out!*)

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$$\rho \frac{\partial \mathbf{u}}{\partial t} + \rho(\mathbf{u} \cdot \nabla)\mathbf{u} = -\nabla p + \rho \mathbf{g}$$

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Now, let us simplify things a little bit for an inviscid fluid it is enough to consider the scalar pressure. In other words it is enough to consider the diagonal terms of the pressure tensor just the diagonal terms where the delta i j is equal to 1 well by definition the delta i j is equal to 1 only when i is equal to j in other words only for the diagonal elements when i is not equal to j its equal to 0. So, for turns out for an inviscid fluid it is enough to consider just the scalar pressure.

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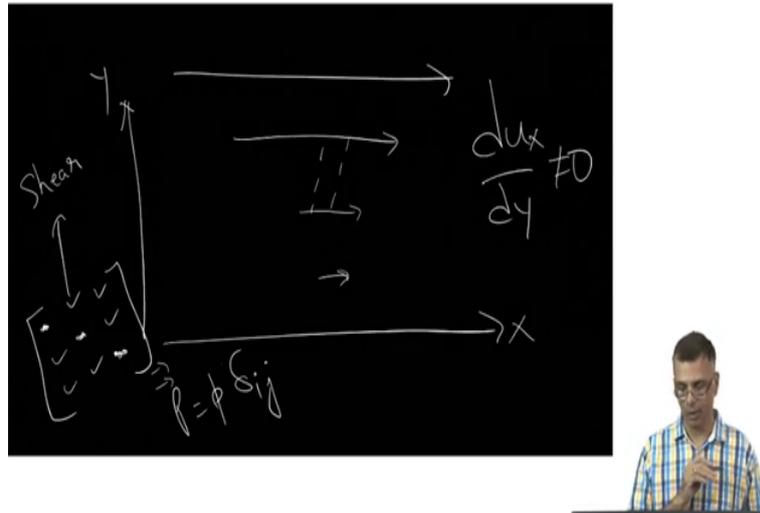
Now, maybe it is worth pausing a little bit to figure out why this is so. You see the whole concept of viscosity arises when you have a situation such as this where you have these would be the velocity vectors of a viscous fluid which is flowing along the X direction and that is Y. So, only when d u x d y is not equal to 0 you need to think about viscosity.

And this happens only when there are shear forces in other words here you see there is a shear, this is the definition of a shear in other words also you can also say that there is rotationality in the fluid ok. Unless this shear there is no point thinking about the viscosity is a irrelevant and when there is shear, shear essentially it represent. So, the diagonal elements would be this would be P x x P y y P z z and the off diagonal elements which are the other ones ok.

These represent shear. No shear no point in any in thinking about viscosity because this shear is you can think about this shear as due to you know little rubber bands connecting the two layers together and refusing to let the layers slide freely no shear no viscosity ok. In other words, no off diagonal elements no viscosity or no viscosity no off diagonal elements in other words when you are talking about an inviscid fluid which is a fluid in other words inviscid meaning no viscosity that is what it means ok.

For an inviscid fluid there is no need to worry about the off diagonal elements it is enough only to consider the diagonal elements in other words you can write the P the P tensor as some scalar times delta i j. So, which is non zero only when i is equal to j ok. So, that is what I mean here and in other words only the diagonal elements of the pressure tensor are non-zero you realize we are simplifying things from what we saw in the last slide in where we retained everything in P i j furthermore we expand this is still the most general equation we expand this.

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot (\rho \vec{u}) = 0$$

Mass continuity equation
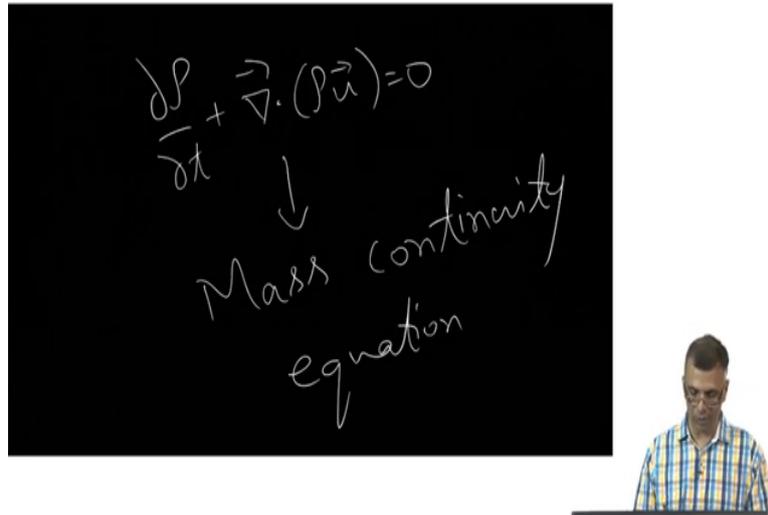
And use the mass continuity equation you use the mass continuity equation which was this is the mass continuity equation right. So, you use this in here. So, this is the mass continuity equation you use this in here you expand this out and you use this in here to get what is called the Euler equation ok. So, there are three things that you need to consider, number 1 the pressure tensor is greatly simplified in that only the diagonal elements are important right and really that is the main thing.

And the second thing is you do need to use the mass continuity equation and then the momentum equation becomes this becomes this and this is called the Euler equation and Euler equation is valid only for inviscid fluids in other words situations where the viscosity is not important and it becomes like this.

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And there is a simpler form of the Euler equation which you will find in Arnab Rai Choudhuris book it follows the Lagrangian treatment in other words you write down an F equals m a sitting on top of a fluid parcel.

And right keep in mind that the Euler equation is valid only for inviscid fluids in other words fluids where the viscosity is negligible and you can really set the viscous terms equal to 0 equivalently the off diagonal elements of the pressure tensor are 0 and only the diagonal terms are non-zero else there are additional terms in the pressure tensor which involve viscosity and you get the Navier-Stokes equation.

So, the Navier-Stokes equation is a momentum equation when you include viscosity and the Euler equation which is this one is valid for situations where viscosity is not important.

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Before we move onto Navier-Stokes..I

- Lets consider flows that are incompressible ($\nabla \cdot \mathbf{u} = 0$) and irrotational ($\omega = \nabla \times \mathbf{u} = 0$)

In other words, it is valid only for inviscid fluids. So, we will stop here and we will consider a few other aspects before we move on to the full Navier-Stokes equation.