Fluid Dynamics for Astrophysics
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Lecture - 52
Magnetohydrodynamic [MHD]: Magnetic flux-freezing

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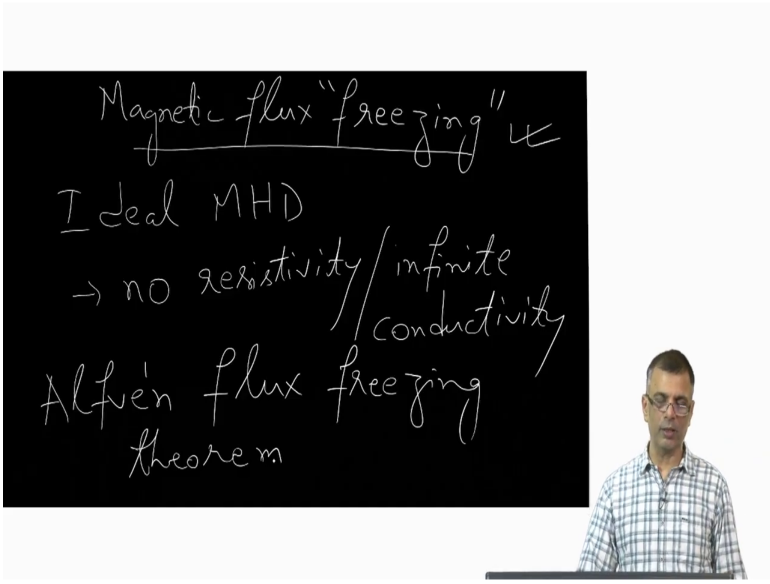

We will now discuss today one of the very important consequences of ideal MHD of which is this effect called magnetic flux freezing ok. Ideal MHD essentially means the main thing to note regarding ideal MHD is that, there is no resistivity its in this sense that the magnetohydrodynamic system is ideal, there is no resistivity or equivalently the conductivity is infinite and yeah infinite conductivity.

So, as a consequence of the of this particular assumption it so, happens that there is this effect called magnetic flux freezing alternatively sometimes called the Alfven flux freezing theorem.

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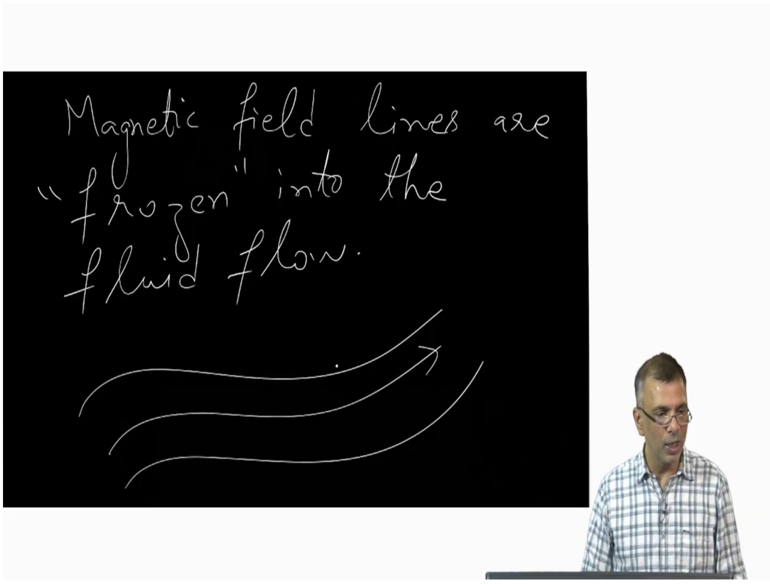

And before going into the detailed discussion of that we should sort of tell you what it really means in practical terms what it means is that, magnetic field lines in ideal MHD as in if the ideal MHD conditions are satisfied, the magnetic field lines are frozen into the flow frozen into the flow into the fluid flow which essentially means that if you have a streamline of fluids like this ok.

That would be the fluid stream line; the magnetic fields also would look like that it would coincide with a stream line ok. So, this is one thing this is one way of looking at it, the other way of looking at it is that the magnetic field lines are frozen in, but more properly though more.

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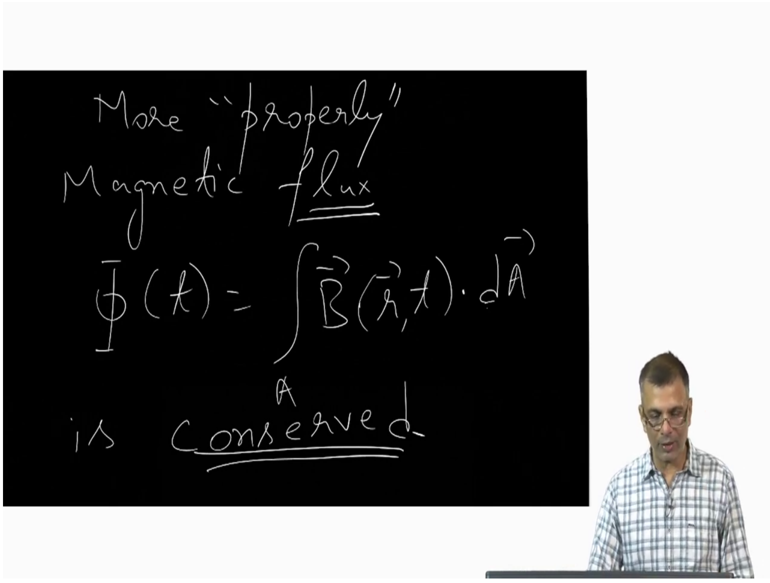

In other words, the preceding definition was kind of a lose statement of Alfven flux freezing theorem. More properly what it means is that the magnetic flux which is which we denote as a phi which is essentially you know the area integral of B dot B of I will say B which is a position of both which is a function of both position and time yeah sorry I sometimes write A and I sometimes write s for the area element and I often get confused between the two.

So, in this case I am writing A yeah. So, this is conserved this is the real statement of Alfven's flux freezing theorem what this means in effect is that, consider a flow right where you know the for some reason that the cross section of a magnetic flux tube is changing for some reason yeah.

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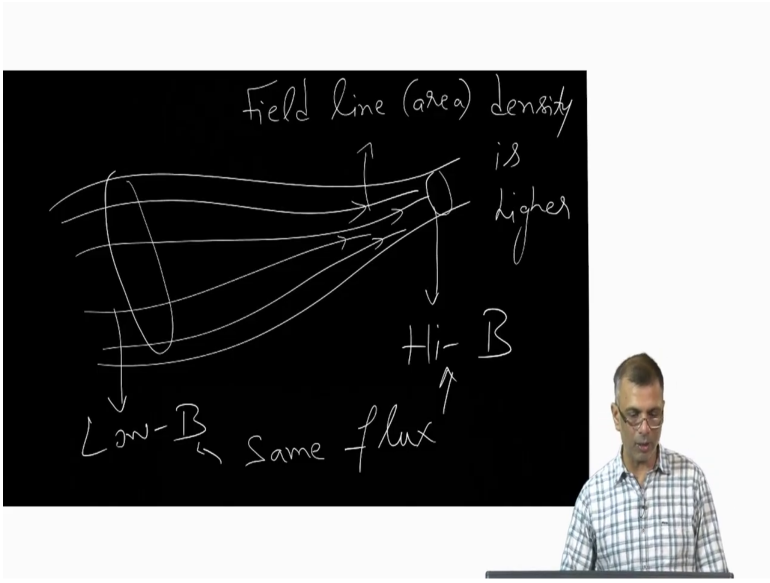

And you know the magnetic field is essentially the number of fade lines per unit area. So, the value of the magnetic field is essentially the number of field lines per unit area and what we are saying here is that, integral of the magnetic field dotted with the cross sectional area is constant.

In other words here you see out here the area is smaller and so, if the area is smaller well then and if the flux has to be constant then the magnetic field has to be larger. You see the area the area is smaller where the flux tube is constricted ok.

Therefore, if the flux has to be the same as what it was out here yeah, it follows that the magnetic field itself out here Hi-B relatively speaking this would be Hi-B and this would be Low-B why? Because same flux here and here ok. The flux is the same that is what the statement is saying the flux is the same yeah. However, the cross sectional area and when we say area we are really talking about the cross sectional area like this.

This area this cross sectional area is larger here than there. So, therefore, there is only one way the flux can be the same here and here and that is if the magnetic field is higher than it is here. Take seen from the point of view of magnetic field lines essentially what its saying is that.

Well you know the magnetic flux tube constricted somehow for some reason I dont know why, but it did that is what the flow is doing and we also said that you know the magnetic field lines follow whatever you know the flow is doing that is you know they are frozen into the fluid.

So, it follows that the B field lines will look somewhat like this. In other words, the field line density, but really the area density number of field lines per unit cross sectional area is higher here as compared to here the field lines are squeezed together they are more concentrated.

That is simply a consequence of the fact you can look at it as a consequence of the fact that the flux is conserved either that or you can look at it as a consequence of the fact that the magnetic field lines are frozen into the flow.

So, the flow is for some reason getting constricted and well you know field lines have to obey the flow then that is what Alfven's flux freezing theorem is telling you um. So, this is one very important property of magneto hydrodynamics, it turns out that that there are many other conserved quantities in MHD well mass is conserved so, on so, forth all this fine.

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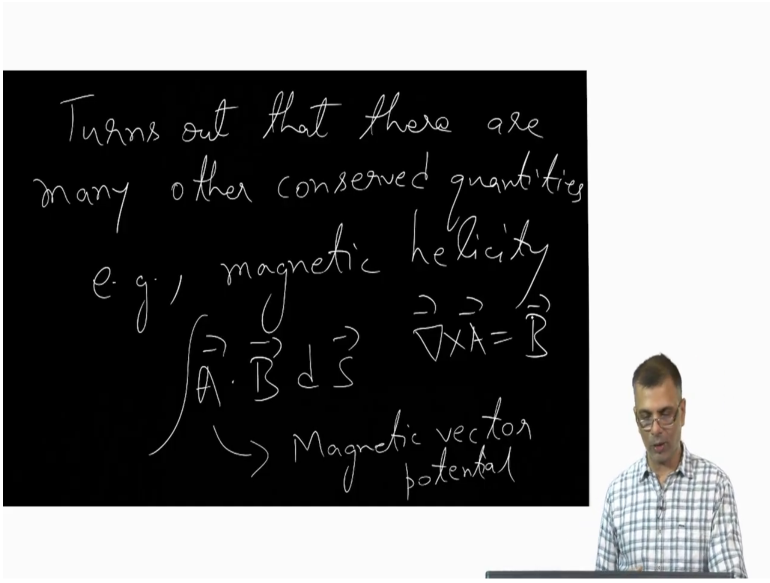

But even with regard to magnetic field and such things ok. For instance, magnetic helicity which would be the quantity A dot B ds or so, on and so, forth where this is the magnetic vector potential which is to say curl of A gives you B we have not really discussed this much, but this is the magnetic helicity, it turns out this is also something that is conserved ok.

There are many other quantities in in MHD that are conserved for instance one important thing one important quantity is the magnetic helicity, we will concentrate our attention right now just on the magnetic flux ok. So, the magnetic flux is conserved in ideal MHD and ideal MHD essentially means that, you know the conductivity is infinite ok.

Now, you remember everything pretty much everything in in MHD has its roots in the magnetic induction equation which is this right.

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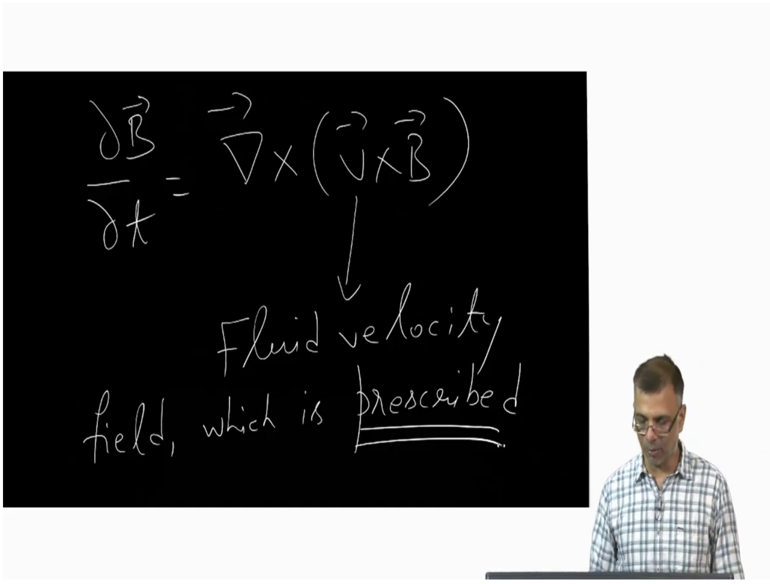

The time rate of change of magnetic field is the curl of v cross B . So, this would be the fluid velocity fluid velocity field which is which is prescribed and I will write this down, but I will I will explain what is what I mean by this word which is prescribed.

In other words, what I mean by this is that I am not solving for the fluid velocity field in this case and this is v and B are the primary variables in in MHD as we have repeatedly emphasized, for instance the current density j which it does not really have an independent place in MHD.

So, to speak its simply a derived quantity, its simply taken to be a shorthand for curl of B right if v and B are the primary quantities of those we generally take v to be a prescribed thing we do not solve for it, we say well this is the fluid velocity field right here ok.

Now, given a certain magnetic field that is embedded in this fluid velocity field, I don't want to know how the fluid velocity field came about I don't want to solve for the fluid velocity field at least as of now ok. The simplest way of understanding these things is to say that the fluid velocity field is just given to me by someone ok.

So, it's prescribed. Now the induction equation is going to tell me how this velocity field evolves with time here how it evolves with time in response to this velocity field now this is ideal MHD. So, there are no resistive terms there is no grad squared B kind of resistive term here there is no so, because you know the conductivity is infinite.

So, turns out that the Alfven flux freezing theorem is a direct consequence of this of this induction equation and let us try to see how. And for that I will need to draw a certain a certain picture and it will take me a couple of minutes to draw this I request you to bear with me.

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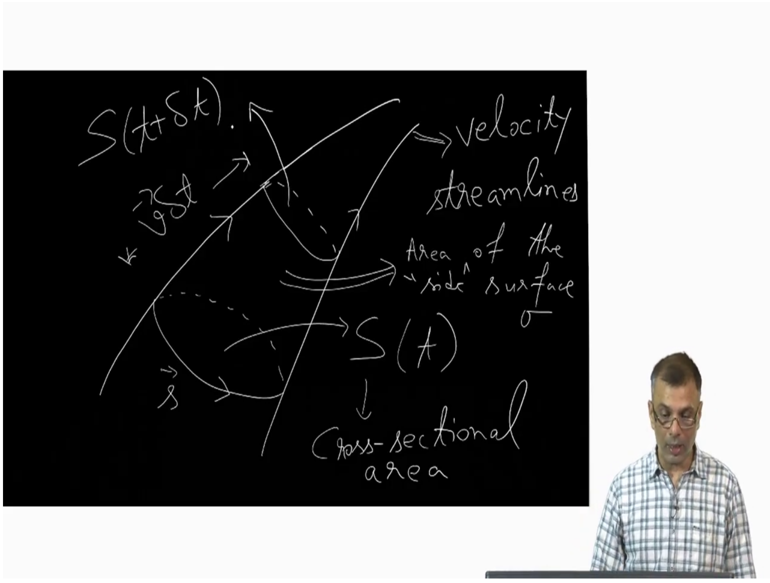

I thought it's better to do this on the board rather than show you a slide. So, this would be these would be the velocity stream lines which for some reason you know you have a larger cross section here and a smaller cross section there for some reason I dont know why ok.

So, now, let us consider a cross sectional sort of a like this yeah and like that I really should be should be dotting it here. If I dotted it there then I should dot it here as well and so, let me do that I erase this. So, the like that that is what I meant.

And so what I am going to do is I am going to call this this length element small s this is bad notation actually the length element this is really it's really should be l, but you know anyhow and so, this would be a cross sectional area S of t capital S ok.

So, the cross sectional area at time t is this is cross sectional area. So, the cross sectional area at time t is some S of t and this little small s is the length element along this perimeter and this is a cross sectional area at time some t plus delta t S of t plus at some time t plus.

So, this would be this is how the area element looks like at time t and this is how the area element looks like at time t plus delta t. In this particular case you know the area element has shrunk with time, but that is not necessarily always true it might have expanded as such that its completely general and the fluid is moving with a velocity v.

So, the difference here is would be v delta t that would be this length that would be this length ok. And I call I call the area of the side surface in other words the area of the cylinder the side I call it sigma and as with everything this delta t is really small. So, although you know there is a there is a squeezing and this v delta t is quite small.

So, you can you can idealize this to be a straight cylinder now as long as everything is really small. So, it’s in the usual calculus kind of sense yeah. So, just to re emphasize these are the velocity stream lines ok. And these two loops are separated by a time interval delta t between this and that. So, this is the loop at time t and this would be the loop at time t plus delta t ok.

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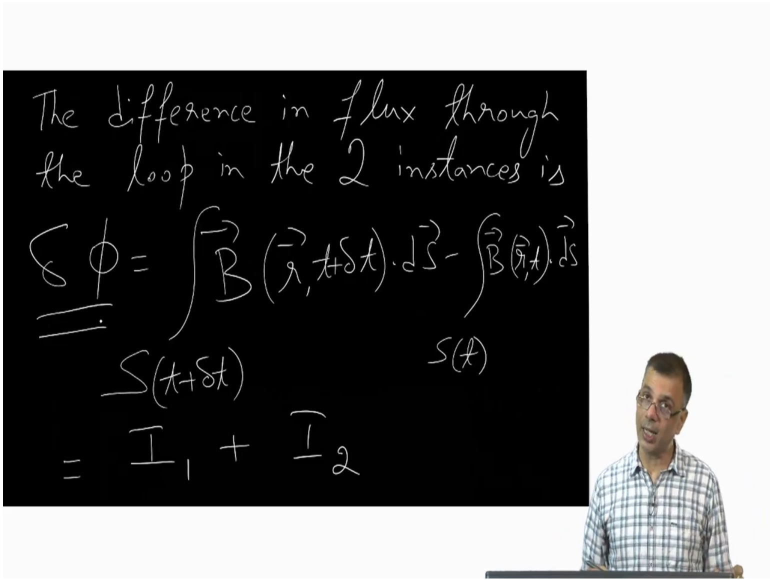

Now the difference in in in flux magnetic flux through the loop through the loop yeah in the two instances in which two instances well this and that ok. In the in the in the two instances is that because I am saying difference in flux I write delta phi yeah this is equal to and by going by the definition of phi, phi is essentially B dot d a is not it.

So, phi and delta phi is integration over the cross sectional area at t plus delta t yeah the magnetic field which is a function of position and time and the time is t plus delta t mind you yeah dot d s this would be the phi at t plus delta t this would be here yeah. So, we are now looking at the difference in flux.

So, the phi the phi at this instant minus the phi at this instant and we have already written down the phi at this instant now what is the phi at this instant its quite simple. The same kind of the same kind of expression except instead of S at t plus delta t I just have s at t yeah then instead of B at r, t plus delta t I just have B at r and t right and its an area integration.

So, what is the difference between these two? Well you know that the cross sectional area has changed and we have allowed the magnetic field to change really that is that is the main difference between these two ok.

So, let us now call this let us say that this is equal to sum I 1 plus I 2 in a minute we will ask what is this I 1 and what is this I 2 ok. So, we will we need to be very careful about this which is why I am proceeding slowly ok.

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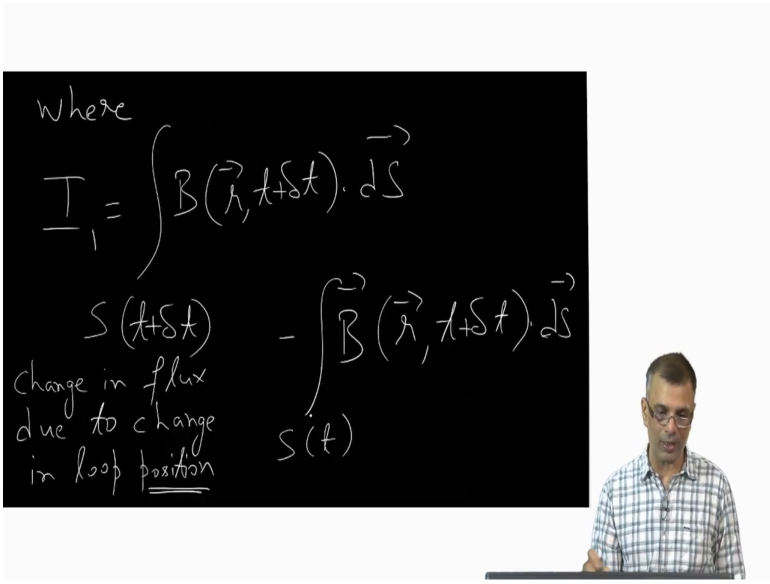

So, where I 1 is equal to I 1 is essentially the difference in in flux due to the change in the loop position and I will write this down and then we will discuss it like that the loop has the loop position is changed r, now what is the difference between this piece and that piece?

You see the integrand is the same the integrand is B r t plus delta t, here also its B r t plus delta t what is the only difference between this piece and this piece? This thing. The fact that the S the area you are integrating it over has changed here it was t plus delta t here is t.

So, this is the change in flux in flux due to change in loop position let me erase this its not very nice. So, due to change in loop position like that. So, the only difference between this and this like I said is the difference in the area over in in the cross sectional area over which its being integrated.

And in other words the difference between this area and that area whereas, in reality the magnetic field is also changed between here and here, but I am pretending that that its the same. Am pretending in this particular case I am saying that the magnetic field is the same as what it would be what it is at t plus delta t.

So, I am taking the same magnetic field here which is the B at t plus delta t and using it here also ok. So, this is not entirely physical, but we will see; we will see why I am writing it down like this its a change in loop. So, because I want to write down one chunk which is exclusively due to change in loop position and another chunk which is exclusively due to the change in the magnetic field I want to split it up like that ok.

So, I 1 is the change in I 1. So, I this this is the really the delta phi and this is the basic thing this is what we want to check and we know that eventually we are going to show that the change in flux the difference in flux is 0 ok.

So, but let us write this down and I am writing this down as I 1 plus I 2, where I 1 is the change in flux due to the change in loop position and I 2 is a change in flux due to the fact that the magnetic field has changed ok. So, we have already written down I 1 right and so, now, let us write down what would be I 2 and I will write down I 2 as I 2 as.

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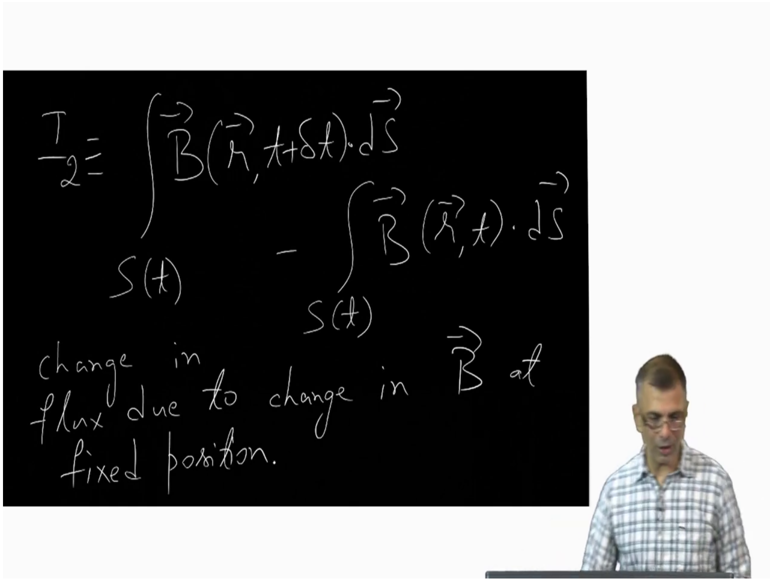

So, you see now the integration the cross sectional area over which I am integrating will be the same will remain the same you will see this ok.

Whereas, the magnetic flux itself is I will change the magnetic flux. So, here it is r B at r t plus delta t that area integration minus I will keep this the same. I will keep the area over which I am integrating I will keep that the same, but I will change the B and so, this would be the change in flux due to change in B at a fixed position.

So, you see the interpretation is as follows, in the first chunk which is this which is I 1 yeah I was taking the same B in both the integrands and I was only changing the area over which its being integrated.

In other words, the changing flux due to the change in loop position and so, the interpretation there would be I am taking the magnetic field here and I am using the same magnetic field for this cross section area although technically that is not correct, but you will see that it all cancels out.

The second one for I 2 for I 2 what I am doing is I am keeping the loop position fixed at S of t in other words I am not changing the loop I am I am just keeping the loop position fixed. But I know that the magnetic field has changed. So, what I am doing is I am keeping the loop position fixed and I am I am calculating the change in flux through the same loop between the magnetic field due to the change in magnetic field.

You see the magnetic field is one value here and its another value here simply due to the change in magnetic field I am trying to see what is the change in flux and really. So, the change in flux can be is because the main thing is that the change in flux is due to the change in loop position as well as the change in magnetic field ok.

And you can see at the end of the day you wanted B r t plus delta t you know d s at s t plus delta t minus B r t dot d s at s of t and if you see if you look at I 1 this is essentially the first chunk here ok. This is the first chunk here and the second chunk is this.

Whereas, in I 2 you see you will see that this is a first chunk and that has a positive sign, this exact chunk this exact thing has a positive sign in I 2. So, when you add up I 1 and I 2 this guy vanishes. So, all you are left with is this and this which gives you what you wanted and so, that is why we are saying that delta phi is equal to I 1 plus I 2 ok.

The reason we are doing is this way in this slightly convoluted way is that, there are certain simple physical interpretations of I 1 and I 2 and its a little easier to arrive at the physical interpretation of I 2 first ok.

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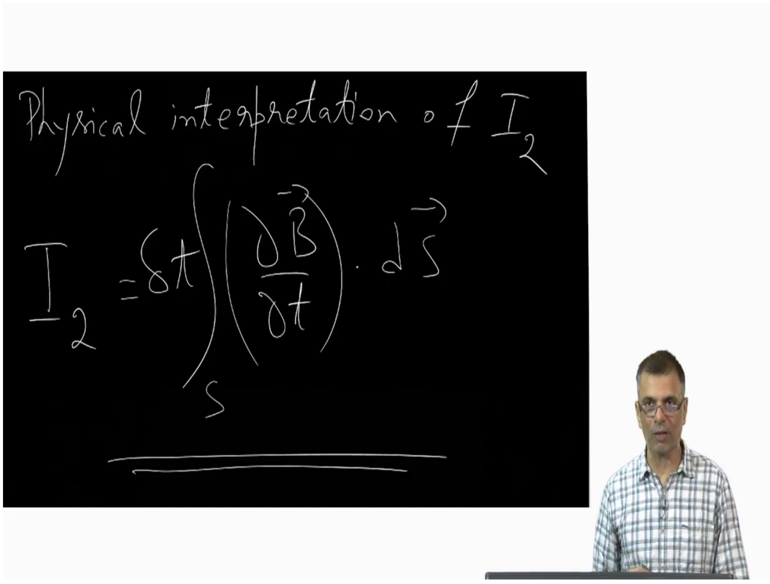

The physical interpretation of I 2 which you remember is the change in flux due to the change in B at a fixed position ok.

So, the time rate of change in B is what? db dt yeah. So, the time rate of change in B is d yeah, but of course, partial because you know the magnetic field is a is a function of both space and time. So, I would write you know. So, the change in flux due to the change in B at a fixed position. So, this is the change in B and of course, I would have to integrate over cross sectional area yeah, like that yeah and of course, this gives me the total over fix.

So, this gives me in the rate of change of flux, but what I want is not the rate of change of flux, I want change in flux itself for which I should multiply by a dt and this is I 2 this is a physical interpretation of I 2 ok. The physical interpretation of I 1 is slightly different, but anyway. So, we will we will leave this here. So, for the time being we will stop here.

Thank you.