Statistical Mechanics
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Lecture – 11
Transformation of Random Variables

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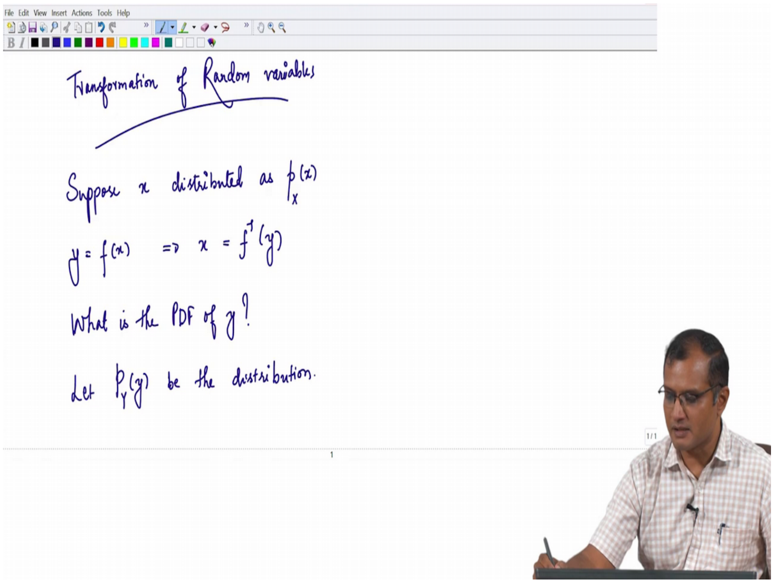

Good morning students, today we will talk about Transformation of Random Variables, which is a very important subject in the context of probability theory. So, this is at least problem is very simply stated. We have a random variable x ok, distributed as some PDF your p of x ok and I am just putting label capital X to remind ourselves that this is PDF of small x. Now, you have a function y that is given as f of x ok and this relationship is sort of one-to-one mapping in the sense that I can invert this equation to get the values of x by simply saying that x is f f inverse of y.

And now, I am interested in asking what is the PDF of y? Ok, so basically I am asking that what is the value or what is the PDF of the distribution of y. So, let us call this distribution as p of y and I am going to put a subscript capital Y, to alert the reader that this is a PDF of y.

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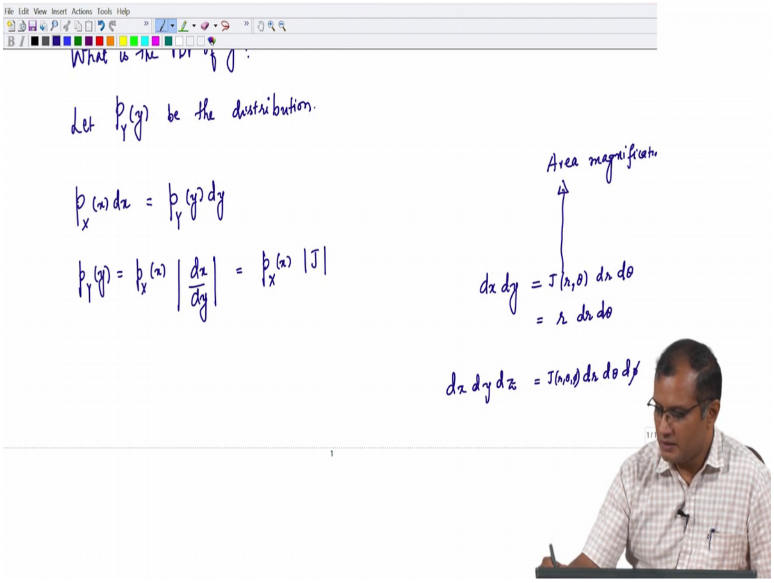

Now, the problem is very simple to solve in this case of you know one-dimensional random variable, you just have one variable that is x and it is map to another variable y. So, I can simply say that the probability to find random variable in the vicinity of x is given by p of x dx and since for each y; each x there is a unique mapping, this should be the same as a probability to find the random variable y in the region y and y plus dy ok. So, then I can extract the PDF of y by simply p of x into mod of dx by dy. And this quantity dx by dy is called as the Jacobian of the transformation.

So, it is simply is a measure of a length amplification factor. In two-dimensions the Jacobian would give amplification of the area, in three dimensions if you have three random variables; the Jacobian would give you a volume amplification factor. You are already probably aware of how to go from Cartesian coordinates x y to r theta coordinates. So, this would be like saying how does the area elements just a miner degradation, dx dy is related to area element in plane polar coordinates.

Now, the left hand side is an area element it has dimensions of length square, if you look at the right hand side, it is just length. So, clearly something is missing here and the something that is missing is the Jacobian of the transformation. So, I am just going to write down, in general it is function of the new coordinates in which I am taking my old coordinates and I will discuss in the following lecture that this Jacobian is nothing but r.

So, the two-dimensional coordinate transformation from Cartesian to polar involves the Jacobian that is r here.

Now, we can say the area matches dimensionally on both sides and the amplification factor is nothing but the radial distance r, which is the Jacobian of the transformation. And if the transformation between new coordinates and the old coordinates is canonical and the Jacobian is unity.; so, in this case the Jacobian is not unity it is r and this is the basically other transformation. So, you can also called the Jacobian as the area magnification factor. So, this is the two-dimensional r event, in three-dimensions you would basically have dx dy dz and this will be equal to dr d theta d phi, if you are going from Cartesian coordinates to spherical polar coordinates.

Again we are running into the same issue that this is not dimensionally consistent. So, I am must attached Jacobian here, which is a function of in general function of r theta and phi; this will give me how the area element or the volume element in spherical polar coordinate this is related to the volume element in the Cartesian coordinate.

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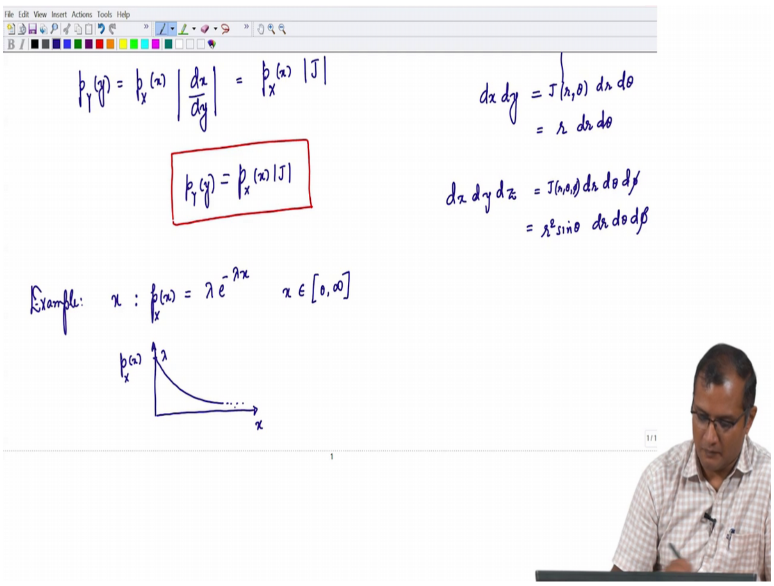

And if you do the calculation, which I will defer it for the assignment this will come out to be r square sin theta. So, in some sense the Jacobian r square sin theta is the volume amplification factor of the coordinate transformation here. So, we are dealing with one-dimensional random variables, so we do not have any area or a volume we have a length. So, our Jacobian simply tells us how length is magnify. So, dx by dy is the Jacobian here or the length magnification factor. So, I can write this as p of x dx into the mod of the Jacobian, because I do not want the probabilities to come out negative, the probability density have to be positive. So, I have to guarded with this mod of J and this will be mu probability density. So, this is going to be my result that is as simple as it can get.

So, we can take one simple example to illustrate the point. So, take this example of random variable x which is distributed as some lambda e raised to minus lambda x, and our random variable x is to find the domain 0 to infinity. So, basically my PDF looks like this. So, if this is my x distribution, then it simply decay exponentially with this value becoming lambda ok.

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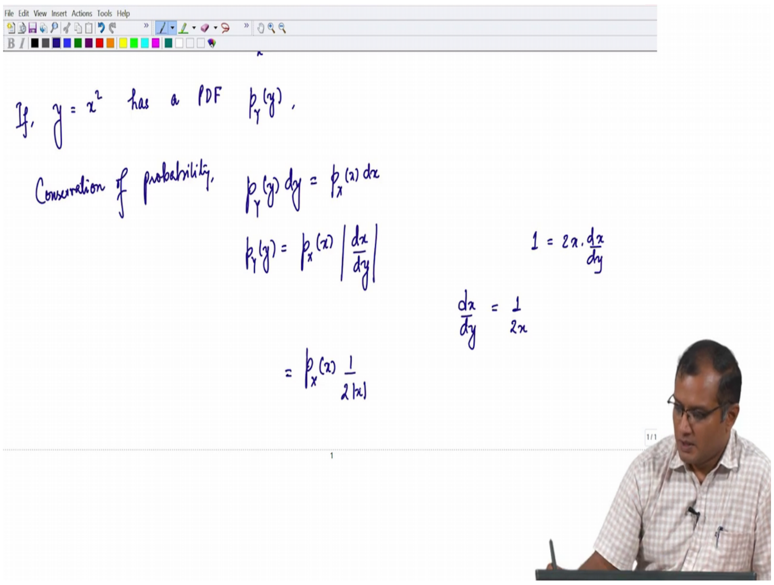

And let us say I define a function y which is x square ok. So, then I have to find out what is the PDF of y. So, you can compute this PDF which is basically we will say that y has a PDF p of y, then I can simply say that by conservation of probability that p of y dy is equal to p of x dx. And by our previous logic our p of y is nothing but the probability as a function of x into the Jacobian, which is dx over dy.

Now, for the equation y equal to x square we can compute this derivative. So, I can write down dx over dy as simply I can take derivative on both sides with x. So, I will with it will (Refer Time: 10:43) respect to y so, I will get 1 equals twice x dx by dy. So, I can write this as 1 upon 2 x. So, basically this is just p of x into 1 upon 2 mod x ok.

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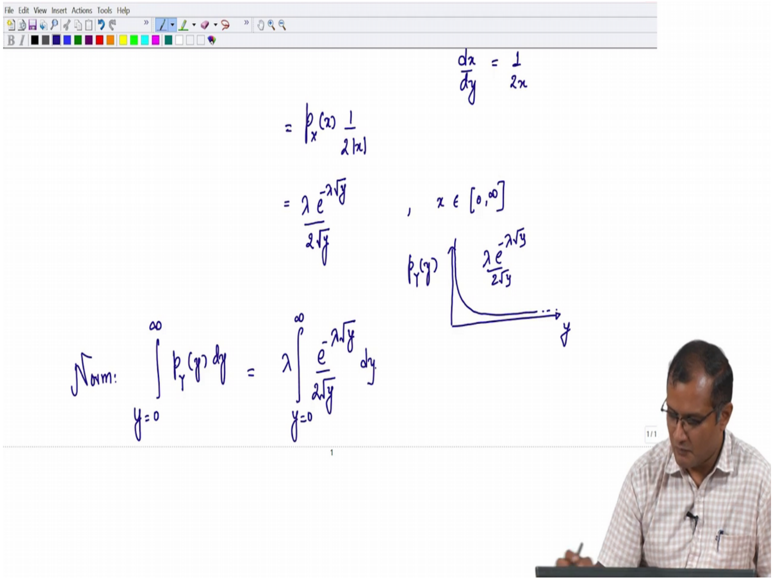

Now, if you want to find out p of y, you can simply use the fact that x is always positive; it is define in the range 0 to infinity, so mod x is just x. So, I can rewrite the PDF of x as lambda e raised to minus lambda and I will write x as square root of y ok. I will not take x as minus square root of y, because x is defined in the range 0 to infinity. So, I am going to take only the positive value of x.

And I have in the denominator 2 times square root of x, because recall that my x was defined in the range 0 to infinity; correction this is square root of y ok. So, this going to be my PDF and if I plot this PDF, you can see that the this has a singularity at x y equal to 0, because this PDF is lambda e raised to minus lambda y over 2 square root of y ok.

So, the y goes to 0, the numerator goes to 1; denominator diverges to infinity, so denominator divers to 0, so the PDF becomes infinity. So, there is a singularity at the origin, but it does not pose any problem this is an integrable singularity. And by integrable singularity I mean, I can immediately find out the norm of this PDF and show that this is well behaved. So, if you find out the norm which is the conservation of total probability, since y is a x square, you do not have negative values of y ok. So, the domain will be from 0 to infinity; y is x squared, it cannot have negative values. Let us see that our PDF just obtained is well behaved or not. So, I am going to integrate this and factor lambda can come out ok.

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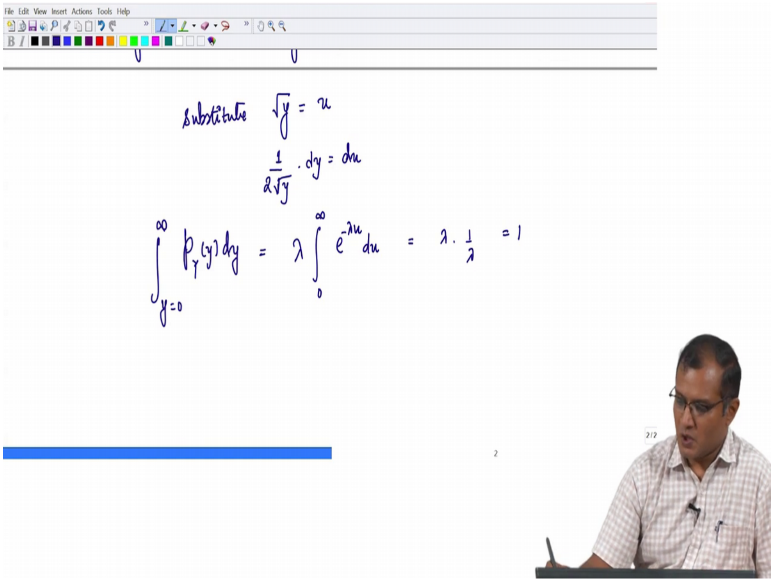

And if I substitute square root of y as mu, then I can just write down this 1 upon 2 square root of y dy as du ok; and the limits will transform from 0 to infinity, 2 0 to infinity, because that is the relationship square root y equals to u. So, I can write down that my norm is basically lambda times integral 0 to infinity e raised to minus u du, which e raised to minus lambda u d 1 ok. So, then you can see that if you integrate this, you will get lambda into 1 upon lambda which is 1. So, this PDF is also well behaved.

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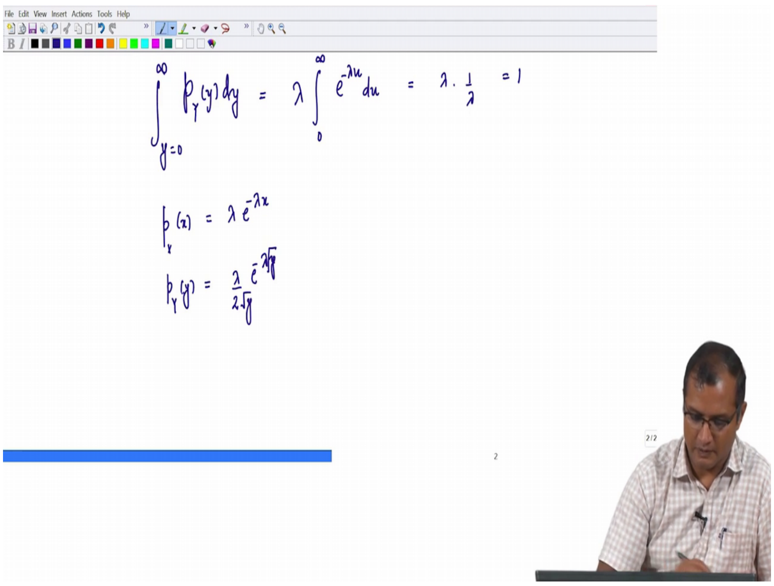

So, our original PDF was yes.

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So, let us write down the PDF or original PDF of lambda e raised to minus lambda x and our final PDF is basically lambda over to square root of y e raised to minus lambda y; lambda square root of y, this is going to be PDF of y fine.

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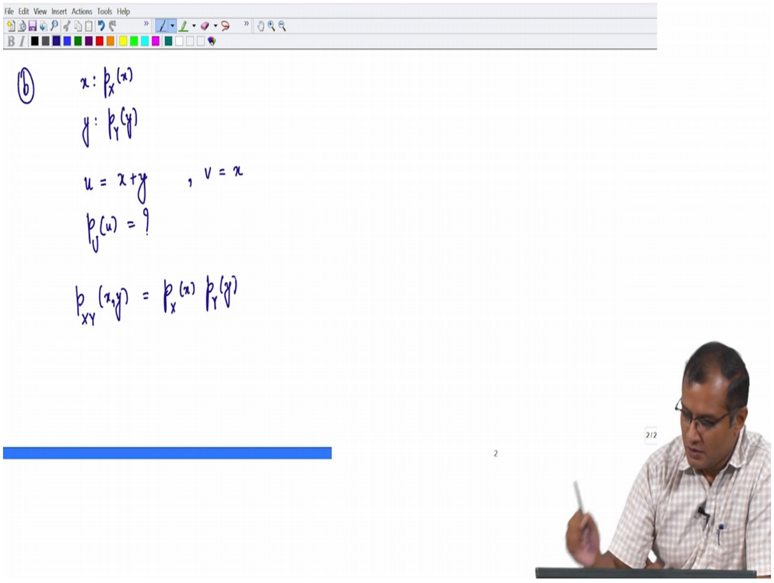

So, now let us proceed to a slightly more general case that if you have a some of two random variables, let us say you have x and y as two random variables distributed as a p of x ; and you have another variable y which is distributed as p of y. And you define the new random variable u which is x plus y, then you want to know how is this random variable u distributed ok. You are in position of the random variable distributions x and y, you would like to know how is u distributed; now there are two ways of doing it, one way is the method of Jacobians that I will describe the shortly, the other way is to attempt this problem by the method of characteristic functions. Both methods are will give you the same answer, but let us extend our present method of Jacobian in this problem.

So, going by the previous discussion if you have the two variables x and y, and you want to know how x plus y is distributed, you can introduce one more variable let us say v, so that from x y you generate a new pair uv, which you can call as just x ok. So, what is happening here is that you have a PDF, I will say that this is an XY PDF of x comma y and because x and y are independently distributed, this is nothing but I will say p of X x into p of Y y ok.

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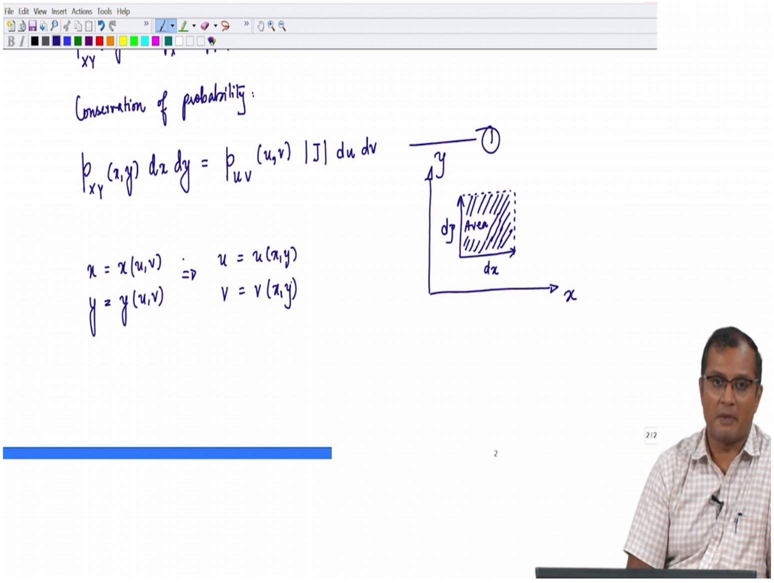

So, you have to find out by conservation of probability we distribution uv. So, we can just write down the probabilities to find the xy in the range x and x and x plus dx, y y plus dy such that the sum x plus y is z or sum x plus y is u in this case. So, I am going to write down this as p X Y x comma y dx dy and basically this should be equal to the PDF uv u comma v into now there is a amplification factor involve, because I have gone from xy to uv so, there is an amplification factor here that is the Jacobian into du dv. So, this is a straightforward generalization of our 1 d case.

If you look at the 1 d case, which is written here I had gone from x to y, now I am going from xy to uv. So, the area element dx dy transformers to area element du dv subject to a Jacobian that is the area amplification factor. Now, this is very simple to drive actually this if we derived the equation 1, then the cases practically solved. So, we can think of a some simple x y coordinate system, wherein you consider two displacements dx and dy ok. So, this is displacement x and that is displacement y ok. And this is the small area that I would like to draw here, I will call it as area x y or let us just call it area ok.

And because our variable x y depend on variables u and v, I can write down x as function of u and v ok, and y as a function of u and v or I can invert them and also write down equivalently that u is a function of x and y, and v is a function of x and y. So, you can invert this math.

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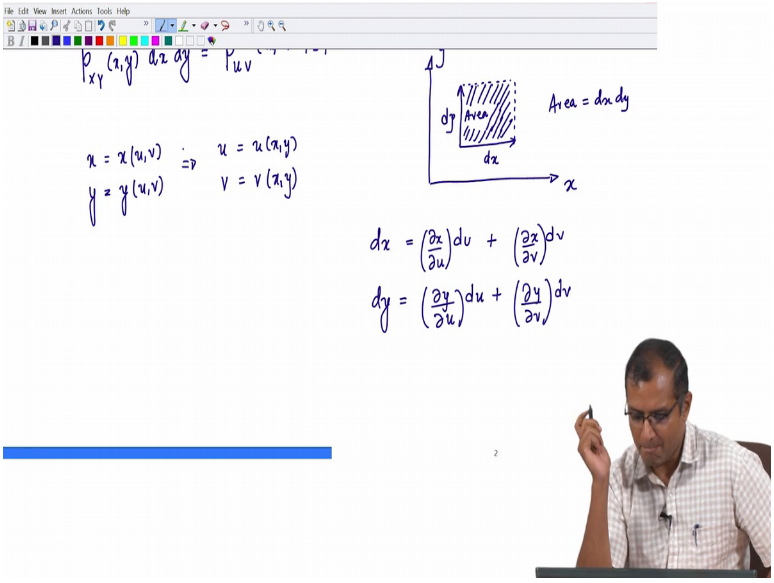

And then say that since displacement in x is affected by displacements in u and v vice versa, you can say that the displacement of u is affected by displacement in x and y. So, I will say that the displacement in x is effected by displacement in u and displacement in v ok. Of course, there are some pre factors here that one has to attach, so that it becomes a displacement in x is prefer as a dx over du and this is dx over dv whether in the total displacement affected in x due to individual displacement in u and v ok.

So, I will rewrite this for displacement along y, then it is easy to see that this is just dy by dv times dv. So, now you can see that x and y we are displacement dx and dy displacement in the xy plane and that give me this small area element dx dy. So, the area element here is basically dx dy. You simply obtained by cross product of dx and dy, but because dx and dy orthogonal to each other, the cross product is just magnitude dx times dy.

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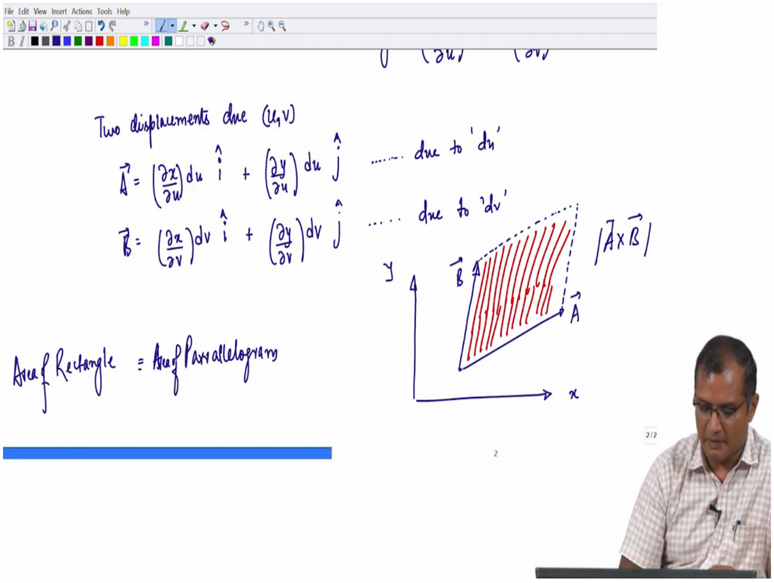

So, I can imagine now two displacements due to the uv pair ok. So, I will call the displacement due uv pair as some vectors A and vectors B displacement due to x and y over dx and dy. So, you can already see the form of this vector A and B. If we look at d dx and dy, you can already pickup the form of u and v ok.

So, we can see that there are first term on the right hand side of dx is basically one term of A. So, I will say this is dx by du times du the direction of i plus the component that is the first term for dy that is a dy by du times du in the direction j ok. So, as you can see I can write down that this is entirely due to du. So, for displacement due to du has been has effected in my displacement vector A.

Similarly due to dv I can write down the other two parts, which is dx by dv plus dy by dv, so this is basically due to dv ok. And similar to this area that was swept by the total basically the region dx dy, I can consider the displacement, but as A and B the corresponding vectors. So, this is my vector A and this is my vector b, their full forms are written already here. So, if you want you can project it along the x axis and get the x component that is not necessary here, the forms are already written before u.

So, the same area which is in terms of dx dy become the area A cross B ok, so I will wipe out this thing and indicate this area as some red color. So, this is basically we shaded region; so, I will say that this region here has the same area as the shaded region here. Of course, the area in the shaded region is given as A cross B and I want the magnitude of the area say I have taken the magnitude of A cross B that is the typical formula for area of a parallelogram with sides that are A and B ok.

So, I will say that in terms of the coordinates what I have done essentially is that I have transformed rectangle into a parallelogram ok. So, either I can work in terms of x y, then my total area swept is dx dy or I can compute A cross B that is the area both are the same. So, I will say that this is basically equal to the area of a parallelogram ok. So, I was say area of this is equal to area of the parallelogram.

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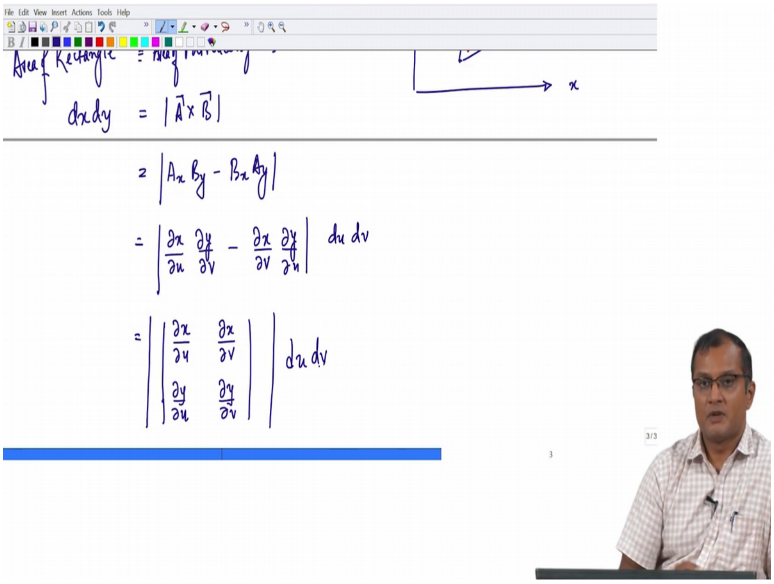

So, then I can simply write down for; write down this as dx dy, which is the area of rectangle that was easy to see as mod of A cross B. Now, mod of A cross B is nothing but I will write it as A x B y minus B x A y that is the form of A cross B, which is in the plane coming out of the display board and I have taken the magnitude of it, so this is the strength of it ok.

So, I can then write this as from this definition here I can write it as, dx over du into dy over dv minus I can write it as dx over dv into dy over du. Of course, I have taken du times dv outside ok. So, then this can be written in the form of determinant, the outer vertical lines for the mod ok.

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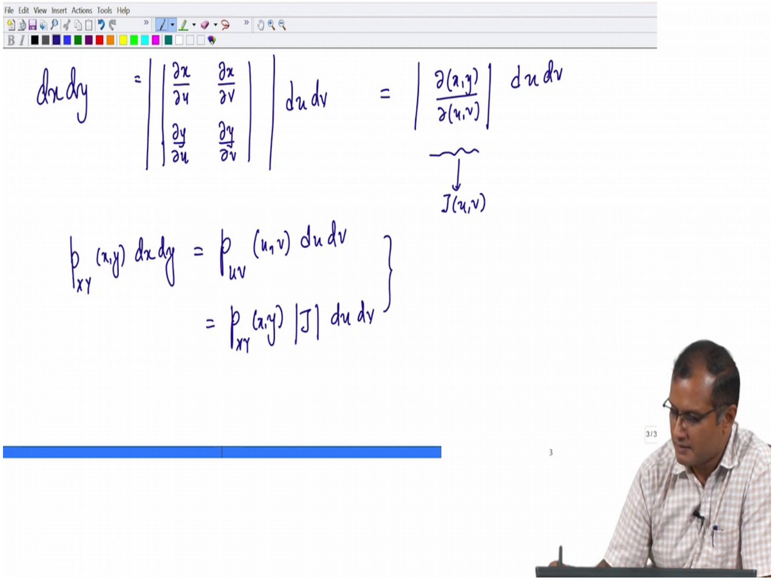

So, this is nothing but the Jacobian of the transformations, I will say this is basically I will say this is del x comma y over del u comma v which is the Jacobian of the transformation times du dv. So, as I the left hand side is basically dx into dy that is my area element in the Cartesian coordinate, du dv is the area element in the uv coordinate system, the amplification factor is Jacobian ok.

So, now you can find out this amplification factor. So, now you can say that the probability density in the x y, which is function of x comma y times dx dy should be the same as the probability density in the uv plane for the variables u comma v into du dv ok, but I know the left hand side can also be written as p X Y times the Jacobian into du dv. So, if I compare these two sides ok.

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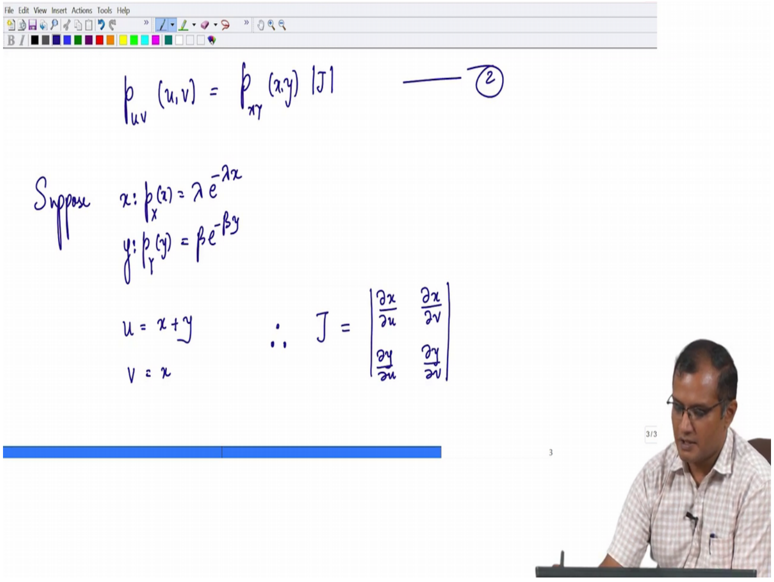

I can come to conclusion that p uv is nothing but p of x y into the Jacobian ok. The first and third equality comes because of the Jacobians, so if I just compare the second and third, I get this relationship just knock of the area element du dv from both sides. So, then for a typical case I can solve this problem, so let us call this is equation 2.

So, now we can apply this concept. So, you can take two random variables x that are distributed as x is distributed as a sum lambda into e raised to minus lambda x. Let us say that the distribution of x, exponential distribution. Take another random variable that is distributed as sum beta e raised to minus beta y. And you want to know how is the sum distributed. So, define a random variable mu from xy we call uv. So, uv is a random variable of our interest which is x plus y and I will also get one more variable which is not necessary, but I will I want to evaluate the Jacobian, so I need one more variable.

Let us say v equals to x, we will keep it as x we do not required. So, we have kept it as x, you can keep it as y you free to do that. So, now let us compute the Jacobian we have gone from x y to u v. So, therefore the Jacobian dx over du into and dx over dv dy over du and dy over dv ok. So, we can call this as equation 3 over Jacobian.

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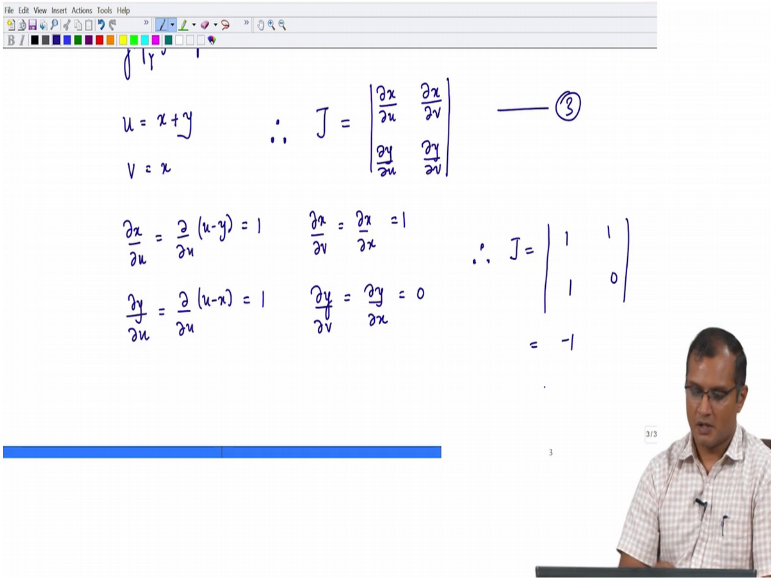

So, compute all these derivative. So, dx over du we can write it as a simply d over du into x can be written as u minus y. So, this is just 1 ok and dx over dv is nothing but dx over dx, because d is equal to x, so that is 1 ok. You can write down for so let me move it this place ok. So, we can write down dy over du as d by du of u minus x which is 1. And we can write down dy by dv which is the dy by dx, because v is equal to x; y and x are independent variables, so this is 0. So, therefore, over Jacobian becomes so you can write down 1 1 1 0. So, Jacobian becomes minus 1. So, then we can write down our from our relationship here that p u comma v is equal to mod of minus 1 which is just 1.

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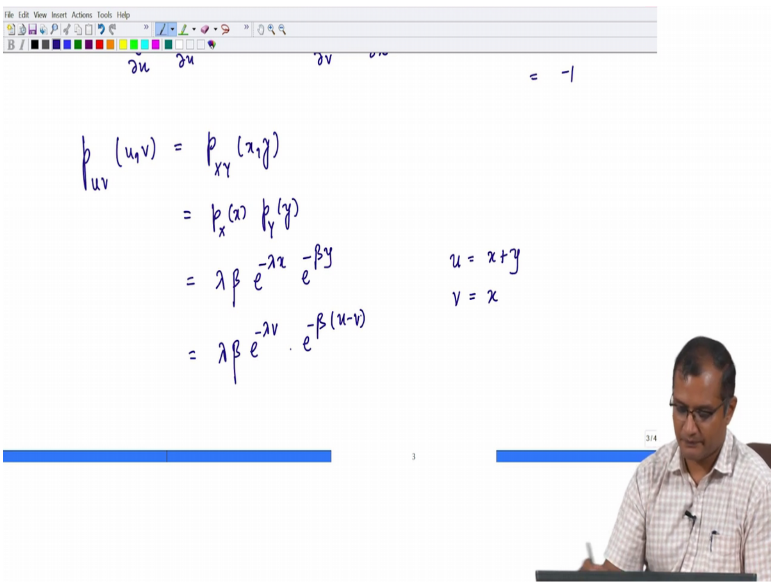

So, basically from equation 2, I can write down a very easily that my u v distribution joint distribution of u comma v is given by the joint distribution of x comma y, but this is nothing but individual distribution products because x and y are. So, now you can write down this as lambda into beta e raised to minus lambda x into e raised to minus beta into y, because those of the forms of p x and p y ok. And let us substitute for x and y, recall that our random variable u was x plus y, and we have taken the new variable v as just x. So, I am going to write down this as e raised to minus lambda v, because x is equal to v into e raised to minus beta, I will write down y as u minus v because x is equal to v.

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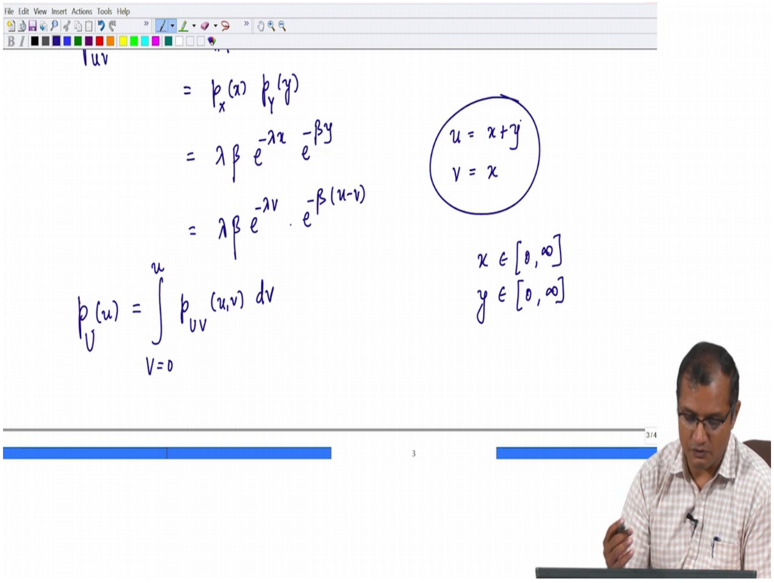

Now this is the joint PDF of u comma v. I am interested in just the PDF of u. This can be obtained by washing of the v degrees of freedom. So, I have to wash out the v degrees of freedom. And this I can do by integrating my total PDF over v. The limits on v has to be carefully chosen by looking at this relationship, I can easily say that the lowest value of v can be 0, because x and y are positive variables distributed between 0 and infinity. The certainly u and v also will have the minimum value of 0. So, in this integral I have to take v as 0 has the lowest value. The upper limit on v will be definitely u, because by this relationship v cannot exceed u, because I want the sum to be x plus y that is with u.

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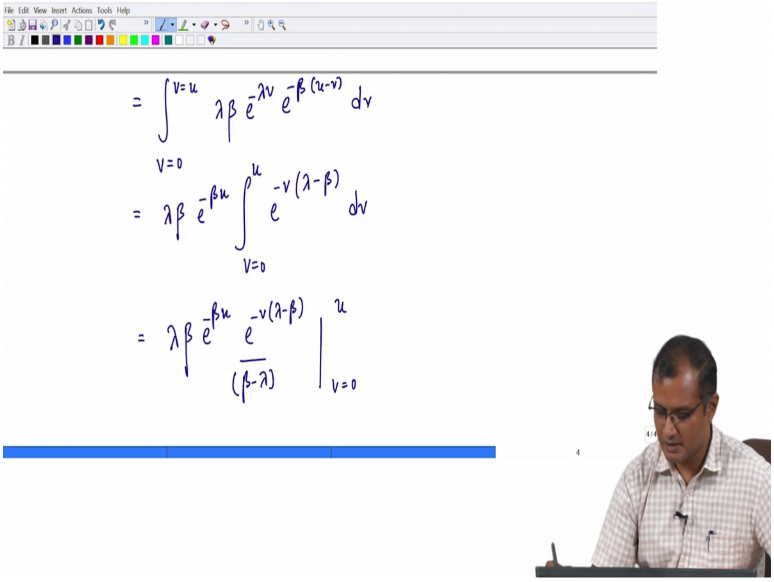

So, then I can write it as e raised to minus lambda v e raised to minus beta times u comma u minus v dv ok. So, then you can see that I will immediately take out lambda beta outside and e raised to minus beta u will also come out of the integral and I have v going from 0 to u e raised to minus v lambda minus beta ok. And then if I integrate this, I will get e raised to minus v lambda minus beta over beta minus lambda in the range.

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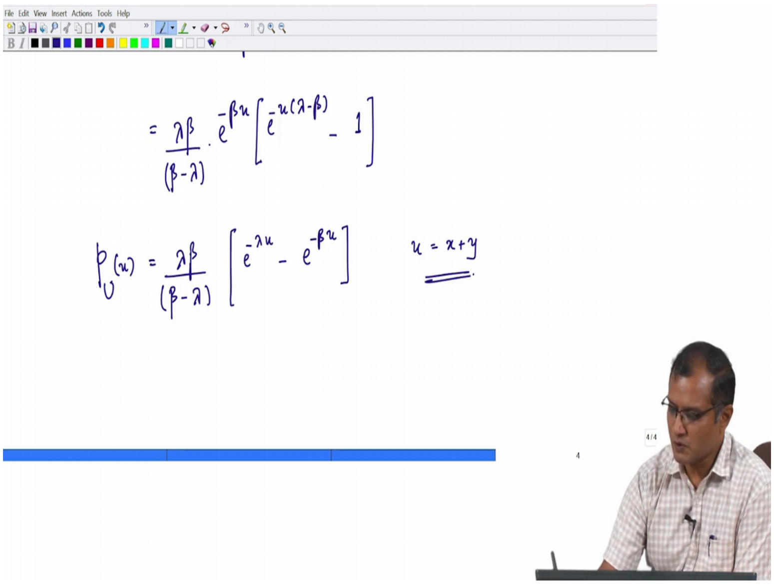

If I apply the limits, this will be basically lambda beta over beta minus lambda into e raised to minus beta u. Just plug the limits the answer is very easily obtained e raised to minus u into lambda minus beta minus 1 all right. So, then this is nothing but e raised to minus beta mu will cancelled what you will have for the first term is minus lambda mu lambda u minus e to the power minus beta u. So, this is the PDF of u. So, this is the PDF of this sum. So, this is how the sum is distributed. So, I will say this is the PDF which tells how x plus y distributed ok. So, this brings us to the end of this variable transformation.

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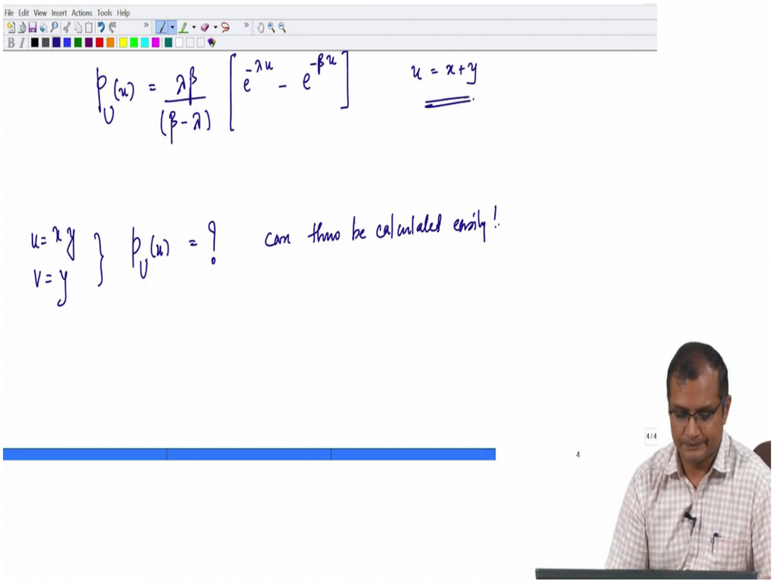

You can also do similarly to just extend method in compute how x into y will be distributed. So, you can take u as x y, and v as y and then say how is u distributed by the techniques you know discuss the above can this be calculated easily ok. So, we end the lecture here, we will in the assignment we will get some problems that will use different distributions the, we have taking the example of the exponential distribution you can generalize this method to compute other distributions as well.

So, you can take x and y that are normally distributed, and comment on how x plus y is distributed x into y is distributed or x upon y is distributed. Please take care of the fact that when you are using the method of Jacobian, it will only give you an area amplification factor. Once that factor comes to your; comes in front of you, you have to ensure that the joint PDF needs to be integrated over one variable to get the PDF of your interest. And when you do that the integration limits have to be very carefully chosen. In this case, we have to ensure that v is not independently moving, it is moving keeping in the constraint that the sum x plus y is always fixed to u, so that is needs to be properly taken care of. So, we will stop here and proceed in the next class.