Non Equilibrium Statistical Mechanics
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Lecture 36
The Wiener process (Standard Brownian motion)

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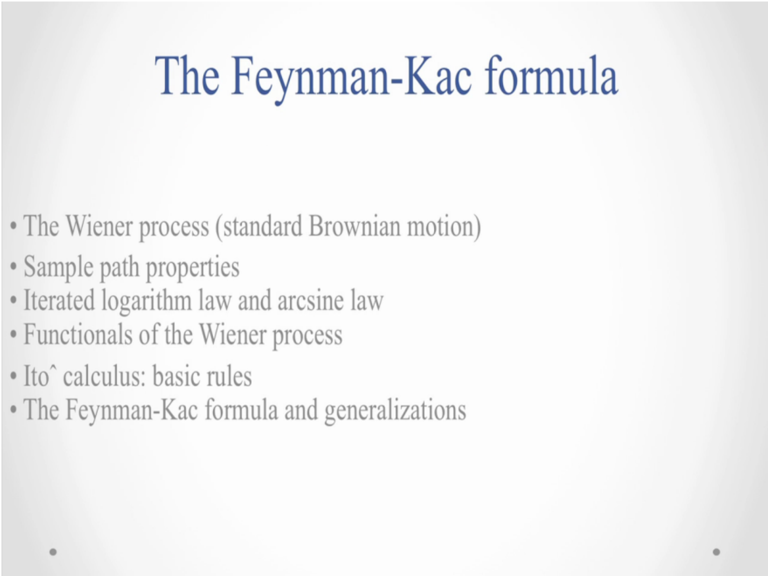

Today I will discuss the topic which should have been discussed the little earlier than we did Langevin dynamics but which is somewhat formal and mathematical and not directly connected to the rest of the topics that we have done earlier but it is very crucial now becoming more and more important in the context of non-equilibrium phenomena specific stochastic models of them and therefore I thought I should at least mention briefly what it is all about.

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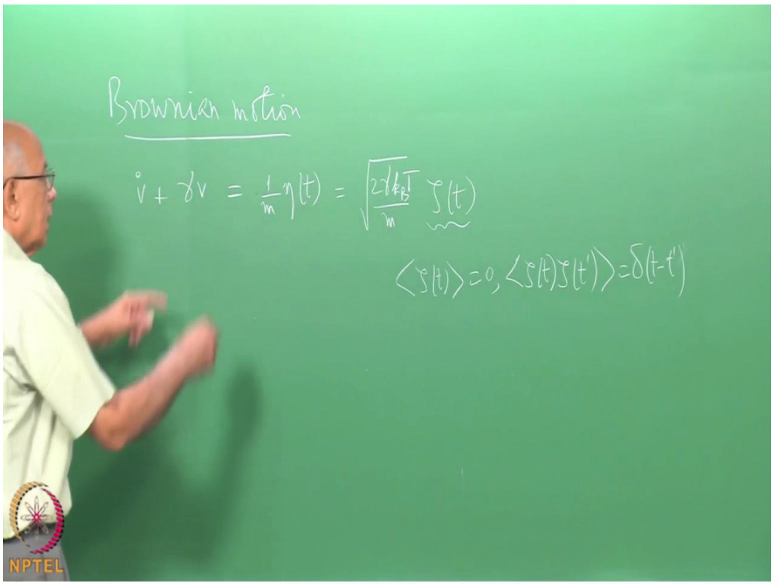

It has to do with our old friend Brownian motion and if you recall, so let me start by recalling a few facts about Brownian motion. If you recall we wrote down the Langevin equation for some Cartesian component of a particle of (()) (1:06) in a fluid and to quickly recapitulate what we had. We had a formula like V dot plus gamma v is equal to 1 over m eta of t something like this but this was Gaussian mark of stationary white noise.

And we also discovered there was a relation between the correlation the strength of this force and the friction constant gamma. If you put that in we could write this then we have a capital gamma here, so this would go to 2 gamma k Boltzmann T over m times zeta of t where this white noise had the following properties. It had 0 mean and t zeta of t prime was equal to just a Delta function. Delta of t minus t prime, okay. I slurred over the fact that this quantity is not very well-defined; it is too singular in some sense in a strict mathematical sense. So we will come to terms with that now and do this little better than what we did earlier.

Now of course in the diffusion regime happens is, your long times t compared to gamma inverse and then the effect of this inertia term gets negligible and this term dominates here.

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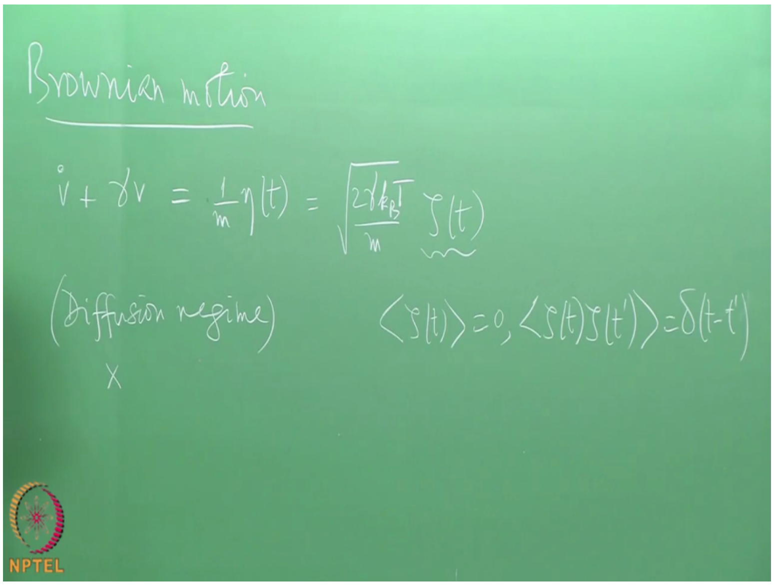

So let me go straight away to the diffusion regime and write the equation using the diffusion, what we call the diffusion regime or equivalently high friction, very high friction this equation gets replaced by x dot equal to square root of 2D times zeta o t because recall that d was kT over m gamma if I divide through by gamma after neglecting this you get precisely 2d here. So all the fact has a right.

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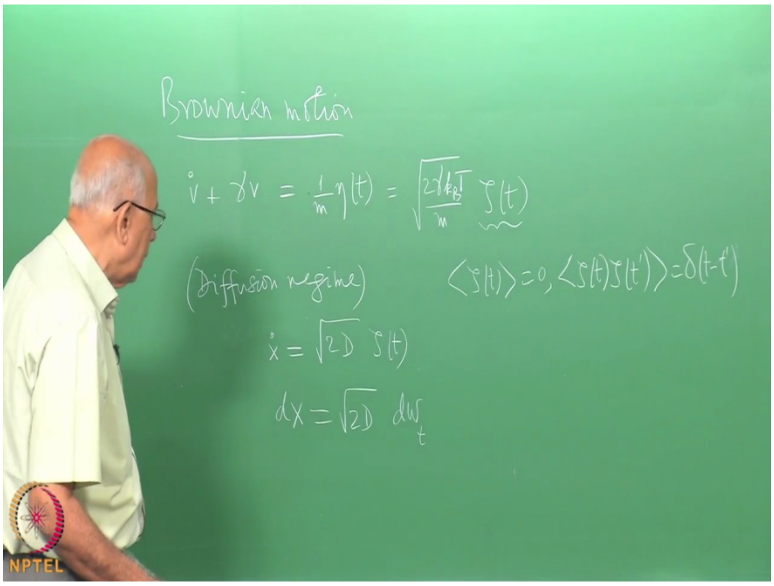

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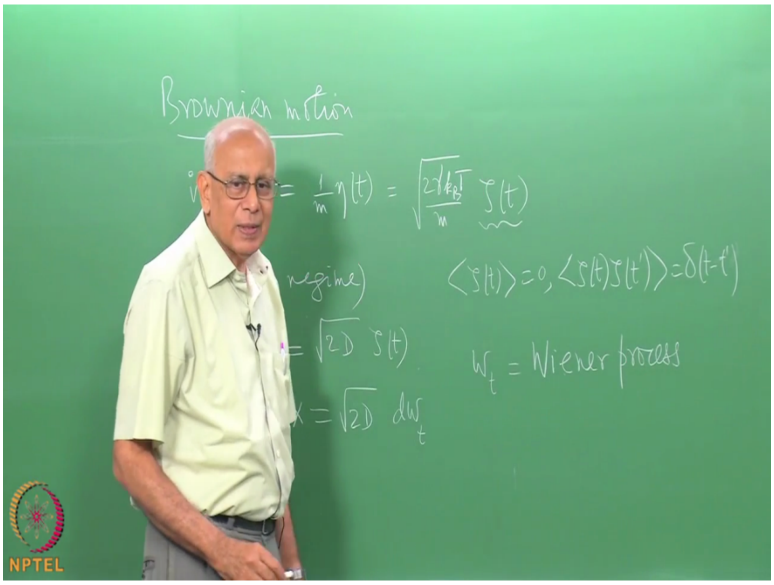

Now this is not a very comfortable equation to work with because it is too singular an object. So mathematicians like to write this in the following form and I will define this term, so this is written as dx equal to square root of 2D times dw t this is called a Wiener process therefore the w of t is called the Wiener process, let us write that down. I should put up t inside the bracket really as (()) (3:59) would but mathematicians like to put it as a subscript because it is more noticeably easier to handle there.

And the pack is, standard practice is to set 2D equal to 1, so release scale matters in such a way that 2D is equal to 1 and then it is called standard Brownian motion.

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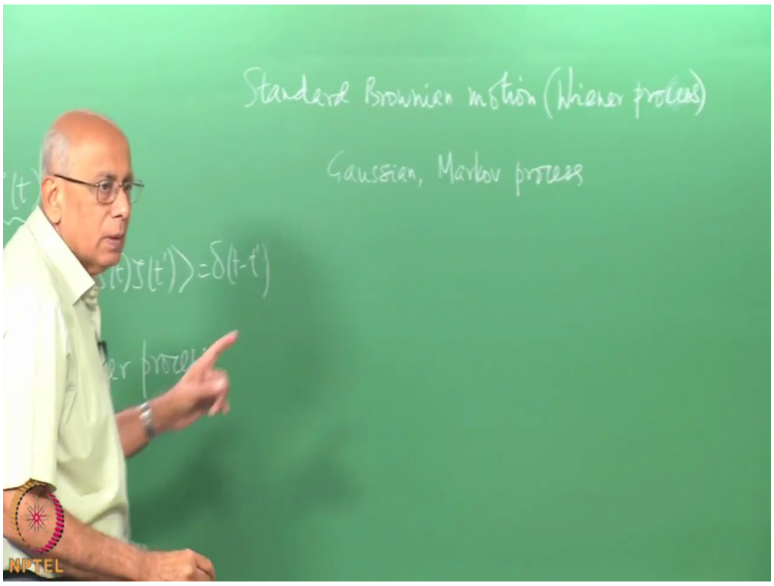

So standard Brownian motion or equivalently a Wiener process it is a Gaussian mark of process that is Delta correlated. That is correlated in a specific manner which I will derive in second, okay. This Gaussian Markov process eminently has most notable properties that is not stationary. Non-stationary, it is a continuous process that is important, okay. With the following property essentially it is this x and x we recall is the position instantaneous position of a particle diffusing on the x-axis with a diffusion constant D.

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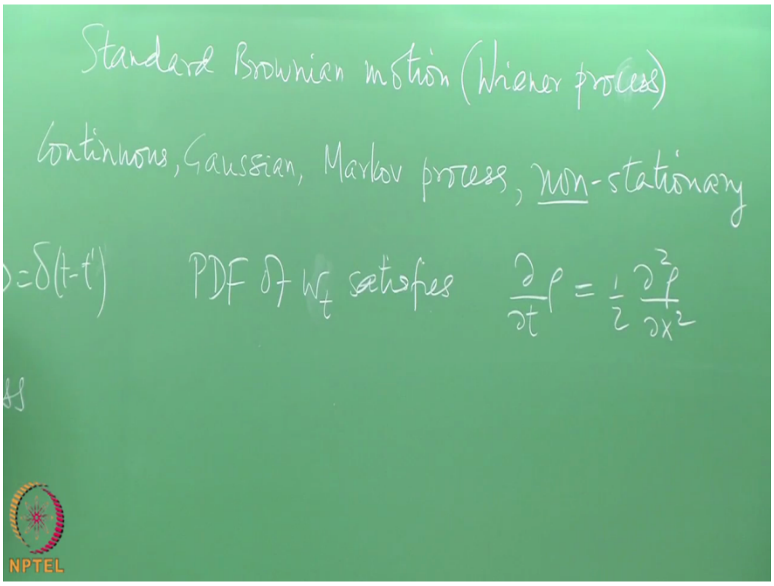

And the probability density function of this x obeys the standard diffusion equation delta p over delta t is D times delta 2p over delta x2 but for this process here we have said 2D equal to 1 it is clear that for this W, so the PDF of x satisfies of W say satisfies delta over delta t let me just call it rho equal to1 half delta 2 rho over delta x 2, okay. Pardon me.

"Professor -Student conversation starts"

Student: (()) (6:37)

Professor: W of t is not white noise, it is the integral of white noise because in a crude sense (()) (6:44) dWt equal to zeta of t dt, so W of t that is what I did here wrote this as dWt, we have to be careful about what I mean by this differential and that is the whole point, right? So in this heuristic way of looking at it this w of t is like the integral of white noise because that is what zeta of t is.

But the problem is that zeta of t is not a well-defined mathematical object. It has got a singular covariance; it has got singular correlation function Delta function which is not pleasant, okay. Now think of what so w of t is essentially x the position of the diffusing particle, okay. That is it apart from this constant here but think we know already what this guy does we know that it is not a stationary process and if the particle starts from 0 the origin at t equal to 0 than the position is a Gaussian and distributed by a Gaussian variance increasing in a limit in etc.

“Professor-Student conversation ends”

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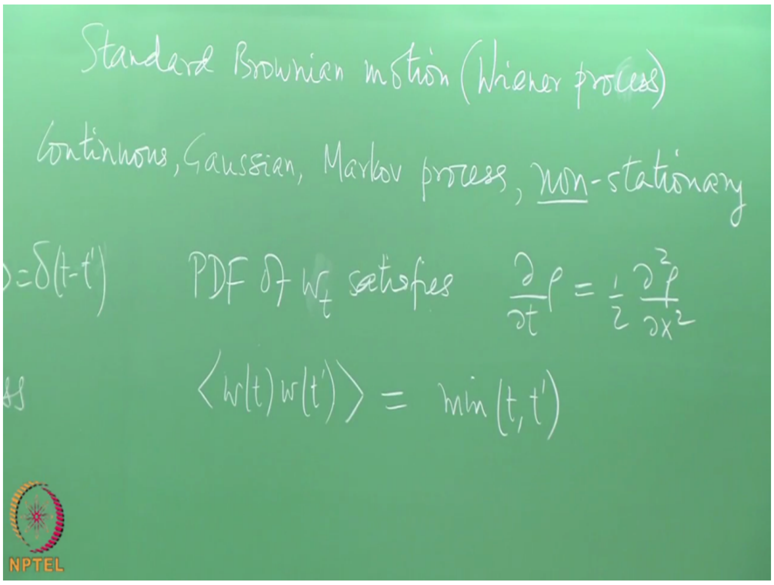

And we also know that this process has an auto correlation function which eminently is non-stationary which tells you, so wt wt prime the average is equal to you recall what it is? Yes, the minimum the lesser of the 2. Now we have said this 2D equal to 1, so with that normal assumption it is minimum of t, t prime. We are only considering positive value, so t and t prime, okay. So it is clearly not stationary because now the correlation pensioners this variant here.

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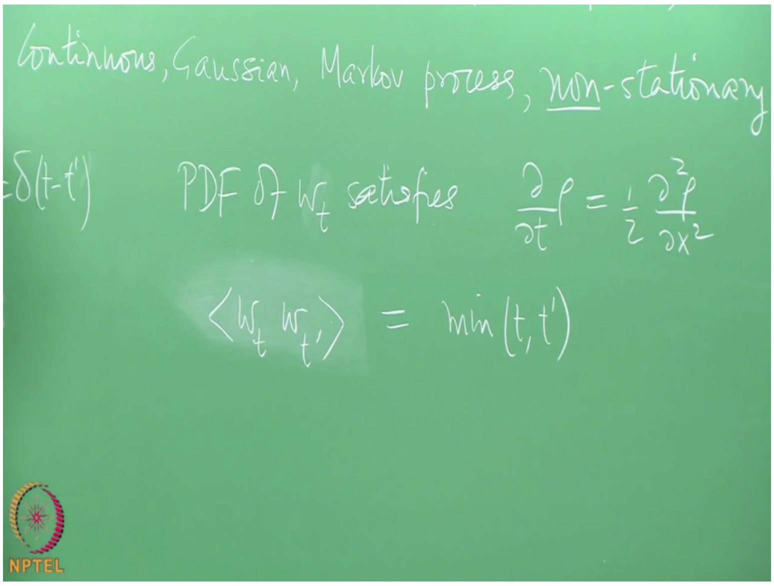

So let me stick to standard notation write this in this chap here has got the standard Gaussian solution if you recall this is implies fundamental solution rho equal to 1 over square root of 2pi t e to the minus x square 2t I have set again that 2D equal to1, okay. Now let us try to understand this little better. What this thing is, this is the covariance of t and t prime that is the word for auto correlation function in statistics.

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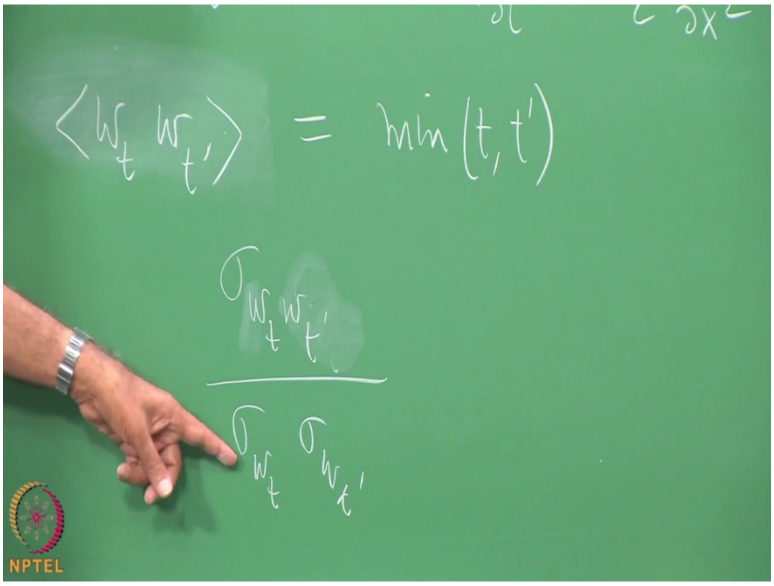

If you divide it by the covariance of this and the standard deviation of this and that, so as to normalise it, so if you write it as Sigma x Sigma t, so how should I write this? Sigma wt Sigma wt prime wt prime here over Sigma wt Sigma wt prime where this is the standard deviation of wt and that is of wt prime this should be Sigma square here by definition it is equal to this quantity.

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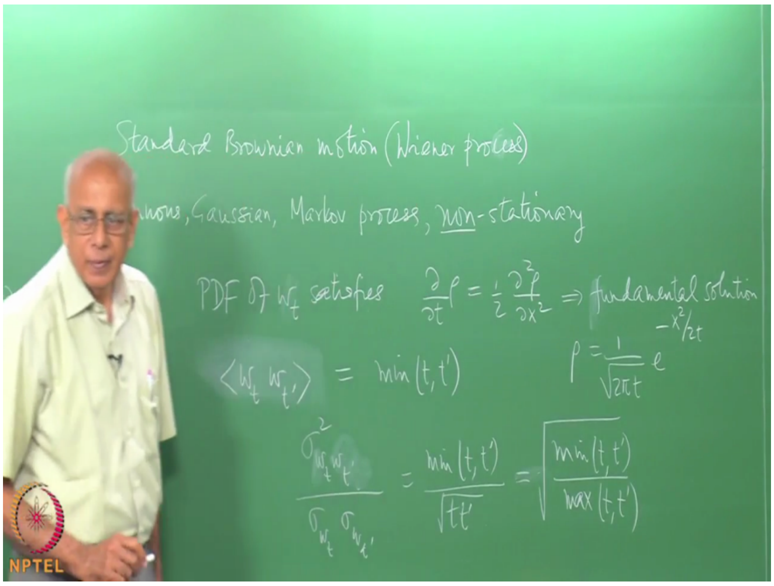

Then this is of course equal to min t, t prime over and we know what this is. Square of this is 2dt and 2d has been said equal to 1, right? So this is equal to t square root of, yes, square root of t, t prime. This is 2d t and this is t prime and we are taking the standard deviation, so root of t, t prime and it is easy to see that this is equal to min t t prime over max t t prime the whole thing square root. That is a convenient way of writing, easy way of remembering this.

Now what can we say about the paths of this wt, okay. It is clear formally that the Brownian particle has got, the Brownian motion has infinite velocity because x squared scales like t not x and therefore delta x over delta t is formally infinite you need to another delta x on top in order to we need 2 finite value, right? Okay.

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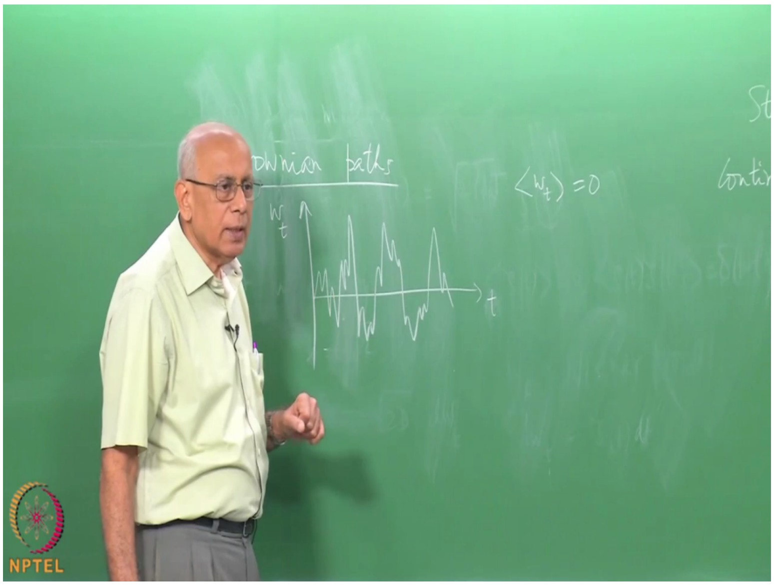

So what does these parts looks like? That is what looking at and a number of results are known in 1 dimension. Brownian paths are sample paths of Brownian motion, we are talking about Brownian motion starting at the origin, so here is x, I am sorry here is t and Wt the Wiener process, starting here this process takes off and does this kind of crazy thing it goes up and down etc.

And the following can be established rigorously from the properties of the Wiener process. The average is 0 we already know that, so overall such sample paths they take the average it remains 0 here but the question is, is it differentiable or not? Is this differentiable? And the answer is no, I will just quote number of results here.

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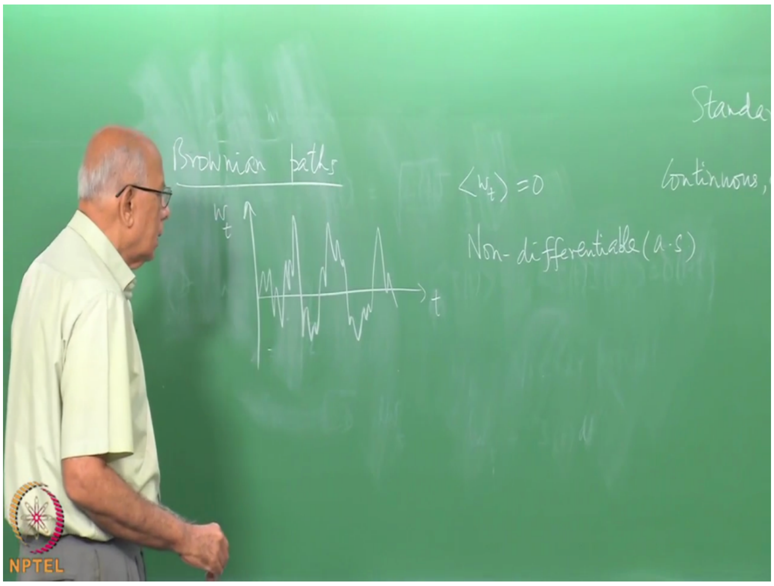

So nowhere differentiable almost surely, so non-differentiable almost sure. How badly is it nondifferentiable? And there is an answer to that, it is a continuous process that is a very very important to remember. In fact had we dropped this and had we require it to be stationary? What would the process B? Pardon me. (()) (13:53) that is the only continuous Gaussian stationary Markov process and it follows a (()) (14:01) exponential correlation and vice versa the converse is also true in that case.

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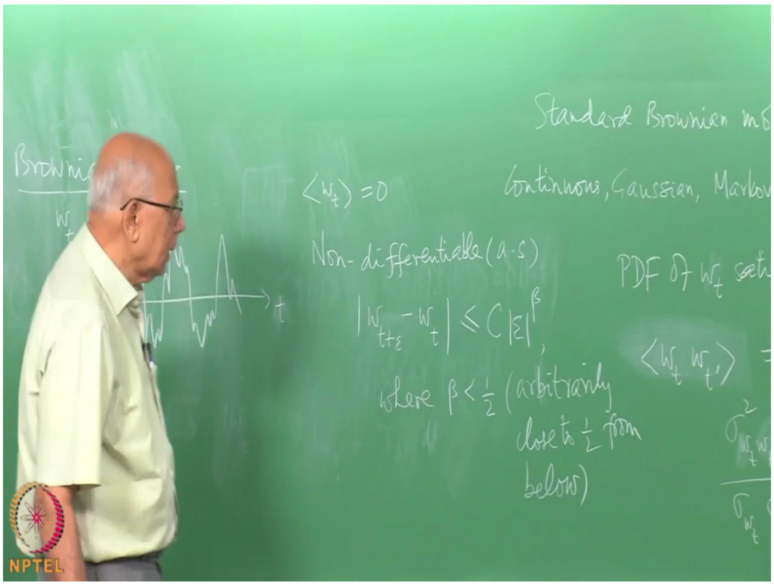

But the moment you make it non-stationary but require it to be continuous then you end up with Brownian motion, okay. It is not differentiable but it is continuous therefore there must be some kind of holder continuity for this process. That turns out that the continuity looks like this W of t plus Epsilon minus W of t modulus is less than or equal to some constant times mod Epsilon to the power beta.

(()) (14:44) beta is less than half but can be arbitrarily close to half and not half except on set of measures 0, arbitrarily close to half from below. So that sort of tells you how jagged these parts are, it is a measure of how jagged it is because you can see that this will imply the function is not differentiable because for differentiability, yes if you divide by Epsilon, so it is clear that W t plus Epsilon minus Wt divided by mod Epsilon tends as Epsilon tends to 0 it tends to infinity because of this property you are going like half, so you divide here you are going to get a half, little more than half in the denominator and it is going to go to infinity as Epsilon goes to 0, so it is therefore not differentiable.

But this weirdness that it is holder continuous with an exponent which is arbitrarily close to half from below means that increments in this process they are acting somewhat like square root of time this Epsilon is like the increment in time, right. So it is acting like the square root of dt the infinite decimal dt one can formalise that and then you have to be little careful but by your clever amendment of the rules of calculus one can actually handle this process quite rigorously.

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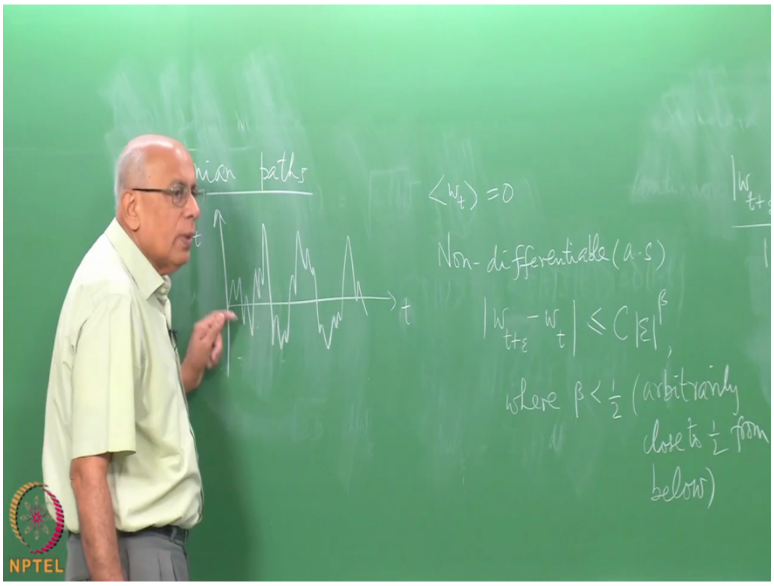

Here is what happens, we continue on another property of the sample path. You can ask when does it cross 0. In a given time interval how long does it stay above the axis and how long will on average and so on? In fact you can ask where is this particle going to be most of the time and it turns out there is a law of the iterated logarithm this is Khinchin's law of the iteration.

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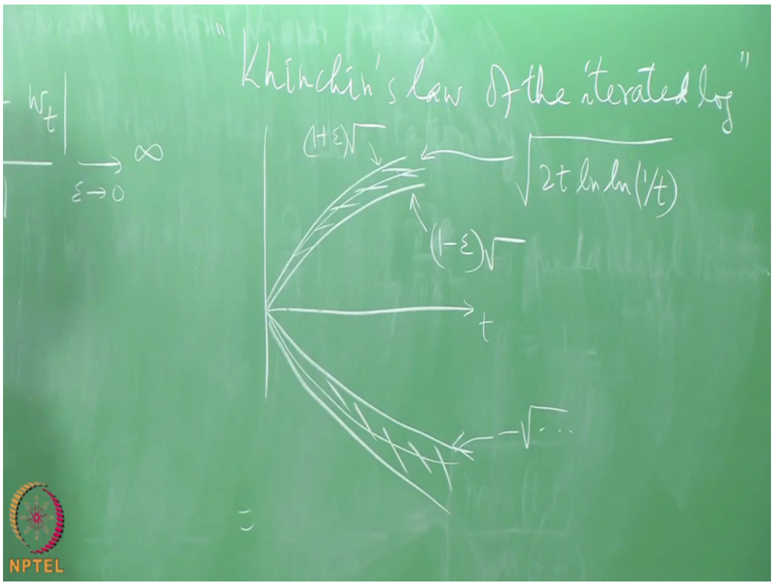

So if you plot the boundary motion starting at P equal to 0, you plot the following function this function is square root of 2t log log 1 over 2 the small t sufficiently small t brought this, okay. And to plot the negative of this function this is minus the square root of this stuff and if this is 1 plus Epsilon times at function and that is one minus Epsilon times at function. So there is 1 times square root and this fellow is 1 plus Epsilon and this square root and similarly here.

Then the statement is almost surely the particle is either here or here in this range for arbitrarily small Epsilon, okay. So you not only at the statement about the probability density function of this x as a function of t or w as a function of t but you also have a statement about where it actually is what this part is?

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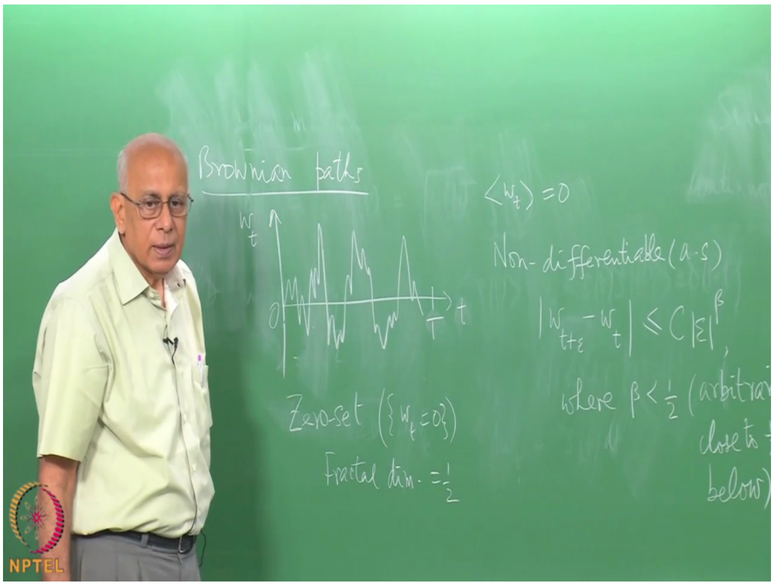

And yet the particle process the axis an infinite number of times, it does so in any arbitral small interval and you can in fact ask what is a measure of the 0 set, the set of points where it crosses the axis, so zero set this would correspond to the set wt equal to 0 here. That 0 set in the limit has a fractal dimension which is a half. So while it can go arbitral is far away the 0 set is such that, by the way this thing the fluctuations obviously they are going to go way up there we down here and that is why the variance itself will change increase with t.

There is a fluctuation such that it is diffusing although the min remains 0 always. You can ask further interesting questions. You can ask what is the given some t here? What is a fraction of time for some capital T starting at 0? What is the fraction of time that dispense above the real axis above there on the positive side? And what is the fraction it spends on the negative side? That is a random variable, it is a, yes but it is a random variable, one can ask what is the probability distribution of that random variable?

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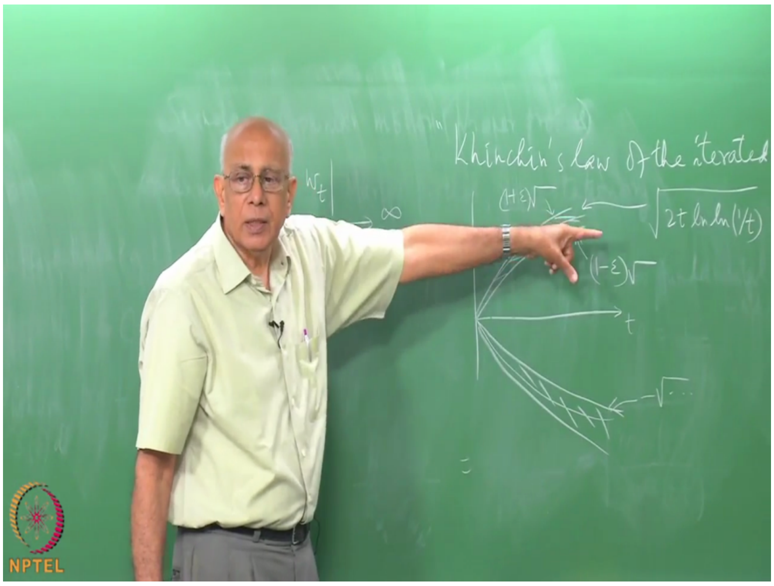

So let us suppose probably an incoherent, you could ask what if t is very large? You have a it create a logarithm for T large as well, all you have to do is to replace that log log 1 over t by t, okay. But that the crucial point is not that, the crucial point is this process is reinventing itself and every instant of time. It is what called a martingale I am not going to talk much about that but we will see some consequences of what it does?

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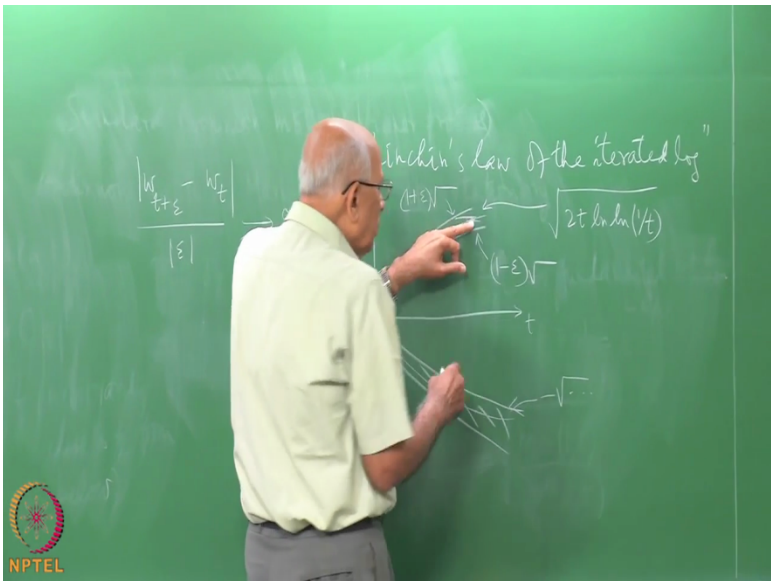

It means that if you are here at this point say you are here at that some instant of time then the Brownian motion is as if it starts from there at this instance of time and it again behaves precisely the same fashion as it behave earlier here. So at every point there is a law of the iterated logarithm every time. The process has the memory is very short some markov process, okay.

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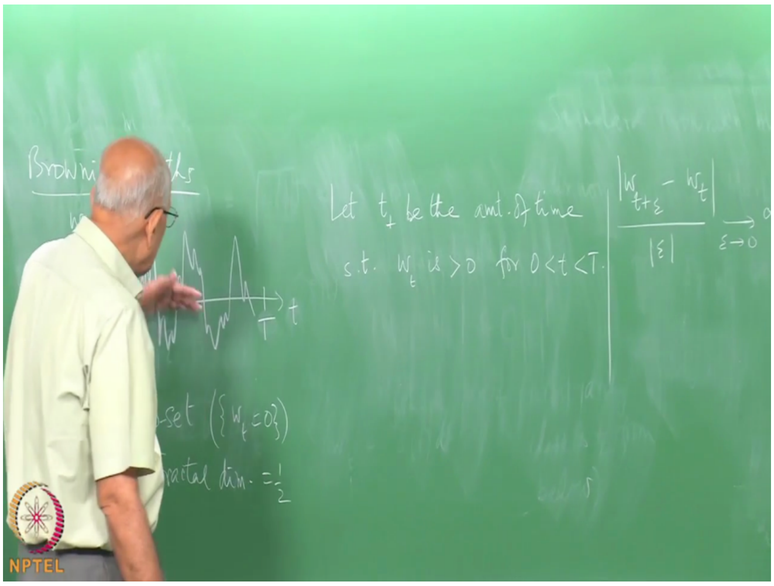

Now one can ask the probability distribution function, so let t plus be the amount of time such that wt is greater than 0 for 0 less then t less than capital T, the. So we will look at the Brownian motion up to some capital T and ask what is the total time spent where it is on the positive side? And what is the total time spent where it is on the negative side of the x axis?

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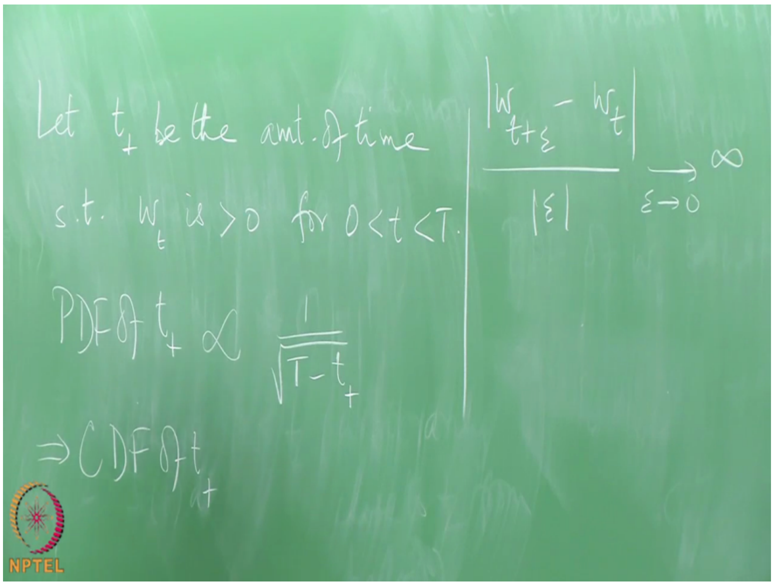

The PDF of t plus and similarly as you said t minus it is completely symmetric, PD of t plus is proportional to 1 over square root of T minus t plus, okay. Which implies that the cumulative distribution function CDF of T plus.

“Professor -Student conversation starts”

Student: It looks like the oscillator.

Professor: It looks very much like the oscillator, yes the time instead of, if you have a regular oscillator and you ask dt, t is distributed uniformly over a period, you ask what is the distribution of the angle? It is precisely this 1 over square root, okay. Very set distributions.

"Professor-Student conversation ends"

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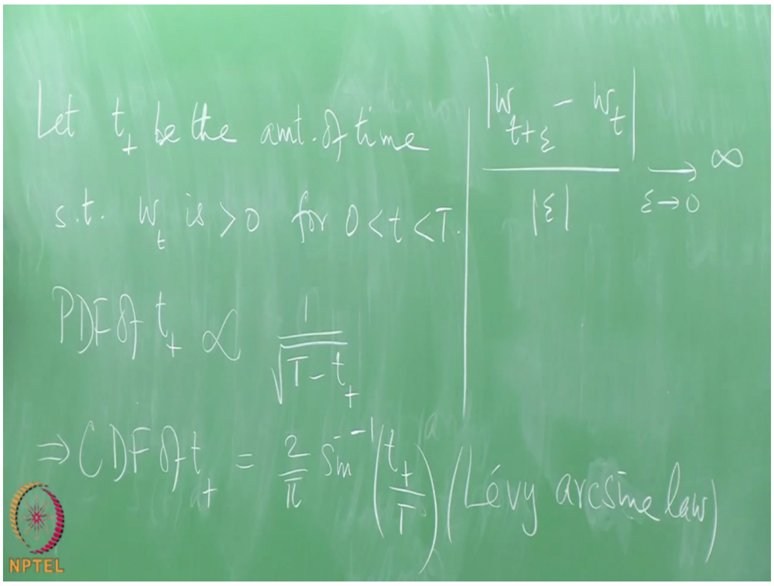

Now the cumulative distribution function namely the probability the t plus is less than equal to some given value PF, this thing here of course now is equal to, you normalise it and you integrate this fellow to 0 to t plus you get 2over pi sine inverse t plus over T, issued of course be 0 when t plus is 0 and it should be plus 1 when t is t plus equal to capital T, total probability must be equal to 1, okay. This is called the Levi arc sine law.

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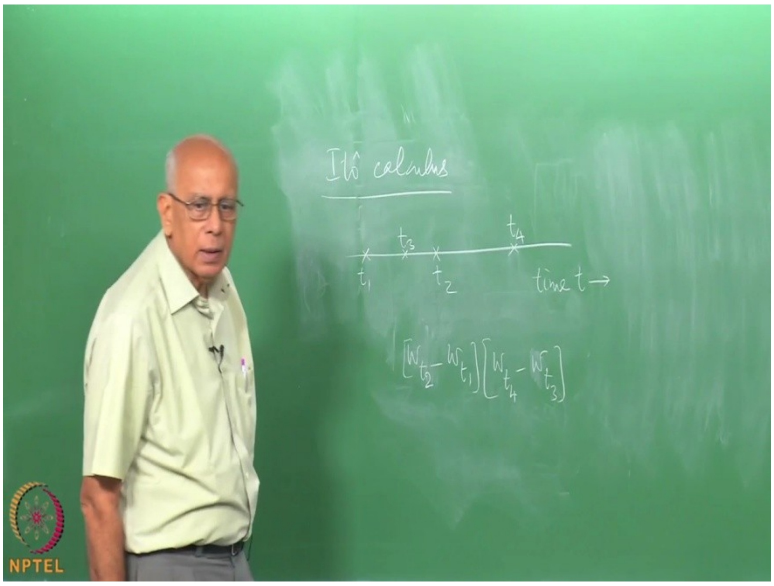

Okay, now let see what fact that the sample parts are regular, very jagged, they have a very specific kind of irregularity essentially characterized by the fact that the holder condition is half essentially half, okay. That says that the following property can we rigorously proved and this is part of what is called Ito calculus. We have mentioned only the rudiments of it essentially 1 formula and it is the formula.

What one can prove is that the Brownian motion or a Wiener process if you have in the time axis any 4 points, so let us start with t1, some point t2 and then some may be t3 then some t4 and you ask what is this quantity? W of w t2 minus w t1 this minus that and you multiply it by Wt4 minus Wt3 this in here the value of this product here is equal to the overlap between the 2 interval.

So you have one interval from t1 to t2, another interval from t3 to t4 and the overlap between these 2 is this right here. So this is equal to overlap, length of the overlap between t2, t1 and t3 t4. It has an immediate consequence this by the way this statement can be proved by looking at the correlation the auto correlation and then it is a simple proof here. Now this has immediate consequences.

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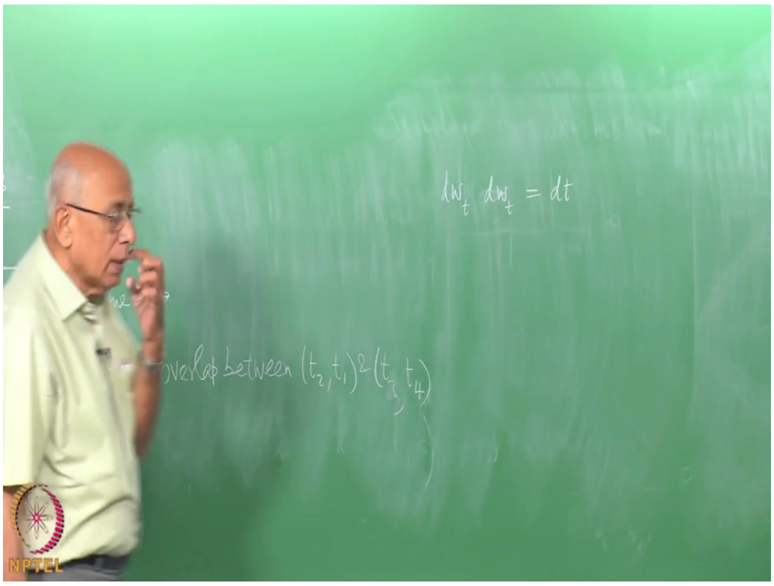

The first consequence is that dwt dwt equal to dt because imagine the completely overlap infinite decimal interval, okay. The length of it is dt in that square of dwt. So this is the one that formalises the fact that this dw the increment in the Wiener process is like the square root of the increment in time, okay.

This immediately has the following consequence there are several ways of doing this. Now what we are interested in is asking for the behaviour of the properties of functionals of Brownian motion x is a random variable in the normal diffusion problem and you ask what does F of x look like? Where F of x is some function etc. so we are going to consider what happens if you had a functional we just write it as F wt you could extend it to the case where it has a explicit t dependence as well we will do that in a second.

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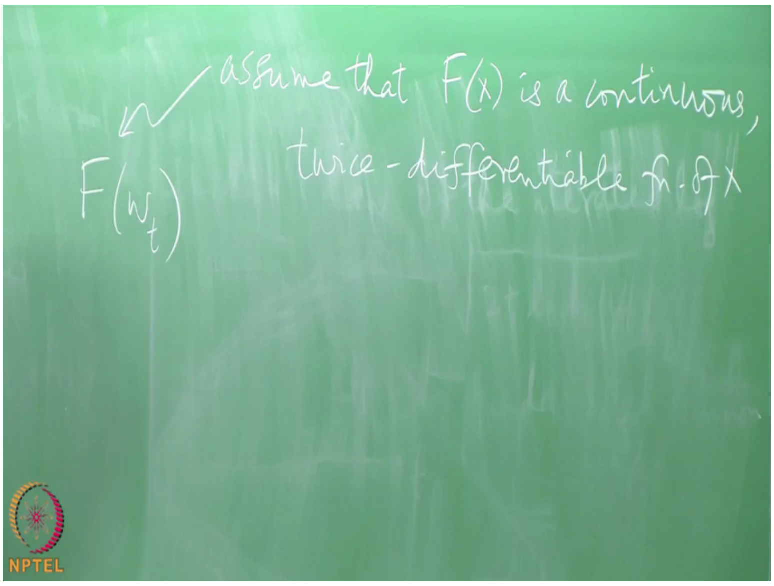

What happens to this functional and what is its differential look like, okay. What you have to do? Is to do a tale of series about any particular point and keep track of the fact that this thing here is like dt, okay. So we will assume that F of whatever argument F of x is a continuous twice differentiable, at least twice differentiable we will consider functions which are at least twice differentiable functions of the argument.

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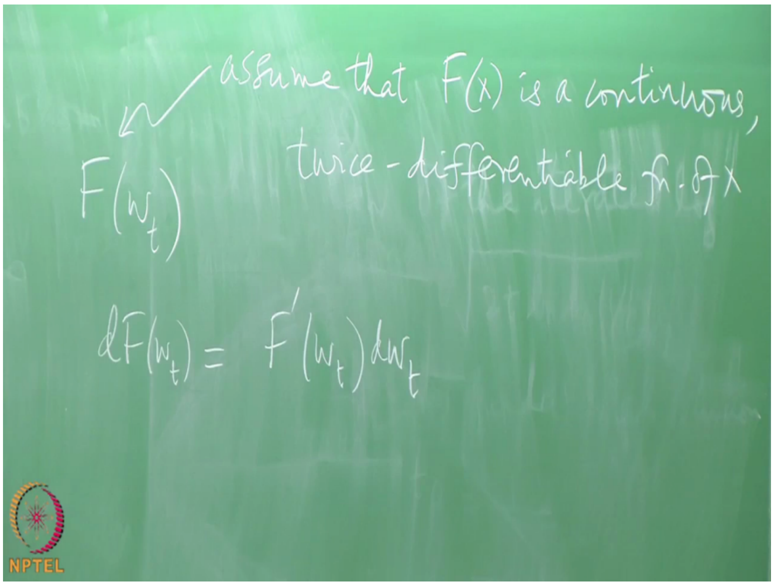

Then we could ask what is dF wt this is equal to F prime wt d wt it is the first, right? But that is not enough because it is a function of a function that could also be an explicit time dependence here.

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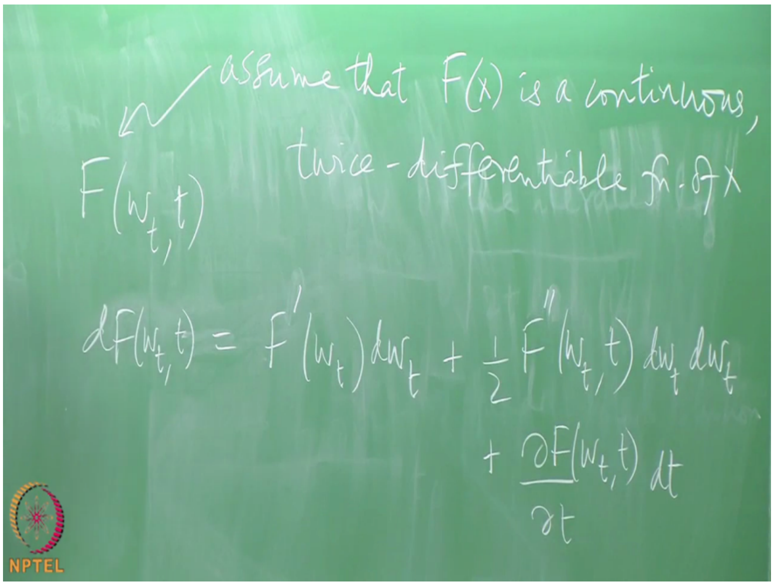

So let me in fact write that dependence, let us look at the more general case where you have this then this, well, let me use the following notation F prime, now I will explain this notation in a second, so dF of wt t equal to this plus the portion coming from the fact that this second term in dwt will still be a (()) (30:00) dt. Now that term turns out to be 1 half F double primeWt, t dwt dwt plus delta F of Wt, t over delta t dt.

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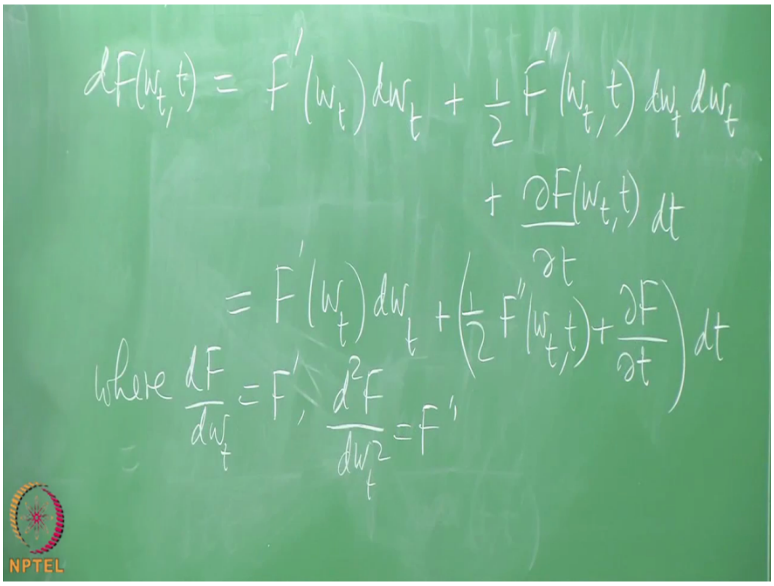

But we know that this is dt, so it gets added to this term, okay. Now this is the fundamental rule of Ito calculus where dF over dwt is equal to F prime d2F over dw2t, so you differentiate with respect to wt alone those are the 2 derivatives, you already assumed that well, I should really write F of x, t this is twice differentiable in x, okay.

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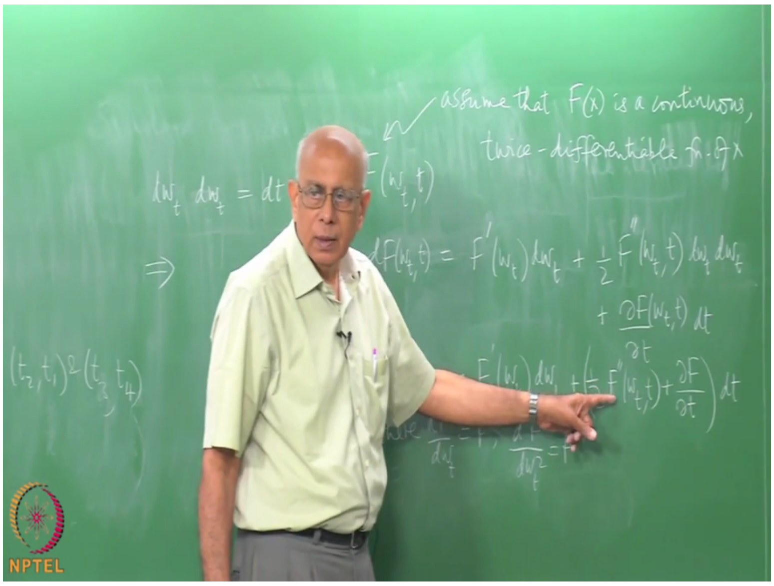

Now that addition of this extra Ps here helps you that is the amendment that the rules of calculus d in order to be applicable to as singular objects as Brownian part as a Wiener process because this is a differential form, the integral form of this will tell you what the direction is in normal cases.

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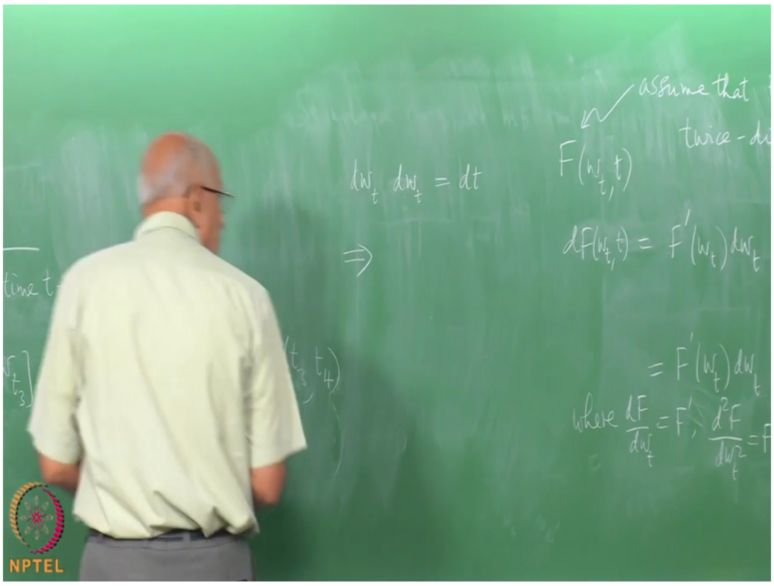

So remember this whole, this here.

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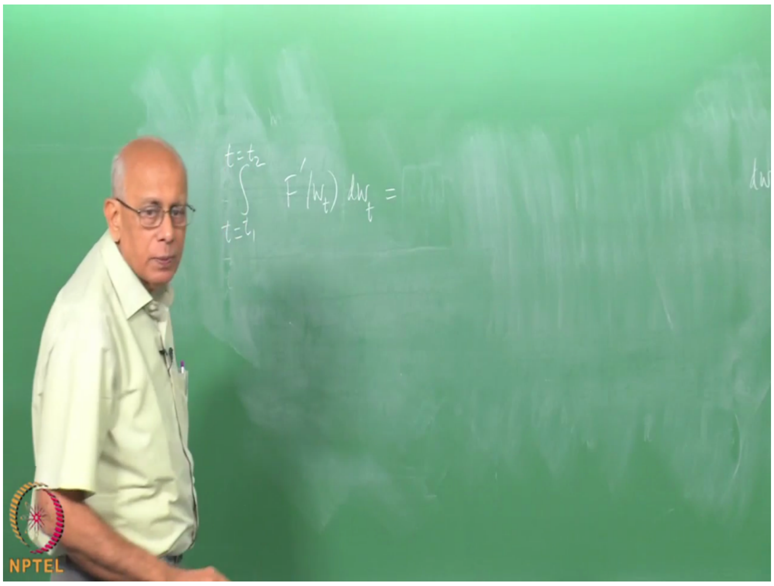

Now let us look at what is the integral of say t 1 to t2, t equal to t1 to t equal to t2 of something like F prime of wt let us look for the moment at functions which are not explicitly time-dependent. So you see what the correction is, dwt what is this guy equal to?

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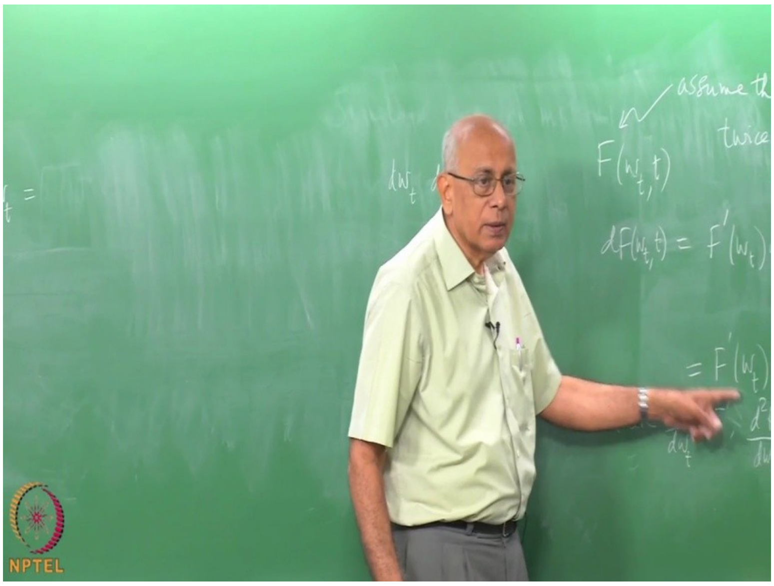

Well they are integrating both sides of this equation here and we are asking what is this fellow equal to? So it is this integral definition it is a differential therefore you can write this as clearly F wt2 minus F wt1 and it is certainly true.

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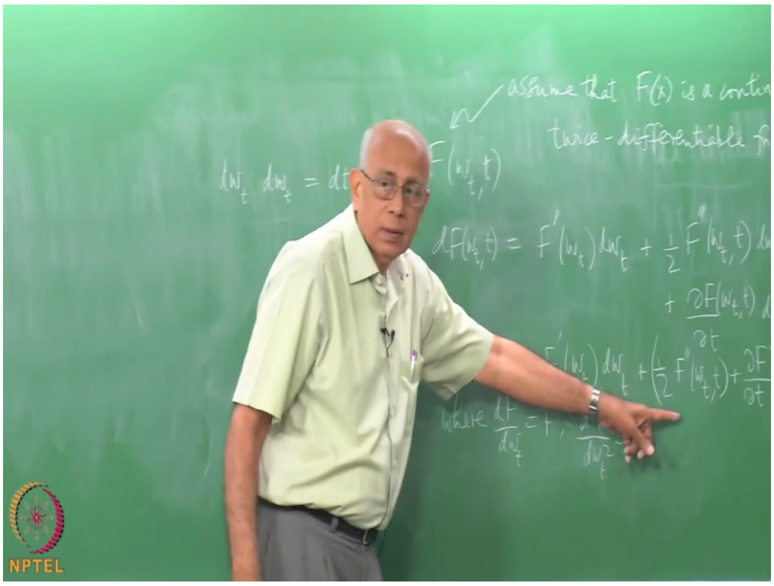

But then there is a correction due to this piece here.

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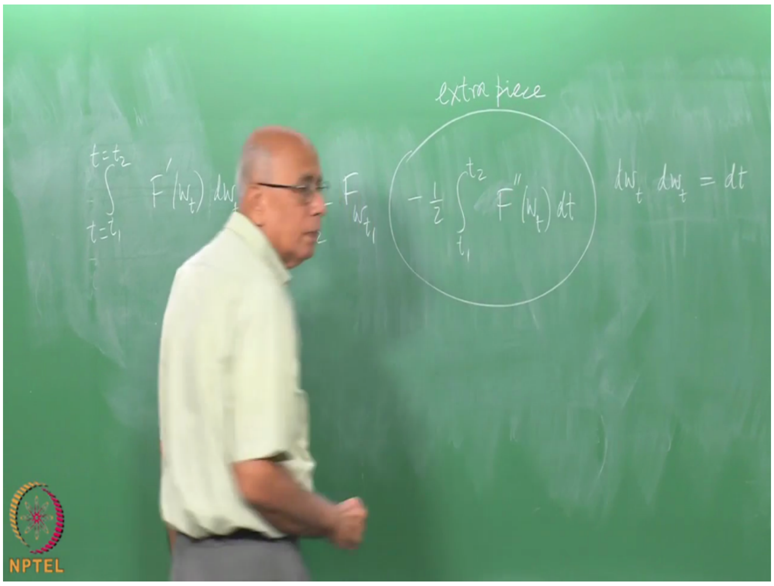

So it becomes minus half integral t1 to t2 F double prime of Wt dt, so there is this extra piece, okay which you require, alright. So let see what this is doing for us? What it implies? let us first look at a case where this F prime of Wt, let us say it is the Wt itself than it else you add integral t1 to t2 in time Wt dwt then we should write t equal to t1, t equal to this, is equal to by applying this theorem applying this blindly.

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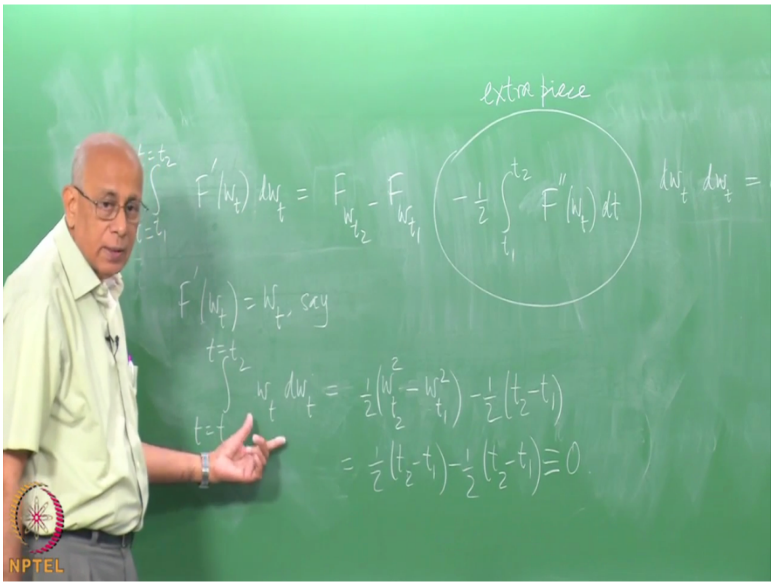

It’s the integral of this fellow which is the Wt squared over 2, so it is half wt square minus half, half comes out minus Wt1 square minus half. Well, if double prime is 1 in this case. So it is just t2 minus t1 but this is equal to t2, right? We already had that overlap rule and so the square of the Wt square is t itself, so this is equal to half t2 minus t1 and its half t2 minus t1 is equal to 0, is identically equal to 0, okay. Because of the way this Ito calculus works.

Now what does that actually mean? I mean this is an important result because it tells you this thing tells you that there is a specific interpretation being given to this, this process here, this integral here.

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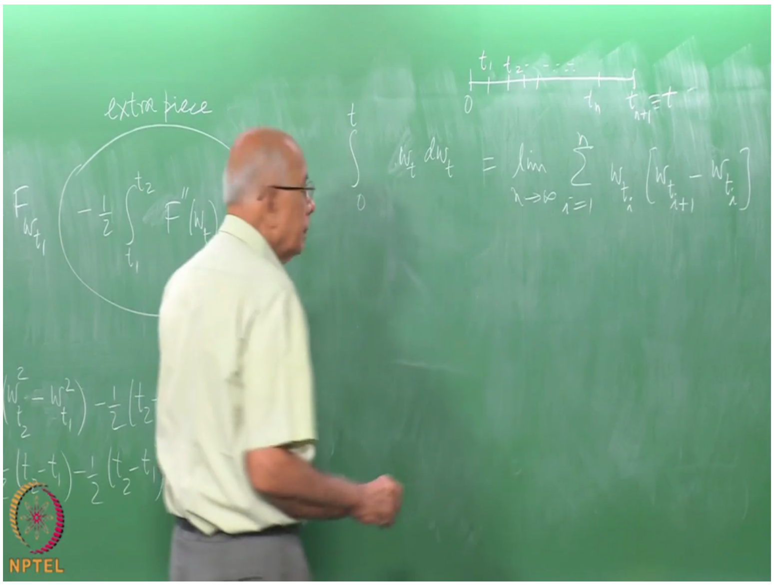

So it says that what we mean by that integral, so since we can write t equal to t1, t equal to t2, Wt dwt can be written in the following way, so I start with time 0 time t and then break this up into n parts, so there is time tn and time tn plus 1 is equal to t and this is 0, so this is t1, this t2 etc break it up.

Then this is equal to limit n tends to infinity summation from i equal to 1 to n times Wti times dW but this dW is Wt i plus 1 minus W t sub 1, it is the forward distance, sorry, I am sorry, yes, I chose it to be 0, yes I started with it here. This is the general result.

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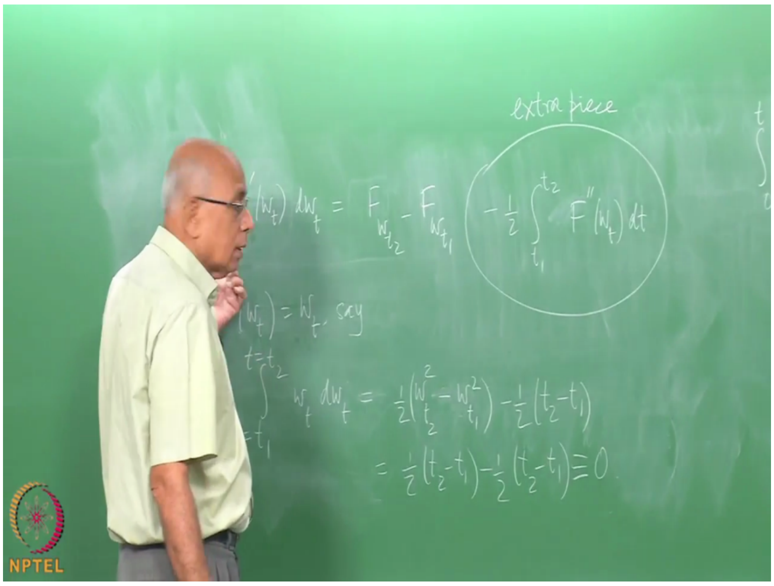

This fellow is a general result here but you can start at any point in this process.

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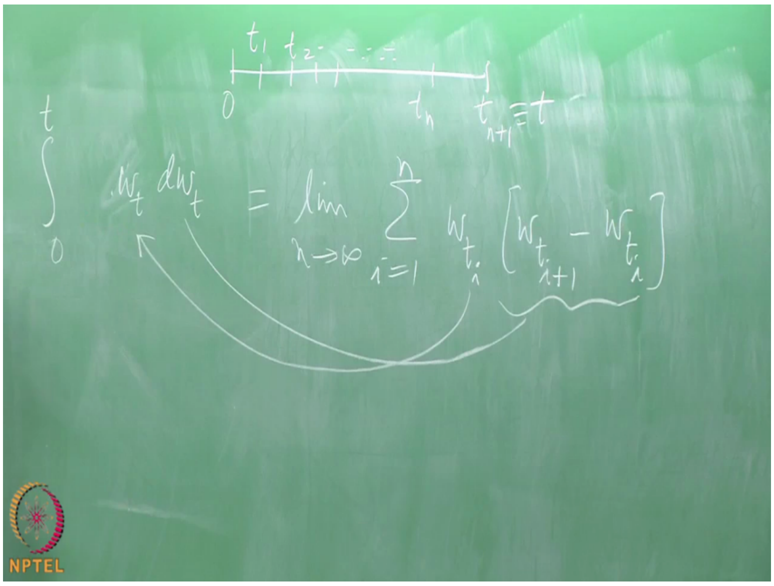

I start from 0 and there is the meaning of this. So it is this quantity that has been written there and this fellow has this difference here in the limit. How is the number of intervals number of subdivisions becomes infinite?

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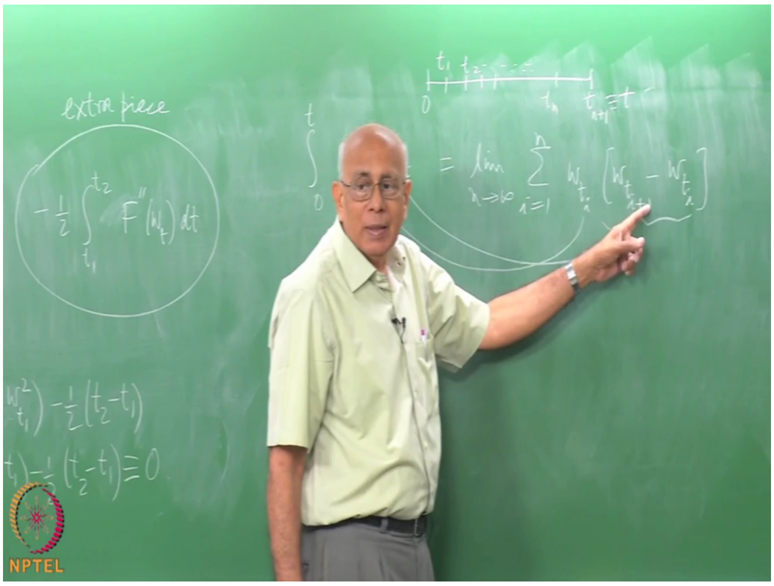

Now this is the forward increment and there is a current value of the variable. We know that the overlap between these 2 is 0 because this goes up to ti and this goes beyond that, there is no overlap between these 2 integrals and these 2 factors therefore by our basic rule for Brownian parts they are independent this is 0.

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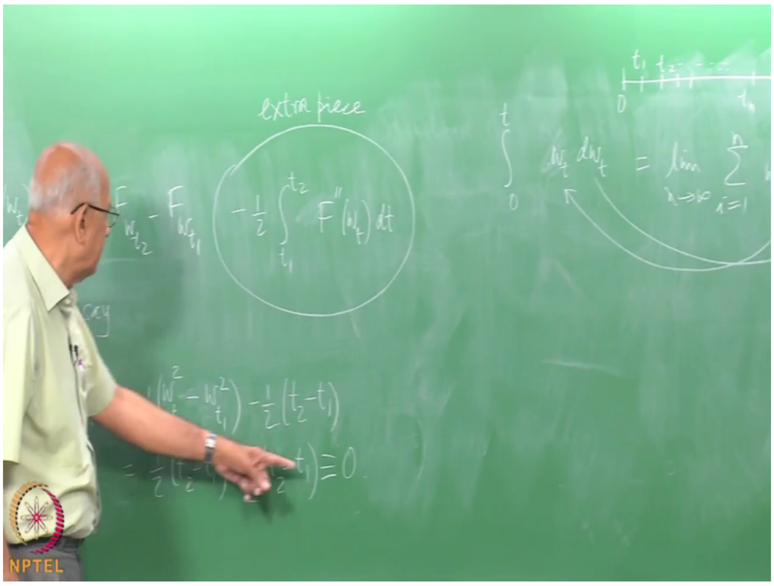

So that explains why you get 0 here, okay. Now you could have interpreted it differently when you do normal integration from a summation to convert it into integration there is no reason, so you normally write F of x Bx etc.

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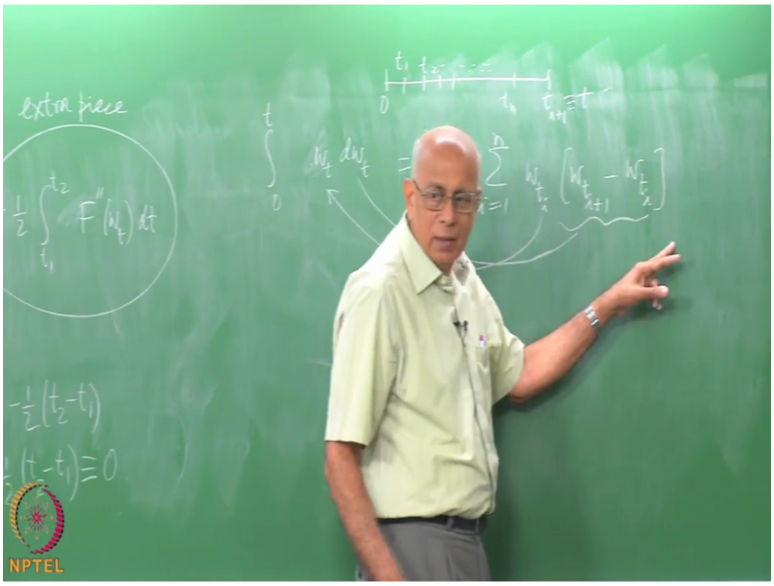

Now you write it as a sum, this dx is say x plus delta x minus x, right? But this quantity here F of x is x at which point? It could be at x plus delta x over 2, it could be an any combination of F of x and F of x plus delta x. This is a specific choice of the way this integral is integrated, the interpretation.

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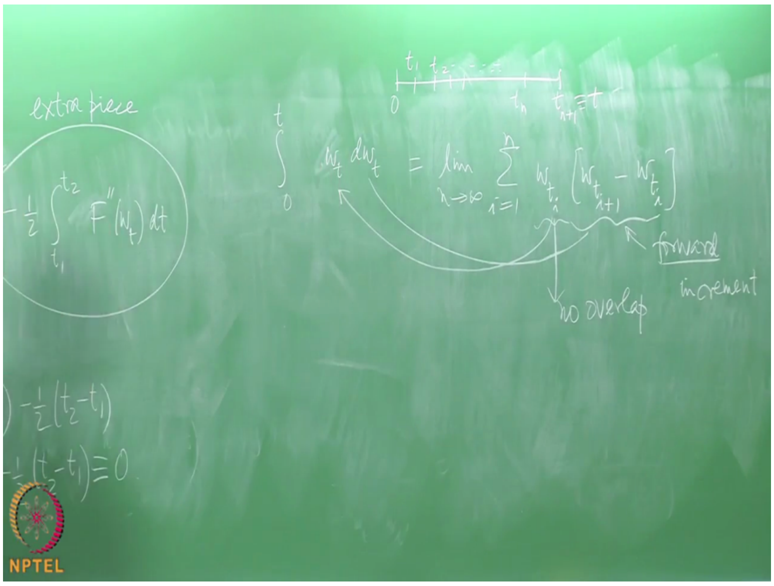

So this is the fact that these 2 factors no overlap and this guy here is the forward increment, it is this that gives you the Ito calculus and all the other properties that I wrote now. So clearly it says that there are other choices possible, you could choose this to be ti plus 1 for example you could choose it to be ti plus ti plus 1 divided by 2. So they have any number of choices here and each time you get stochastic calculus, okay.

Now if you choose the Ito interpretation, you have this feature then and only then does it turn out that the interpretation we have for the general Langevin equation or general Ito equation now.

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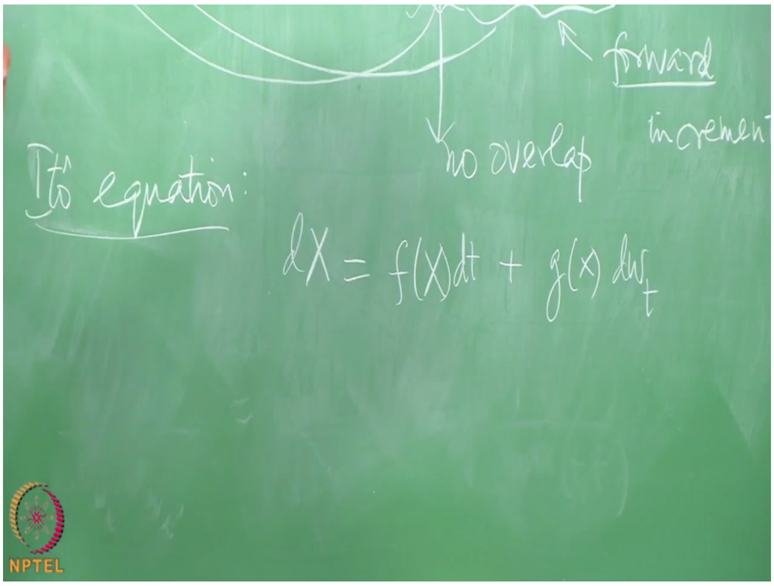

So our general Ito equation stochastic equation, I should not call it Langevin anymore this is of the form some process dx is equal to some f of x dt plus g of x, remember the Langevin equation that we wrote down with a drift term and a diffusion term there was a g of x times we called it white noise and put this as x dot but the correct way of writing it is with a dt here and this is dw this function. This is an Ito equation this is to be understood in the sense of Ito.

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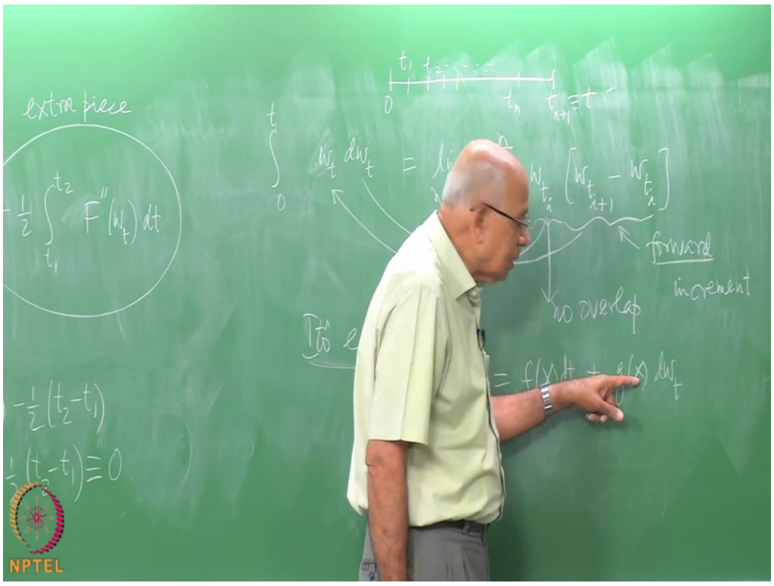

So the x here and if you (()) (40:45) time-dependent the t is such that this is the x at instant t and this is the forward increment in the Wiener process, okay. We need that interpretation otherwise it is ambiguous multiplicative a stochastic process is definitely ambiguous, okay. Question is you have a noise in physical term, so you have a noise that is dependent on the state variable itself.

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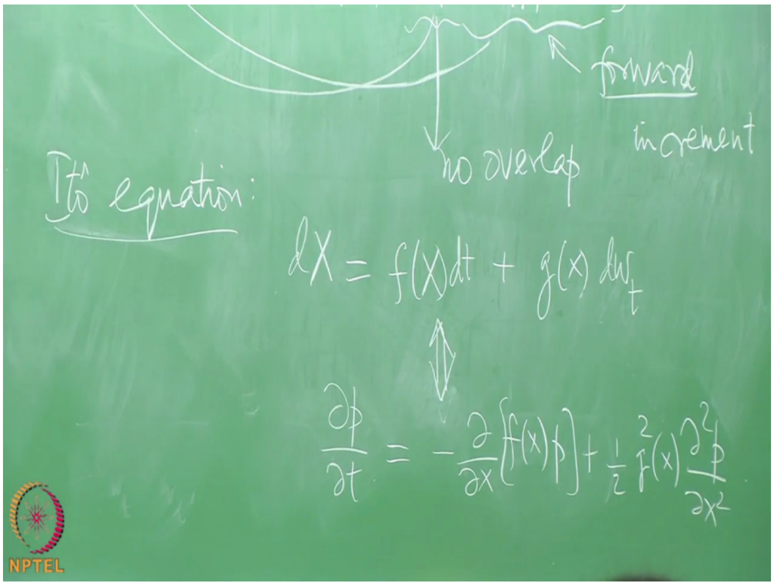

The question is state variable at what time? It is the state variable when you write the increment in X is a state variable at time t where this dwt is the forward increment between t plus delta t and t, okay. Then this implies and is implied by the Fokker Planck equation that we wrote down for this process. It implies that the probability density of this thing is delta P over delta t is equal to this fellow here is equal to minus delta over delta x type of xt plus half g squared of x delta 2p over delta x2, okay.

So when you have a nontrivial noise a multiplicative noise, you have to interpret it into the Ito sense and then this is true, they will start (()) (42:21) interpretation where this would be ti plus 1 would give you a different Fokker Planck equation for the same stochastic differential equation interpreted in that sense. The point is that the physics has to be the same in both cases.

So what would happen there? Is that you would have what is called a spurious drift term, so in going from the, you go from the engineering equation to 2 different stochastic differential equation. So as to get the same master equation once again for random variable because its moment are physically measured.

So you start with an equation between averages the so-called engineering equation you add noise to it, if it happens to be stochastic multiplicative noise, you have to then specify is the equation, this equation in the Ito sense or the Stratonovich sense or anything in between there are other interpretations? And then each time the prescription to go from a stochastic differential equation to the master equation changes in such a way that the final master equation is exactly the same forgiven physical process, okay.

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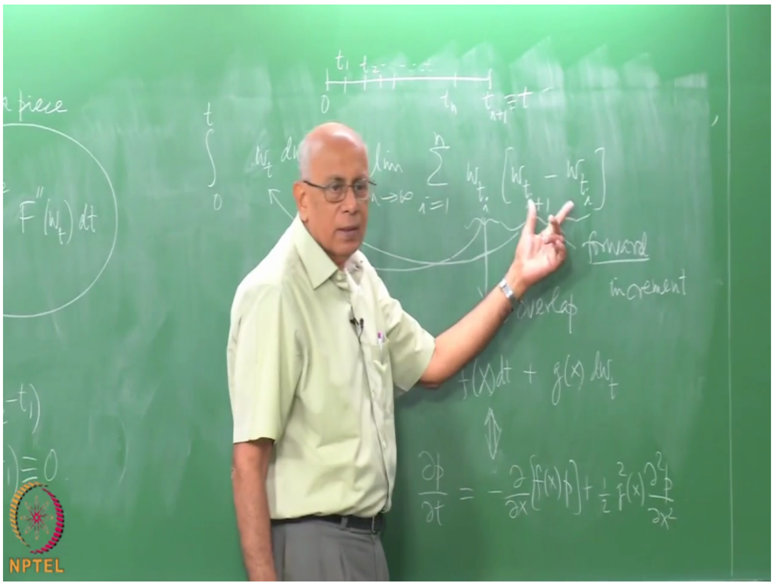

We have used all the (()) (43:33) while I have used the Ito interpretation, it is all satisfying because of this that is not anticipatory, it is a current dependence on the rate of the current on the state variable at the current time and then the forward increment, it is not sacred it is not sacred. In fact in physics very often one uses the Stratonovich interpretation, okay.

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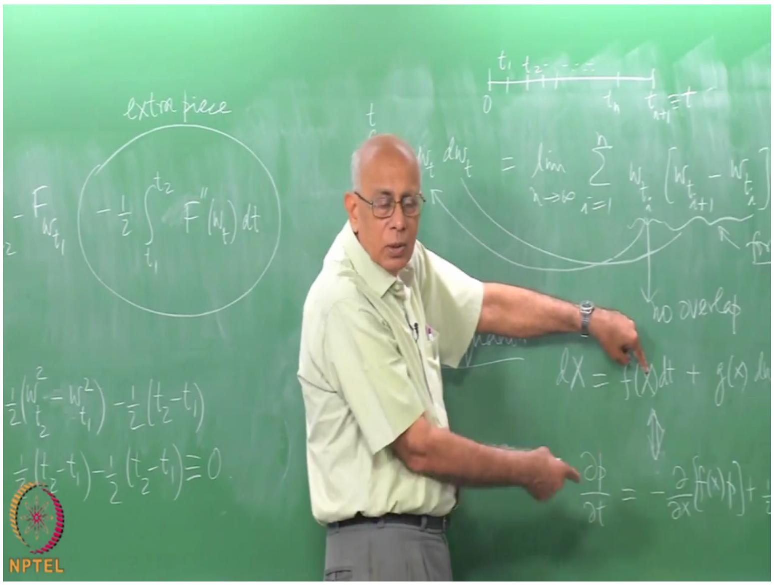

But then you have what is called this spurious drift term, there is a correction to this fellow here and the prescription to go from here to the stochastic to the Fokker Planck equation changes into 2 pieces we have consistently use the Ito equation, okay. Now of course if you if you use the calculus carefully then questions of uniqueness comes, for solutions of this process question of uniqueness arrives and here is an example due to Ito himself.

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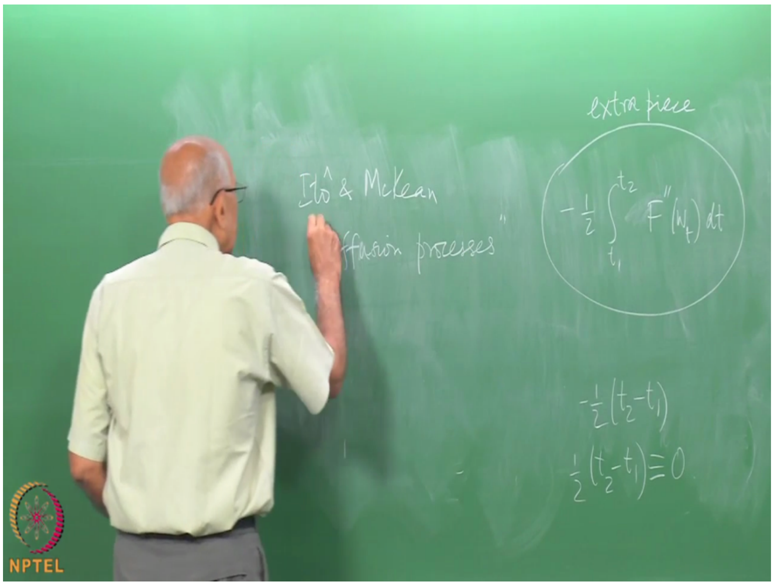

By the way I should give a reference to this whole business. It is the mathematical treatment can be made very rigorous and one of the best places or it is the original book by Ito himself, so Ito and McKean it is called diffusion processes.

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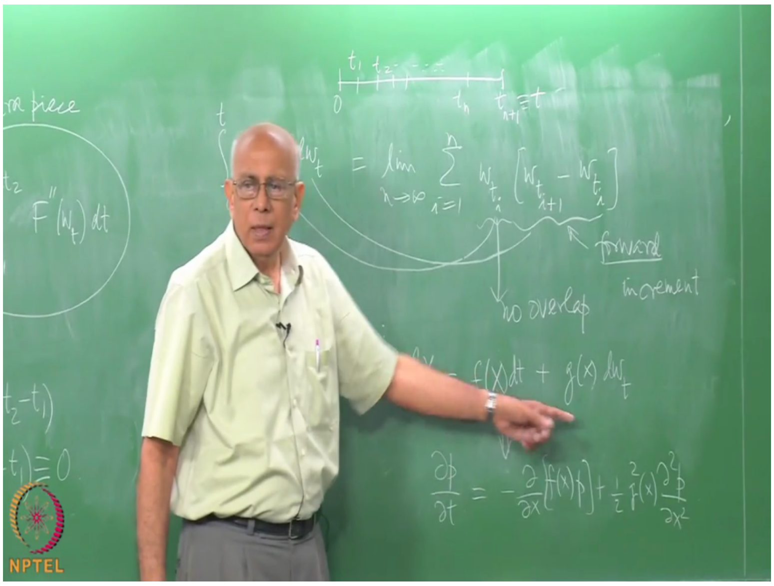

We call that an equation like this like diffusion process, we called x a diffusion process and you have to you have to worry about the uniqueness of the solution to this sort of equation this all this for the answer (()) (45:17) in that case it was completely trivial this was a constant, that one was a linear in X and then it did not matter we wrote down the solution but if you do it rigorously properly then you will have to be little careful because you could have non-uniqueness and here is an example due to Ito himself.

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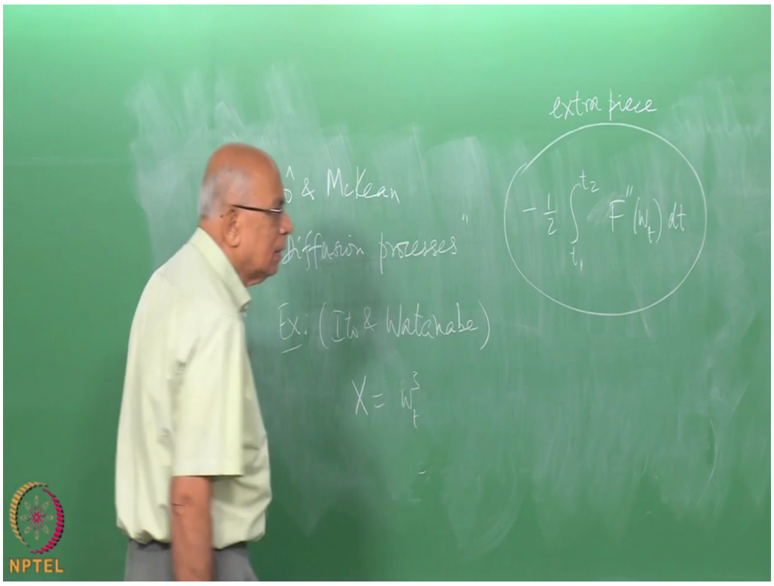

We are going backwards to write the equation down, so if you have example this is due to Ito and Watanabe and the equation is the following X equal to Wt cube for example, there is a nice function of Brownian motion, right? Then what does the equation give you? It says dx equal to f of x that is 3Wt squared, so it is 3 times x to the power 2third dt plus the second derivative you have to differentiate this, this is 3x square 3w squared and that is equal to 6w divided by 2 and W is X to the one third and the 2 goes away in the calculus.

So you get plus 3 times x to the one third dwt, sorry that is a coefficient of, how did I write this? It is the other way about this is F prime and then you had rho dwt, right? So this is the correction part. So if I ask what is the solution to this equation? We know this because we work backwards to get this equation. So if I ask what process is this x? You would say it is wt cube from this guy but you could also put 0 there and gets you as your solution.

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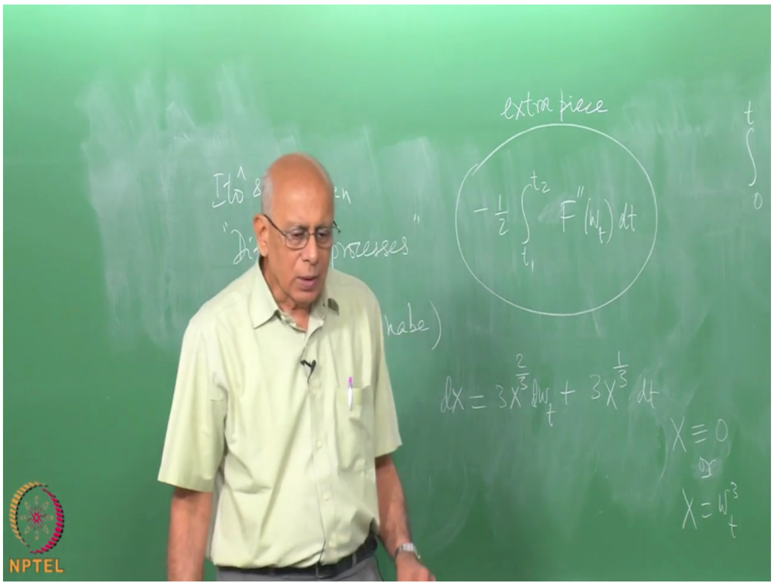

So you have this or that therefore the uniqueness of solutions is not guaranteed. Fortunately this is a kind of academic example, fortunately it turns out that if you put in sufficient holder conditions on the continuity of this process then you get unique solutions for instance if you can show that the mod X at sometime t minus X at time t prime is less than, sorry the function.

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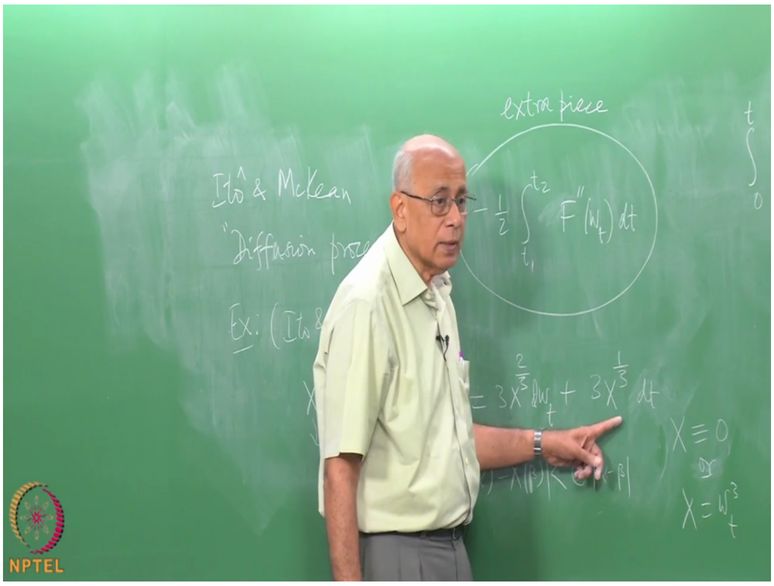

So if this fellow is some function x of wt, if you can show that the x of in some Alpha minus X or some beta modulus is less than some constant times modulus of Alpha minus beta for all alpha and beta all argument then that suffices show that the solution is unique, okay. This is not, this does not satisfy that, these 2 guys are not smooth enough at the origin, there are 2 singular at the origin but if you have something milder like this then the solution is unique and that is what happens in most physical examples, okay.

Now you can ask in this thing the generalized? But before that let me find out we may run out of time but I should point out what is the final (()) (49:33) formula is at the very least, okay. This requires a bit of a preamble, let me again motivate this on physical grounds.

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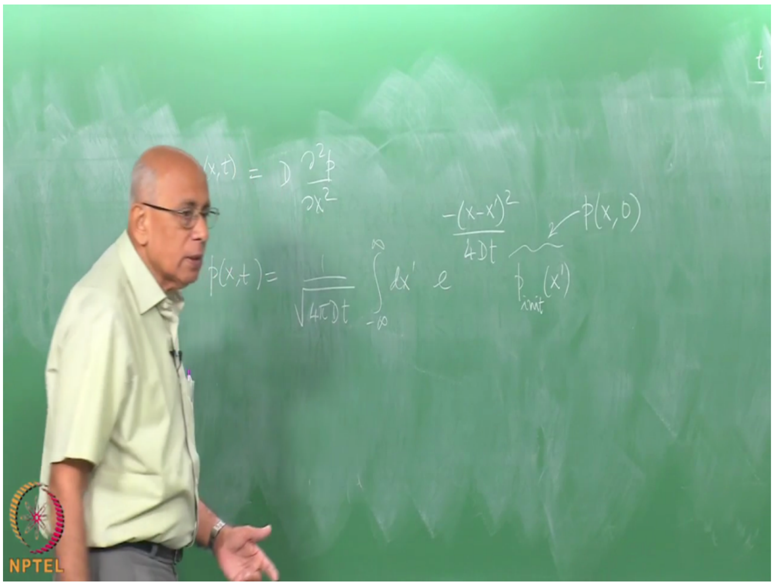

So let me give a little preamble to this, we call that the original division equations, so let us write it in the physics notation. Our original division equation was of the form delta P over delta t for the position of a Brownian particle D d2p over dx2, you can ask what is a solution to this equation for some given initial distribution, not necessarily a Delta function. Then of course you take the original Gaussian solution use that as a kernel of the green function for the differential operator and you integrate over the initial distribution whatever it is.

So you would write the solution p of xt equal to 1 over square root of 4pi dT when integral dx prime over all x prime e to the minus x minus x prime whole square over 4Dt, it could have started at any point t minus t not the initial time would have replaced this t minus t not, we are talking about processes starting from the specified at x equal to 0, this multiplied by P whatever be the initial, P initial of x prime this was P of x and 0. It acts as a green function here.

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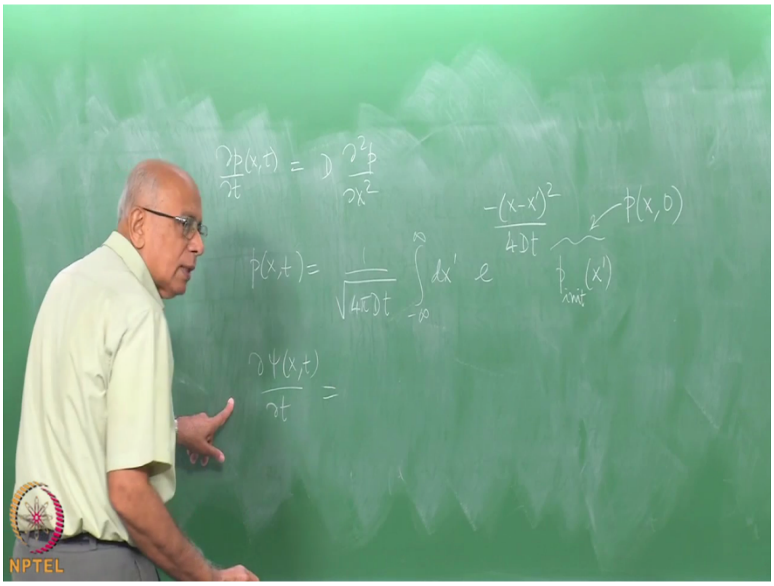

Now the Schrödinger equation for a free particle has exactly the same behaviour. So there you have delta psi of x, t over delta t equal to ih cross times that this is minus h cross squared, so let us write that as minus ih cross whole square, so this is equal to ih cross by 2m d2 psi over dx square this too has a similar solution except there is this i, oh it is as if that d were imaginary quantity and it is not clear if this is an oscillator function in that case, it is not clear that if this thing converges.

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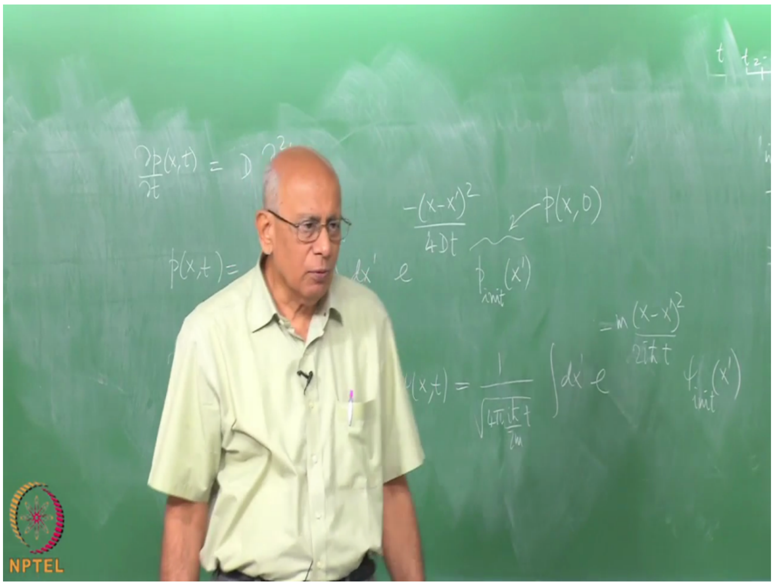

But formally if you do a weak rotation here in time and go to imaginary time i times t than the 2 problems are mathematically equivalent and well-defined. So you have exactly the same sort of solution this would imply that psi of x, t is equal to 1over square root of 4pi and then instead of d you would replace it by it cross over 21 t integral dx prime e to the minus x minus x prime whole square divided by whatever it is.

4 times d, d is this, so twice, 2 ih cross t and then an n or something like that times psi initial x prime and if you recall this is the starting point of the path integral formulation of quantum mechanics then you do time slicing and stuff like that and you get the path integral formulation of quantum mechanics, right?

Now Mark (()) (53:30) and said, now this operates into different, gave a clever formula which essentially says that you can use this fact the fundamental Gaussian solution in order to write down solutions to some deterministic equations or conversely starting the deterministic differential equation which is essentially the division equation with a potential term, alright.

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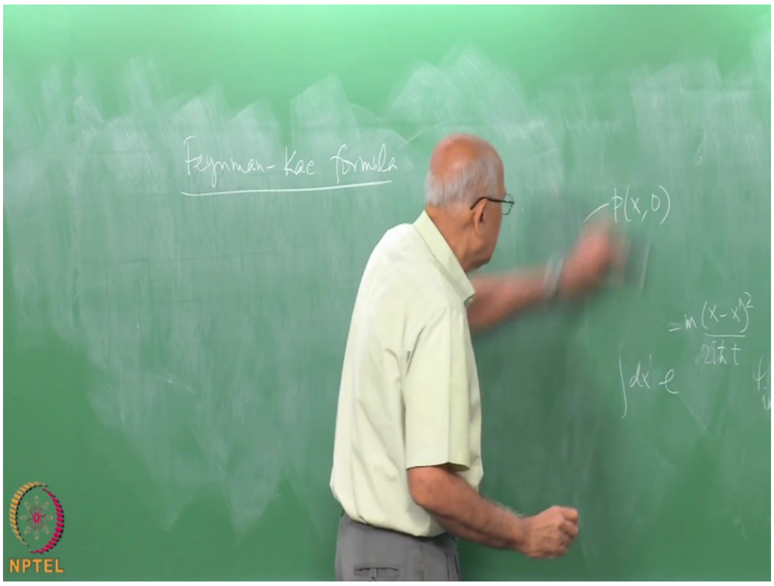

You can find expectation values of certain functionals of Brownian motion. It works in both directions and in the simplest form it looks like this. So this is the so-called final carts formula and let me write it in the simplest form, it has got all can the generalisations higher dimensions etc but here is what it looks like.

Suppose you have a partial differential equation of parabolic partial differential equation in one space dimension in 1 dimension equal to let us write the standard diffusion equation down, okay. This is the PDF of the Wiener process you like but along with that let suppose there is an extra term V of x times u of x, t. If this were the heat conduction equation it is like there is some external cooling which is state dependent x dependent here.

Or in the context of Schrödinger equation like a potential then the question is what is the solution? You have to specify an initial condition and the initial condition is, so some conditions are put on this, this is a nonnegative continuous function and U of x, 0 is equal to some specified initial functions, so this is u mod of x and we will assume this to be continuous bounded. So it is a continuous bounded function.

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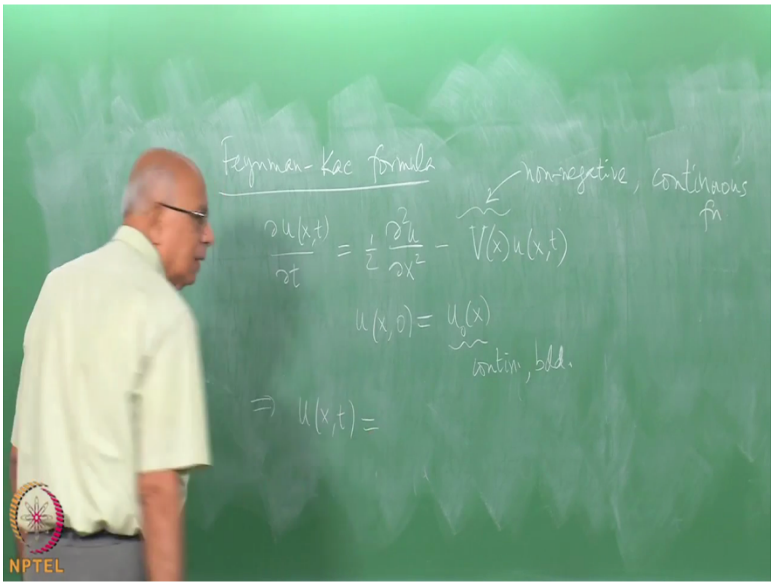

Then the statement is that the solution of this, the unique solution of this has the following form. It is equal to the average value of e to the power minus integral 0 to t V of Wt dt times U not of Wt but I have to say what is Wt and what this average is over?

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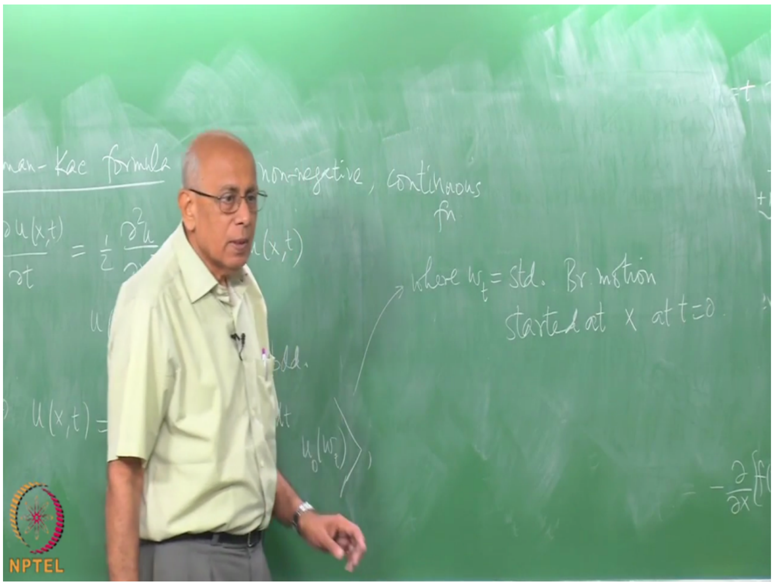

Where Wt equal to standard Wiener process or Brownian motion, standard Brownian motion started at x at t equal to 0.

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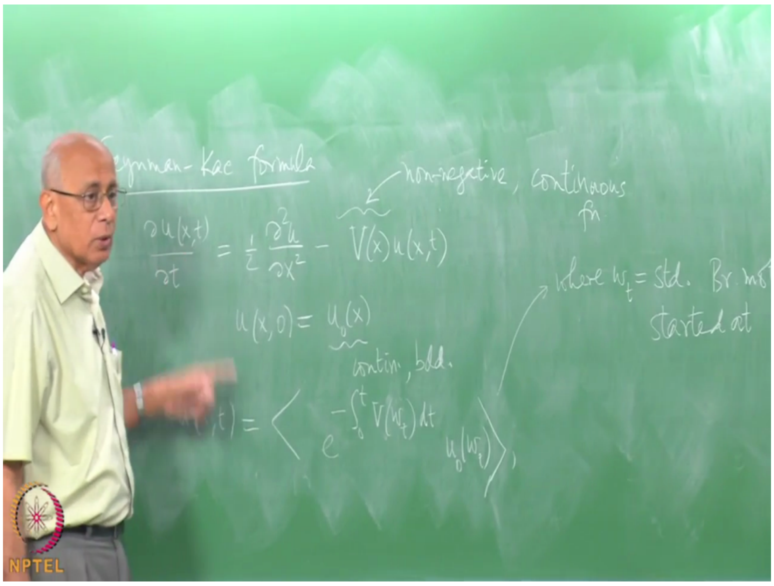

So your Brownian motion started from wherever you want dissolution whichever point you want.

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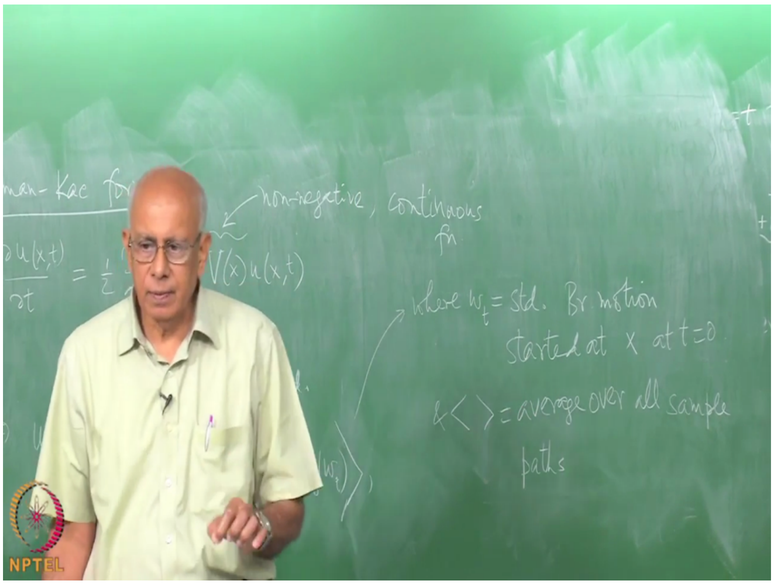

And then you let it go and equal to average over all random paths, all sample paths subject to the above conditions then they all Brownian motion starting at x at t equal to 0 at any (()) (58:24), no starting at x is going forward, right in time and then you do this integral, okay. So this works both ways where you have an arbitrary functional out here and here some given functional then you have this back again, where the simplest case would be you V equal to 0 then you see where this is coming from.

You put V equal to 0 this goes away, you are going to take the initial point and then you are going towards the Gaussian kernel and you are going to integrate it would be one way of computing the (()) (59:05) that is exactly the solution we wrote down to the diffusion equation but this generalisation to the case of arbitrary functions here subject to very mild conditions, okay.

“Professor -Student conversation starts”

Student: (()) (59:21)

Professor: If you wrote it back it is just the (()) (59:24) for the potential V of x, exactly.

“Professor-Student conversation ends”

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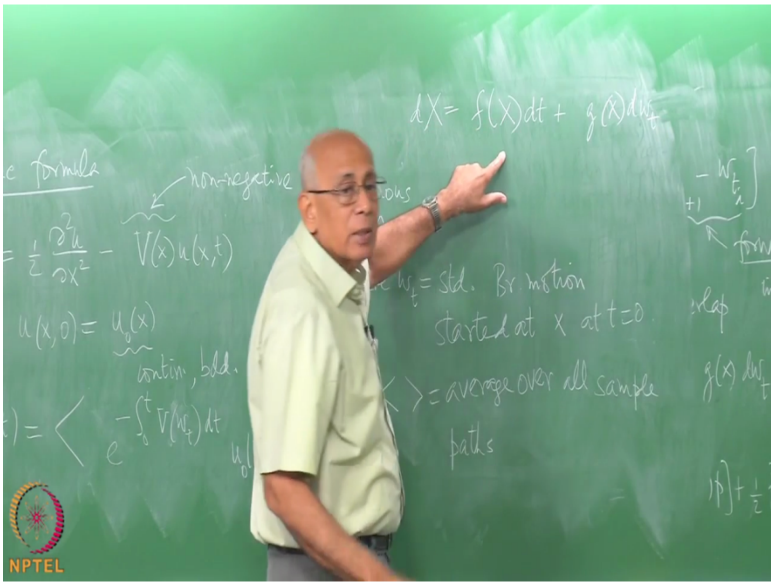

But what is interesting is this is capable of enormous generalisation, first of all to higher dimension Brownian motion for which we know how to write you have a general equation of the form dx equal f of x dt plus g of x dwt where this is an end dimensional vector, so is this. This is a new dimensional noise and this is an n times the matrix suitable conditions on the g. We know how to write the Fokker Planck equation down.

Now for the process X, the general diffusion process X you have an analog of this formula does not have to be standard Brownian motion alone that would correspond to the case where f is 0 and g is 1 but you can instead of Wt you can use x and you can read a generalize formula once again for a more complicated equation here which involves the first derivative of this U with respect to x with that drift term and the second derivative with that diffusion term, okay.

So instead of this operator you replace it with g square times this guy plus f times f of x times the first derivative (()) (61:06) equal to x and it is still true. Why dimensional is more complicated parabolic equations it still works in this form and people are using it for long time.

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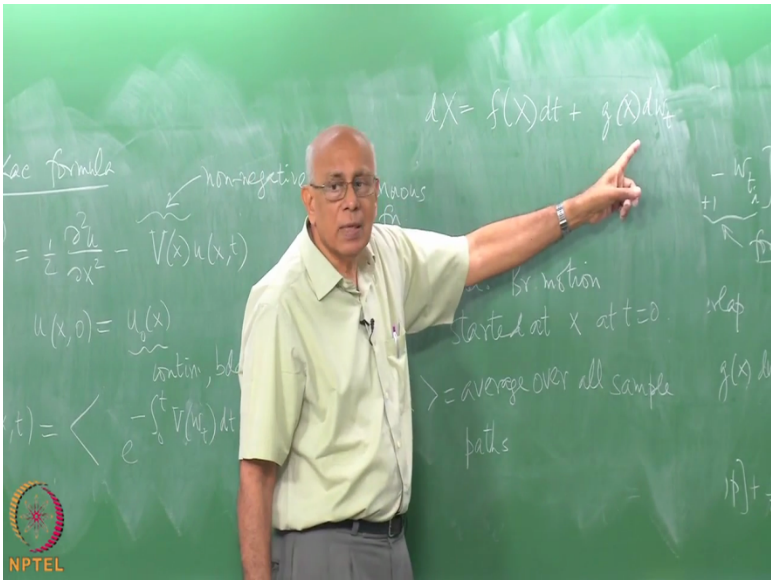

I should finally mention that the case where this fellow is linear in x and that too is linear in x is the famous Black Scholes model that is used in financial mathematics and it corresponds to exponential Brownian motion. So the random variable is e to the Wt if you like times some constants, okay. And then an explicit function of t you take care of Ito calculus, okay. That is what I wanted to say here in this context.

We have this entire course we have sort of started with very simple Langevin dynamics and come back to it in a very general form here. On the way we went through all kinds of applications of Langevin dynamics that is a paradigm that is a model and I try to point out very briefly that is a model is generalized to the case of fields, auto parameter field and then it has connections to all the equilibrium and non-equilibrium phase transitions, how they help of the (()) (62:26) function you thought handle all non-equal is a critical exponents including dynamic critical exponents.

We did not take in excursion into physical phenomena per se that is a separate topic for itself but we did study some linear response theory on the way both classical and quantum and some topics we run out of time and most notably chemical kinetics and thermodynamics are very small systems all the fluctuation theorems these are 2 notable exceptions perhaps at some future date, okay. I will stop here.