Non-Equilibrium Statistical Mechanics
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Lecture-34
Critical Phenomena (Part-6)

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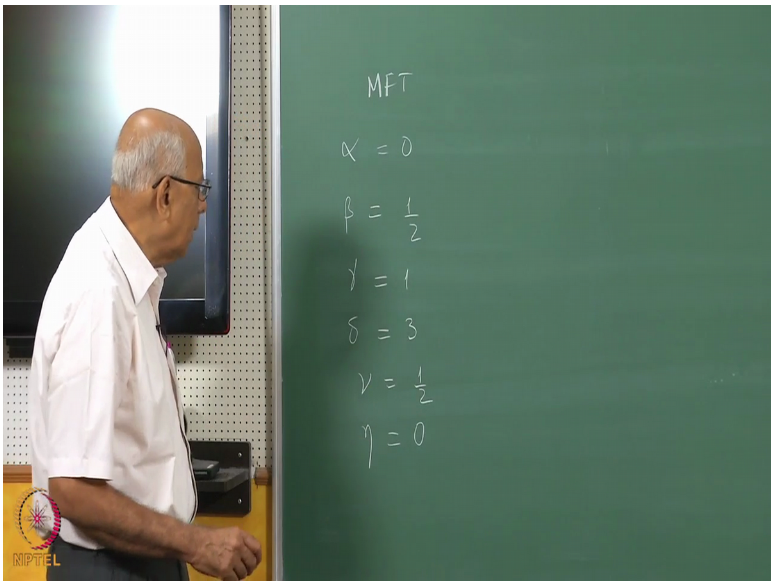

Right, now I want to talk today about the way the scaling relations arise between critical exponents and also give you a little bit of an introduction to the Ginzburg Landau theory of phase transitions and then go on from there, so let us first go back take a look at what we deduced from mean field theory in the Ising class, we found that I did not calculate this but this exponent, the specific heat exponent is 0 in mean field theory, I did not compute this, it is a trivial calculation but I did not do it but its alpha equal to 0 is the result in mean field theory.

The magnetization exponent or order parameter exponent is a half, the susceptibility exponent is 1 and the critical isotherm exponent is 3, okay, remember that here m is h to the one third right so that was the relation here or m goes like h cube that is the relation sorry h goes like m cube was the relation and then I introduce two other exponents, one was mew for the correlation length and this was a half and this exponent was 0 essentially although again we did not talk about it in great detail.

And I said there are relations between these exponents here. Now the way they were arrived at originally was empirically, completely empirically and today we have an understanding of why these things are the way they are from mean field, from field theoretic approach to it, from the renormalization group but in the sense it is ultimately experiment, it is ultimately basic experimental evidence okay.

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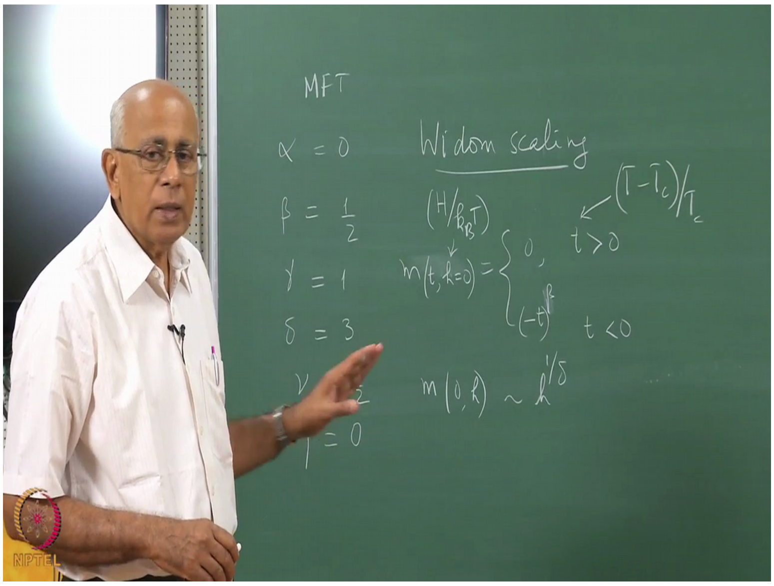

So through the 1960s people very patiently collated a lot of data and then finally Widom proposed the following scaling function, remember the equation of state we had for the Ising model and his point was the following so its, it was followed shortly after that by other proposals for scaling more sophisticated proposals by other people such as Kadanoff and so on culminating in the renormalization group approach of Wilson and Fisher and others to calculate critical exponents is a function of dimensionality and the number of components of the order parameter.

Okay that is a sort of crown jewel of equilibrium statistical mechanics but the idea of Widom Scaling was very simple, his point was that if you took the magnetization m in the critical region then this was equal to on the one hand 0 for t greater than 0 even by that t is T minus Tc over Tc and it was equal to minus t to the power beta apart from some constant functions etc in this fashion for t less than 0.

So this was the magnetization and I should write this down here in the absence of a field so one has statement for m at t and h equal to 0, on the other hand if you look at m at 0 and h for small h, oh let me mention here, I should have done this that this is H the field divided by k

(())(04:23) T it was rescale the magnetic field by kT and that I call little h, I did it implicitly but since you pointed out that there is a 1 over t let us call it little h that is a standard notation for it.

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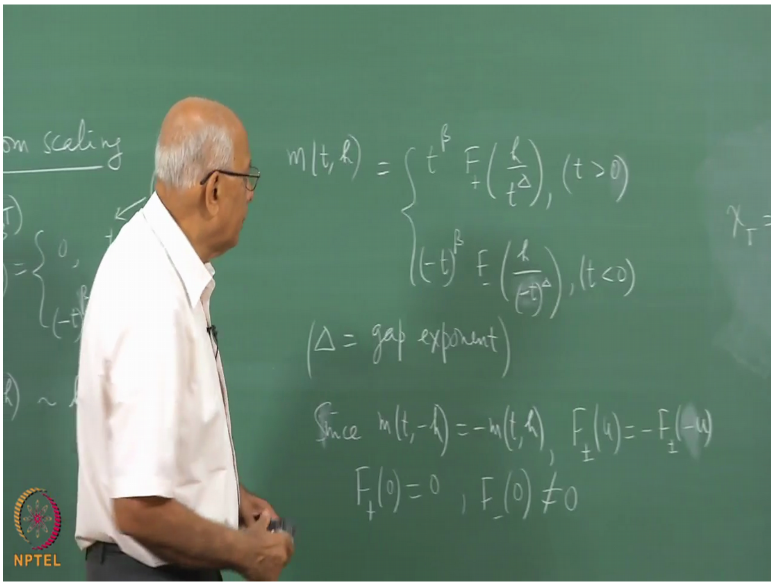

Now you could ask what happens if you on the critical isotherm, this stuff iself this goes like h to the power 1 over delta and delta was 3 in mean field theory okay so is that a way to combine these two relations and get them out in one formula then one expression and that is what Widom did, what he did was to show that the huge pile of experimental data fitted the following scaling rule, the function m of t and h which should ordinarily be a function of two independent variables becomes now dependent only on 1 particular combination.

So this thing here is equal to t to the power beta times something called a scaling function so this f and this is for t positive so let us called it F plus h over t to the delta for t greater than 0 and it is minus t to the beta so this is just a dependence on the modulus some other scaling function of h over minus t to the delta for t less than 0 and this functions F plus and minus are called scaling functions so his point was asymptotically close to the critical region where m is 0, h is zero, t is 0 asymptotically close.

All the data collapse onto this kind of functional form okay where this is some power, this is some power, this is some power etc okay, now capital delta is nothing to do with little delta, he said there is a power delta such that you fit it in and you choose a proper delta and all of the data collapse onto this sort of formula, the question is what is this capital delta etc, so delta is called the so called gap exponent.

But now let us see if it fits this, how do you get that, you set h equal to 0 and then you immediately see that, oh first of all it is clear that since m of t minus h equal to minus m of t h if you reverse the field the magnetization simply reverses at a fixed value of the temperature. It is clear that both F plus or minus of whatever the argument is of u equal to minus F plus or minus of u, whatever be the argument here these scaling functions, oh minus.

So the scaling function is an odd function in each case, moreover above Tc for T positive you get a 0 here right, at H equal to 0, so it is guaranteed if you take F plus of 0 equal to 0, pardon, not necessarily that is the point not necessarily, think about this it is not, the odd character does not force it to be 0 because there could be a finite discontinuity right, I mean if you have a function that is odd then there are 3 possibilities, if it is continuous then at 0 argument it is 0.

If it is got an infinite discontinuity like 1 over x is an odd function but it is got an infinite discontinuity that is perfectly alright but it could also have a finite discontinuity like the magnetization has so it could do this and indeed that is what is happening here that is an od function but it is not a finite discontinuity and therefore does not vanish at the origin that is what the magnetization does is spontaneous magnetization okay.

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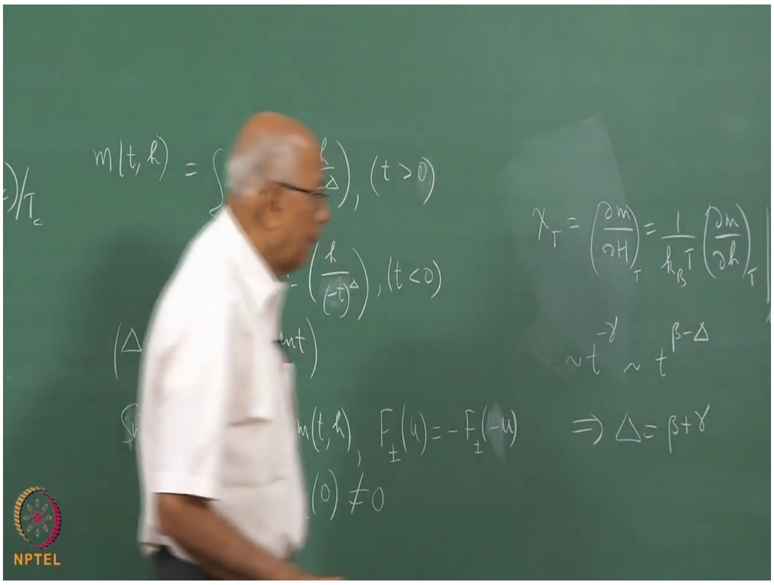

So what we require is that F minus of 0 is not equal to 0 some constant not equal to 0 then you find with this, you get this immediately okay. But the question is what is this delta? For that one looks at the susceptibility, so since we know that chi t equal to delta m over delta h which is 1 over k (())(10:15) T delta m over delta h at constant temperature and at h equal to 0, the zero field susceptibility.

What you need to do is to differentiate this on both these functions here, this will give you the paramagnetic susceptibility and this gives you the ferromagnetic susceptibility in the critical region just below, then assuming that the slope of this functions at zero argument is finite, not 0 or infinity it immediately follows that the susceptibility chi T goes like T to the minus gamma which from here will go like T to the power beta minus delta.

Because if I take out this function and differentiate it to first order and H since this follows 0 at zero argument and I differentiate it, you get a T to the delta and denominator when I differentiate with respect to H so you get this relation so that immediately tells you that this gap exponent is not anything new but delta equal to beta plus gamma.

But now let us look at what happens if H is equal to 0, sorry if P equals to 0, so there we got to be little more careful we want this result at little t equal to 0, so how is that going to happen. You want to make sure that on the critical isotherm at little t equal to 0 you get this cubic curve that is what you want and that will happen provided you see it can only happen cause this fellow blows up at little t equal to 0 right, so the only way that can be saved is that there is a leading way here over here which is some power which vanishes when taken with this.

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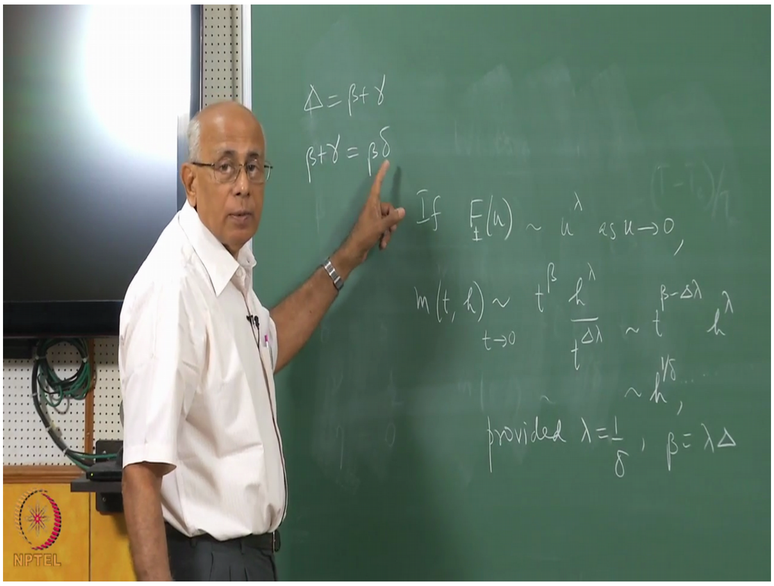

So if F of u, F plus or minus of u goes like u to some power as u tends to 0 then m of t h goes as t tends to 0 goes to t to the power beta and this is h to the lambda over t to the power delta lambda which is t to the power beta minus delta lambda h to the lambda okay, and that is the same as h to the 1 over delta provided lambda equal to 1 over delta and beta is equal to lambda delta so that this goes away, the t dependence cancels out there is only possibility but delta is beta plus lambda so it says, therefore we have second scaling relation or first one was that delta equal to beta plus gamma and now we discover if I put in 1 over delta here beta plus gamma equal to beta delta okay.

So you get this extra relation, therefore these are not independent exponents once you give me the others then this once you give me beta and gamma you know delta okay. This is independent of mean field theory because this is extracted from experiment okay, the actual numerical values of these are irrelevant the point is empirically you will get numerical values for those but the point is across systems, across different values of these exponents you still have these relations, of course the scaling functions will change in different cases and so on that does not matter.

The point is that there is scaling at the critical point in the sense that whenever you have function of several variables if you have a generalized homogenous function then you have what is called scaling because it means you can reduce the number of variables by 1 okay depending on how many combinations you can form you can deduce the number of independent variables and here you reduced it from a function of t and h separately to apart from this power a single function of this combination okay.

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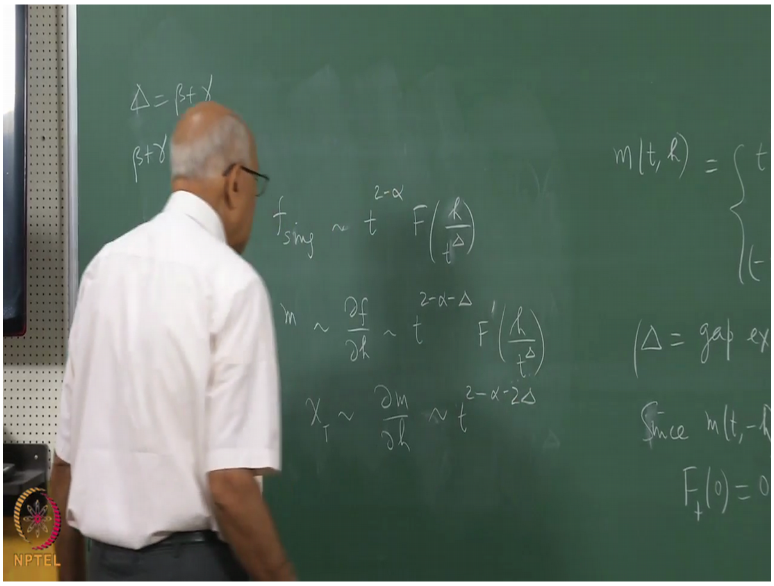

Now one can go on with this one can ask what happens in other, in the case of other exponents for instance if you look at the let us see what are the other exponents one can talk about, if you look at the thing like a free energy, the singular part because it is always a piece added what I called phi not earlier which is not singular uninteresting piece the free energy per unit volume and call it little f suitable free energy.

This guy scales like t to the power 2 minus alpha times again some scaling function let me just call it F once again of h over t to the power delta in the critical region then the susceptibility sorry, then m was like delta f apart from the 1 over t factor etc over delta h okay and for small values of h near 0 this goes like t to the power 2 minus alpha minus delta times F of 0 right sorry, delta h times F prime of h whatever that be, okay.

And of course m will vanish in the critical region on the critical isotherm and h is 0 that is perfectly okay but now the susceptibility chi t goes like, there might be a minus sign here in this free energy region (())(17:49) this goes like delta m over delta h apart from the 1 over t etc and this is now going to go like k to the power 2 minus alpha minus 2 delta times f double prime whatever.

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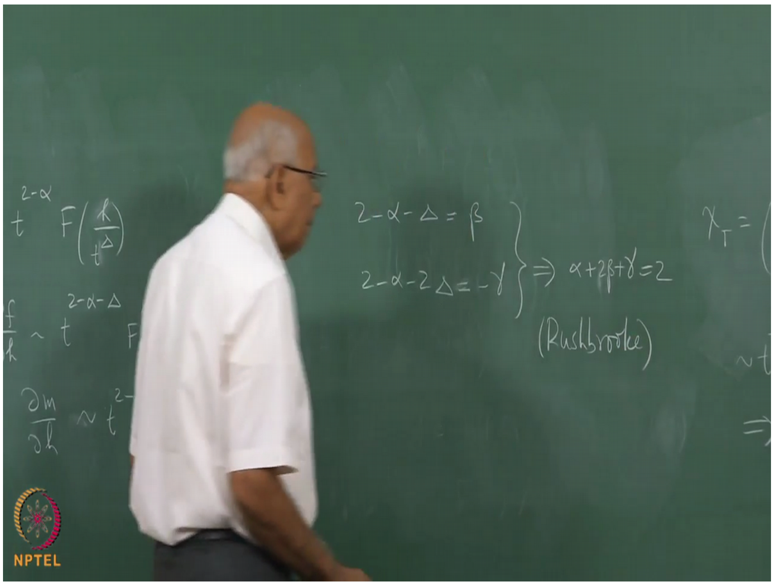

So what does that give us, part of the information we already know that says t minus 2, 2 minus alpha minus delta is equal to beta because that is what the m should do right, and now chi t this thing should be gamma where that is a susceptibility minus gamma, 2 minus alpha minus 2 delta equal to minus gamma or you put beta plus gamma here it does not matter it is exactly the same thing. So these two together will imply alpha plus twice beta plus gamma equal to 2 this is called a Rushbrooke equivalent.

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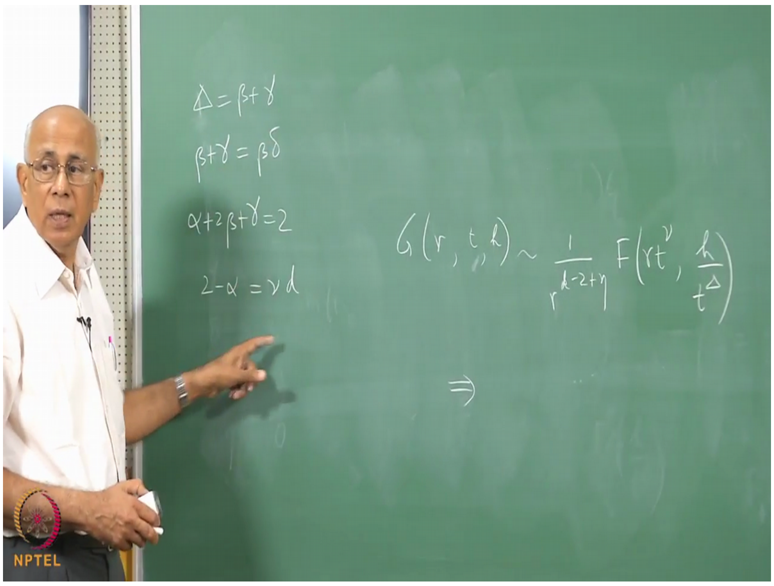

So we will add that to this alpha plus 2 beta plus gamma equal to 2. Then you could ask what about the green function, the correlation function itself because this was really where everything came from turns out that this G that we talked about as a function of r, it is a vector r but we now are looking at the dependence on this variable r, t and h okay, this is the correlation function suitably fine coarse grain and integration etc etc done.

This guy has a scaling form 1 over r to the power d minus 2 plus eta times again this I use the same symbol F for this F it is a function of 3 variables to start with but once you have this generalized scaling it is a function of only 2 combinations, so it turns out to be a function of r t to the nu and our old friend h over t to the delta.

Again empirical elements, so from this exactly as we did before we can extract further relations among the exponents for instance (())(20:50) down you have a result which says 2 minus alpha equal to mew d where d is the dimensionality remember dimensionality is appearing here in this place this is the only relation among all these for which the dimensionality is physically appearing is called hyper scaling but it is a consequence of scaling finally

And also you get gamma equal to mew times 2 minus eta, now mean field exponents will not satisfy this because mean field theory is to only in above the upper critical dimensionality which was Ising classes 4 but if you substitute mean field exponents in 3 dimension they get answers for alpha mew etc which are independent to a dimensionality, if you put 0 here and you put a half here and 3 here clearly it is not satisfied but if you put 0 here, half here and 4 there then it is satisfied because that is the upper critical dimension, all the others will be satisfied.

Wherever dimensionality is involved mean field theory will flunk of course if you are below the upper critical dimensionality you have to go beyond it. So the question is how do you this, how does one go beyond all this? The answer which culminated in the renormalization group is rather long and tortuous but let me show you at least an indication of what the starting point is.

It is as follows, we would first like to include fluctuations in the field but before that we would like to find the systematic way of getting the relations we have such as the fact that for the order parameter m above Tc there is one solution, below Tc there are two stable solution and one unstable solution and so on. We would like to get this from some principle which looks like a variational principle for some minimization of some energy function here, this energy functional is just a generalization of what I wrote down here for the Ising model made a little fancy.

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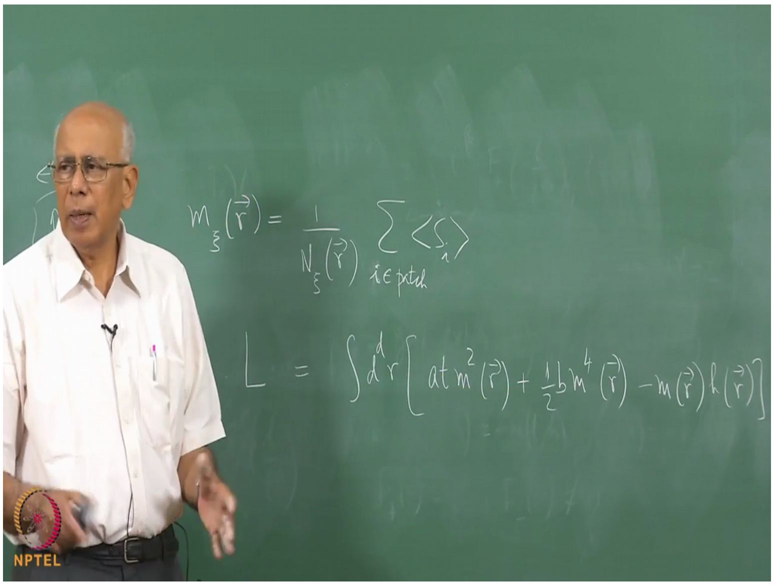

What one does is instead of looking at individual magnetic movements one argues that in the critical region whole patches of linear dimension xi form where the system gets ordered. And in the paramagnetic phase there are as many patches were down as up but as you get keeping as field in the positive direction, as you get near the critical point these follows grow at the expense of other domain and finally the whole system in the thermodynamic implement the correlation length diverges and you have the incipient magnet up, okay.

Now what when one does is to take this m and define a coarse grain magnetization for each patch of this size by writing m the size of the patch is linear dimension xi correlation length of r you define this as 1 over the number of spins N xi at the center at the point r this is the center of it, this is the point r summation i element of the patch or the block Si so that is a coarse grain magnetization you defined, okay.

I am going to drop this subscript here just call it a field and then you construct this land of free energy, I should call it functional of some kind which is geared so as to give me in equilibrium all this solutions that I had if I took the minimum of this function here, so this is equal to an integral d d of r times, you want a quadratic term and you want a quartic term, so conventionally one writes this as a times t m squared of r.

This little t is t minus tc we for reasons which we saw this co-efficient has to change sign from positive to negative as you cross the critical point so you get this splitting of the minimum from a higher order minimum to two minima and a maximum in between, by symmetry there is, we argued already in the absence of a field there is no linear term, there are no odd powers of m by symmetry in the absence of a field plus the next term for stability one writes b m to the 4 of r and if you put in a field you could put in now an inhomogeneous field it does not matter because we still couple the same way, so minus m of r H of r, put a little h, yeah.

In d dimensions and r between number of dimensions this is a positive number a that is a positive number b and its temperature dependency is irrelevant in the critical region, then the statement is the thermo-dynamical equilibrium state is going to be obtain by taking the minimum of this L with respect to m, we will soon convert it to at least briefly convert it to a field theory because what you really want to compute is the partition function which means you must have Z is e to the minus beta times this quantity here this do not act like beta times this acts like some effective Hamiltonian if you like so you do e to the minus beta L and you have to trace but trace over what?

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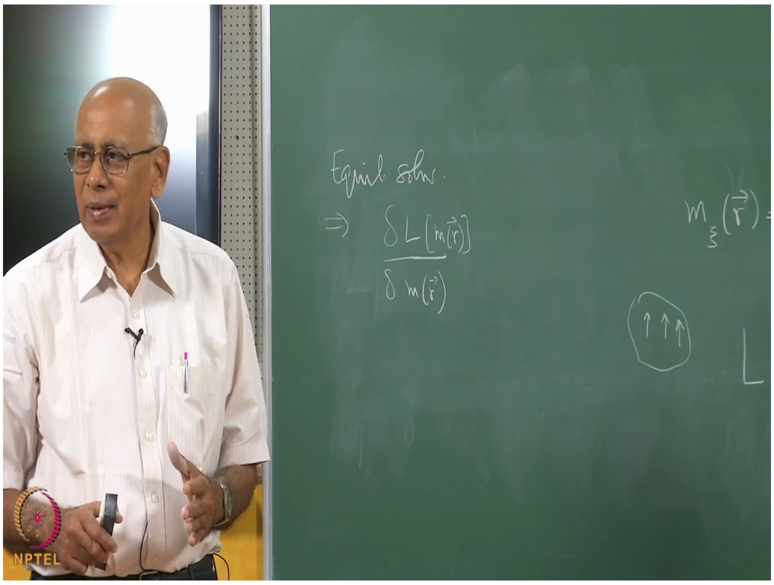

You have a continuous order parameter here so you trace over all configurations of this order parameter, so it will be a path integral or a functional integral over all this guys but before that let us see how to get the equilibrium solutions, so equilibrium solutions would imply delta L over sorry, now I need a functional derivative okay.

This is now L is the functional of m of r and I want this functional derivative okay. A functional is a function of a function (())(28:29) okay and how do you do that well I presume you know the simplest rules for functional differentiation is this familiar or should I mention them can I go ahead and assume them yes? Is there anyone who does not know what a functional derivative is? If so say it now okay.

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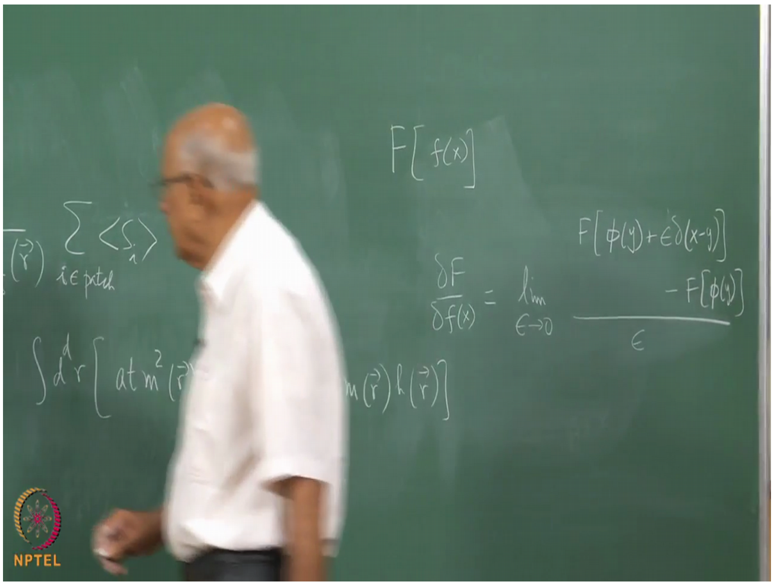

So the properties we need are very very simple, we need to define well let us not make a mystery of it, suppose you have a functional of f of x, function of a function this guy is a functional of m of r, okay then the way you define delta F divided by delta little f of x in this version. If you write this as F of phi of y plus epsilon delta of x minus y minus F phi of y divided by epsilon and take the limit as epsilon tends to 0 and this guy is delta F right.

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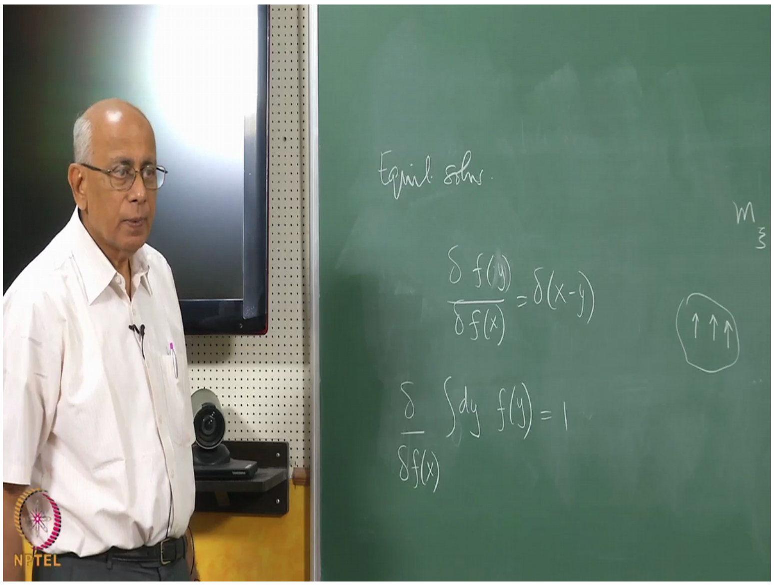

So in particular this rule tells you that, before we find this, this rule tells you that the functional derivative of f of x or f of y with respect to f of x equal to delta of x minus y right, therefore if you have integral dy f of y and you do the derivative of that with respect to f of x here you take this in there, get a delta function and the integral is 1 so this is equal to 1, that is the only rule you need to know.

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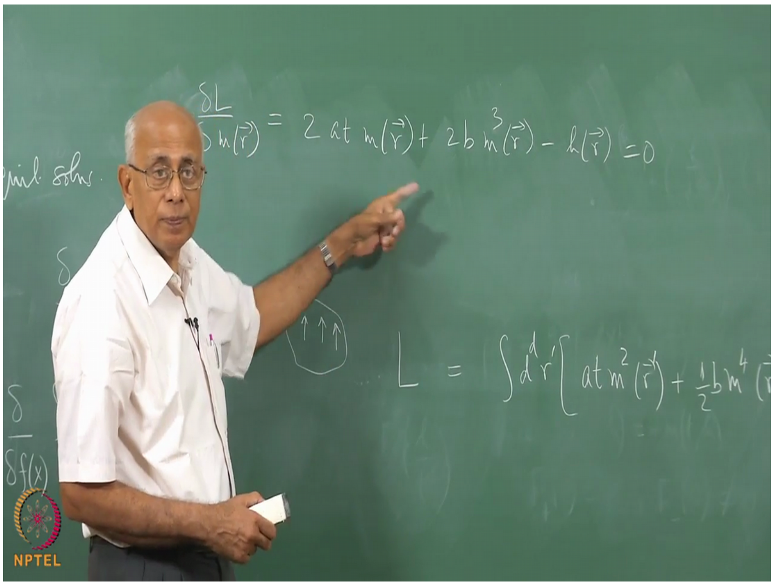

So we can now differentiate this guy okay and what would you get, I am going to save this here, delta L over delta m of r equal to, so let us make all this r primes, r prime, r prime, r prime and start differentiating, so I put it in here, I get twice m of r prime okay and then I need the functional derivative of m of r prime with respect to m of r which is delta of r minus r prime, I do the integral, I get a t times m of r with a factor 2.

So it just like ordinary differentiation, twice a t m of r plus 1 half b m squared of r minus h of r and the equilibrium solution is given by setting this equal to 0 and you have to ensure that this second derivative is also got the right sign so its certain minimum, ah sorry this b m 3 thanks I, yeah 2 on top. So now in the absence of the field out here you want to get your usual properties for a given sign of t, now the point is that this is not enough you neglect, you have not taken into account fluctuations okay, you need to have, two things are going to happen.

If you have an inhomogeneity, you could incidentally get stuck at some local minimum but thermal fluctuations will get you out of there to a global minimum but you need to put in term which will take care of special variations now this is a non-physical model as its stands because it is exactly like saying that I have no connection between neighboring patches at all it is too local.

It certainly costs a lot of energy by way of, when you have a up patch with a down patch at the boundary with the domain wall you have a rapid change of the magnetization up to down so it is going to cost you in gradient energy it is just like taking a string and when you vibrate the string there is a certain cost in energy if the string goes up and down too many times of gradient energy which is the reason why if you have very short wavelength you have a higher frequency those are higher energy modes cost you more energy to excite those modes.

So the same kind of argument but we would have to improve this functional here and how are we going to do that we need to take into account the term which takes not just m at one patch but m at the neighboring patch as well, so some gradient of m is going to appear and what is the simplest form that you can talk about well since it is going to be cannot be linear in m it has to be quadratic at the very least, it is got to be symmetric under m goes to minus m so it is got to be even powers of m and the simplest such term is a gradient squared.

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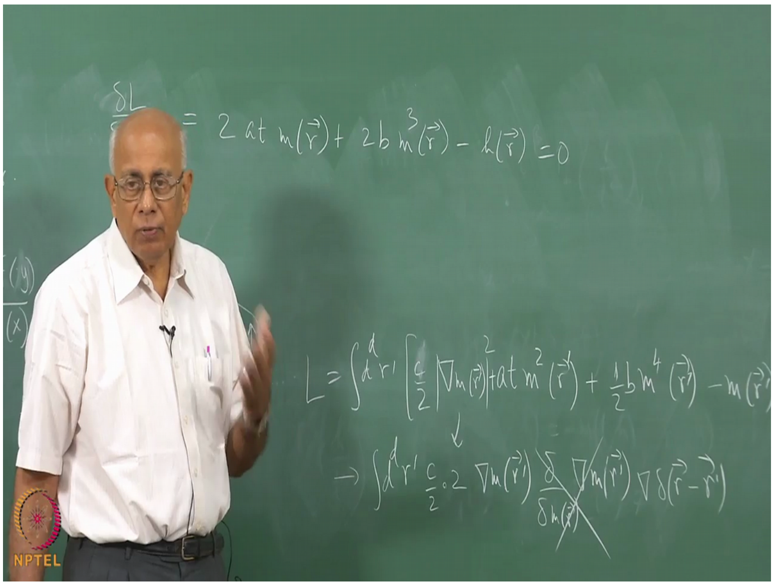

So you add to this a half for our gain for differentiation purposes, some co-efficient gamma gradient of m of r, oh gamma is an exponent right, so let me call it, I do not know what the standard notation for this is, C for now because I am going to get rid of it okay, I am going to scale out with this C in a second this is not the standard way in which this free energy is written, I am going to scale it out. So what is this do to this equation here there you got to be little careful you have to find this is exactly like the energy of a vibrating string except there is a non-linearity here, otherwise everything is linear as you can see.

So what is the functional derivative of this term if I differentiate it, you are going to have twice so it is C over 2 times twice gradient of m itself times the functional derivative of the gradient, so times delta so everything is r prime right, r prime so let us wipe this out, this term here if I take the functional derivative this may be the r c over 2 into d d r prime twice gradient of m of r prime times delta over delta m of r gradient m of r prime.

But these two operators operations commute with each other so I can put this inside there and put a delta function so get rid of this and write gradient of delta of r minus r. But you cannot directly handle the gradient of a delta function so you do an integration by parts and when you do that there is a surface term which vanishes in this case because there is a delta function sitting in this side, the bulk, so with a minus sign this gradient acts on this so it is del dot del on m of r times at delta function.

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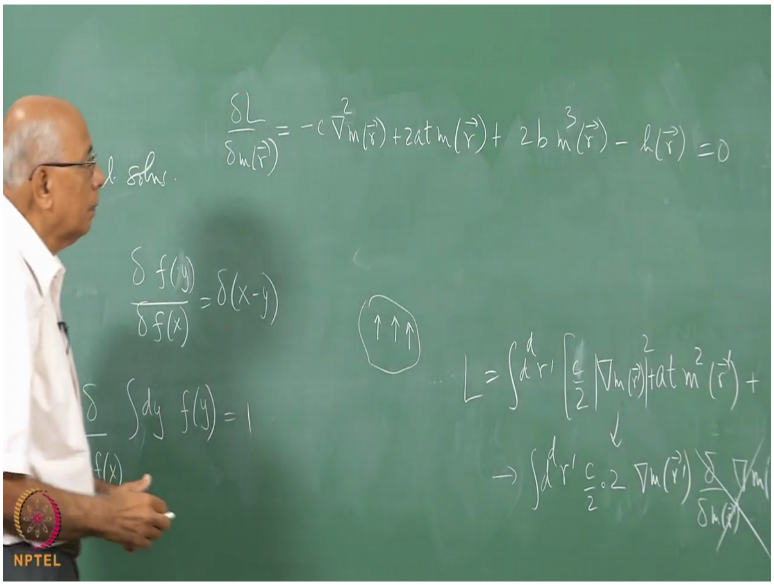

So you have del squared of m of r prime times delta of r minus r prime you integrate you get del squared of m of r itself, so this term becomes, in the absence of a field you get an equation which is got an n cube term otherwise you would normally have del square plus constant times this guy which is like the Helmholtz equation but you know have an extra n cube it is an non-linear equation and you put it in an external field you have further complications, okay.

And the remaining names for this equation but we are still not done because, oh by the way the way it is when you do the renormalization group the scaling that you do you get it of this whole thing is you divide through by c all the way through so you have a t divided by c out here is a co-efficient and you redefine your m by taking you want to make this thing here so you make you want to make this half gradient of whatever this squared.

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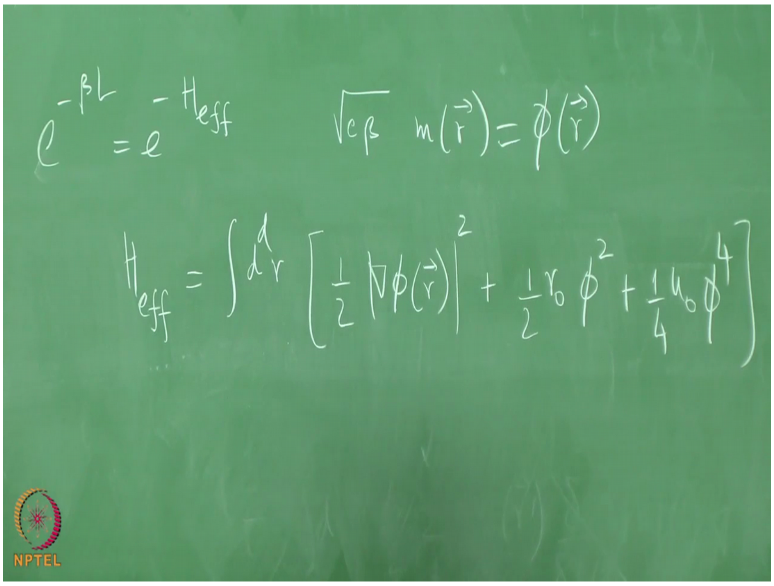

So you set, you want to find e to the minus beta L and write it as equal to e to the minus effective Hamiltonian so you got a beta c so you write square root of c beta times m of r equal to your order parameter phi of r, okay. And then this H effective with identification of constant here let me write it in standard notation, so H effective integral d r of gradient phi of r squared plus the conventional notation as r not phi square plus one fourth U not phi 4 in the absence of a phi, okay.

This is like a phi 4 field theory so it starts at this point and then you can root dimensional analysis, find out what are the dimensions of r not, u not etc etc this guy is like one over length square and so on okay. So there is a piece of algebra I am not going to get into, what I want to talk about instead is how does the system escape from a local minimum and fall into a global minimum that is how what the equilibrium state is okay.

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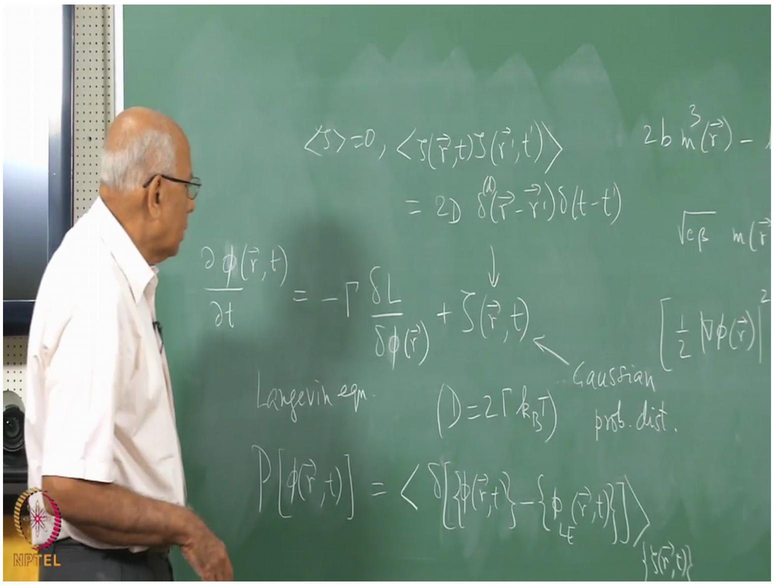

We are just setting this equal to 0 is not going to tell you whether it is a local minimum or a global minimum alright so you argue as follows, you say that there is a time dependence in the problem and now you say you have m of r and t so an instantaneous configuration, field configuration will change with time according to this equal to on the right hand side you say look the further away it is from a minimum the faster it relaxes so this relaxes with some co-efficient let us call it gamma I use up little gamma for an exponent and what is left is delta L over delta m, that is a single relaxation time kind of approximation.

You say there is a relaxation of this if this is 0 then of course this does not change at all but otherwise the deviation of this fellow from the minimum value from 0 as that equilibrium this is 0 will tell you how fast this changes, the m changes okay. However you still have not allot for random fluctuations because thermal fluctuations are random.

So this by itself is like writing a Langevin equation with a friction term but without putting in the random force which you need for consistency so you put in now plus as zeta of r, t the noise okay, the white noise, this is now a Langevin equation, it is called the time dependent Ginzburg Landau equation.

But we know that such an equation we need to make some assumption about this, they are going to give us further information right, so what are the assumptions we are going to make about this noise, it is now a field it is not a configuration but a random field so we are going to assume that average data equal to 0, zeta of r t correlation with r prime, t prime equal to, now some subtlety is creep up here depending on whether the order parameters conserved or not conserved and I am not going into that here now but in this problem this magnetization problem.

This thing is equal to some constant which is 2 D times that delta function of r minus r prime delta of t minus t, so it is really random noise. This is the d dimensional, to go further that is not enough you have to do what you did in the Langevin equation, we have to make some assumption about the nature of the probability distribution of this zeta here.

So you assume that it is Gaussian, that the field configurations are distributed in the Gaussian manner about some equilibrium value some fixed value of the configuration and then you have a Langevin equation but again you discover there is a consistency condition you would have to satisfy between this and this and sure enough you would discover that D equal to 2 gamma k (())(45:49) T this is our old friend the fluctuations equation theory (())(45:55).

But once you make a Gaussian approx for this, Gaussian probability distributions, then they are all set. We could write down a Fokker Plank equation for the probability distribution of the field configurations in sets okay and what would that probability distribution be, well you would have to ask what is the probability of a configuration zeta of r, t.

This is of course equal to, okay what we mean by probability distribution in any case is the expectation value of the delta function of that random variable at any value it assumes so this is equal to the expectation value over what I will mention in a second, of a delta function of zeta of, oh what did I write m the field here we are looking at all this is gone this is phi so we are looking as we scaled out, of phi is delta of phi of r t minus, sorry.

I need better notation, delta function of this configuration minus phi, okay, this is the solution of the Langevin equation for a given zeta, noise and you want this to be equal to that solution so you put a delta function here and this is over that field configuration zeta.

So that is this P and it will satisfy a functional Fokker Plank equation as you can explain, everything that you had is ordinary derivatives will become functional derivative and so on, I am not going to write that down just makes the notation messy but I am just trying to motivate what happens here okay and then you will look at the timescales you look at the critical going down, dynamical indices and so on.

So this is where dynamic critical phenomena start in this point so it is very straight forward extrapolation of whatever we did for a single particle we did the Langevin equation, we had certain basic ideas those ideas are just extrapolated to a field, a degree of freedom being reprised by continuous set field and we do not care how many special dimensions it has or how many components it has.

So the notation gets messier and messier but the basic intrinsic ideas are very straight forward okay so at this level still we write it down as a phenomenal logical thing but now it becomes serious business because renormalization group analysis of this equation to start by identifying physical dimensions doing quote unquote dimensional analysis on this identifying the normalized dimensions and then what happens if we look at renormalization which is a fancy way of saying you try to scale things up, write things in a scaling variant manner and study the flow so to speak of this transformation and look at its fixed points so very in a nutshell that is really what critical point and critical phenomenal analysis is all about.

But the time dependent part is also a straightforward extension of whatever you do most elementary cases. So I said right in the beginning that the Langevin equation would serve as a paradigm as a kind of a model for much more serious problems non-trivial problems and these are examples of such problems here and we did not talk about the Hydrodynamics at all, we did not, we just had a short excursion we took at kinetic theory but we did not get to the hierarchy the BBGKY hierarchy and so on but I wanted to give a flavor for the way this subject is approached at least some aspects of it is approached. So I think I will stop here now.