Nonequilibrium Statistical Mechanics.
Professor V. Balakrishnan.
department of Physics.
Indian Institute of Technology, Madras.
Lecture-25.

The Boltzmann Equation for a Dilute Gas (Part-3).

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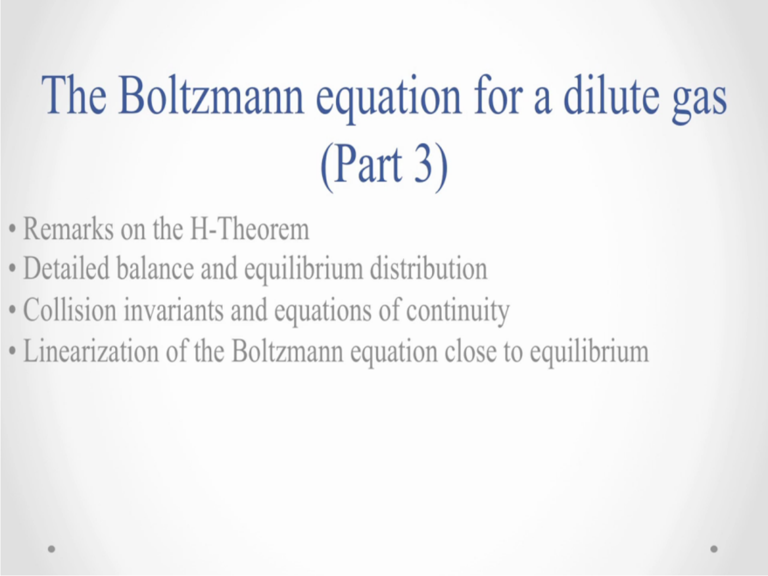
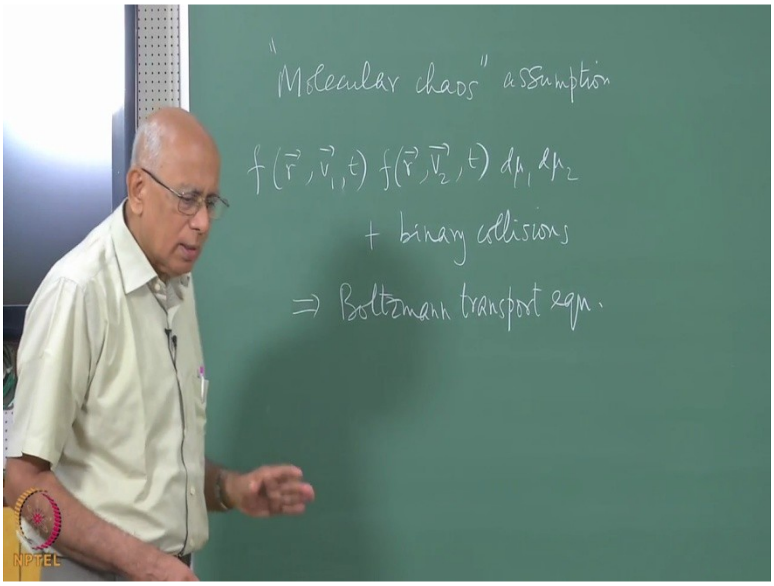


The Boltzmann equation for a dilute gas (Part 3)

- Remarks on the H-Theorem
- Detailed balance and equilibrium distribution
- Collision invariants and equations of continuity
- Linearization of the Boltzmann equation close to equilibrium



Right, so we are going to take a look today at some consequences of the Boltzmann transport equation. If you recall we made one fundamental assumption in writing the Boltzmann equation down and that was the assumption of molecular chaos. In other words, we assumed that in a cell in mu space, in single particle phase space, the probability that you have a particle at position r with a velocity v at time t with some velocity v1, simultaneously with the probability that you have another particle in the same cell with a different velocity v2 was simply the product of the density.

So this was, multiplied by the corresponding volume elements in mu space, d mu 1 and d mu 2. So this was the assumption of molecular chaos, assumption, that together with binary collisions led to the Boltzmann equation, the transport equation. So this + binary collisions, elastic collisions in which you did not lose a particle, nothing got absorbed, this led to the Boltzmann equation, Boltzmann transport equation. In kinetic theory it is usual to call the Boltzmann equation the Boltzmann transport equation because as we will see very briefly today, you can extract transport coefficients from the Boltzmann equation, such as the diffusion coefficient, the coefficient of thermal conductivity, electrical conductivity and so on, viscosity, etc.

So they can also computed, for whatever system you looking at in this case it dilute gas, from the Boltzmann equation with a very systematic procedure. the catch was that the Boltzmann equation was a non-linear integral differential equation in this function F. So that has what made it practically impossible to solve analytical and then there is a huge literature which over the last 100+ years has been devoted to trying to develop systematic approximation procedures for going further and further with towards the exact solution of the Boltzmann equation. Now we are going to look at the very very very oversimplified version of it today, very briefly.

But I would like to go over again since it is so important to the foundations of statistical physics, I would like to go over again some of the key steps that we went through in deriving the Boltzmann equation. I am not going to give the derivation again, I am just going to call the attention to what are the features of this equation. So if you recall the equation itself said that delta F over delta t for a particle with velocity v1, say some arbitrary velocity v1, so by F I mean F of r, v1, t + v1 dot dell in position space dell r times this F + if you had an external force of some kind, then that force F over m dot gradient in velocity space on f.

this is a differential operator acting on the v dependence of F. this was equal to the famous collision integral which is delta F over delta t, I use delta F over delta t just to define this collision integral here through this equation here. this is what the dimensions of F divided by time which is why I correct this, but very often in text books you will see it is called the collision integral, and that is it, and a separate symbol is used for it altogether.

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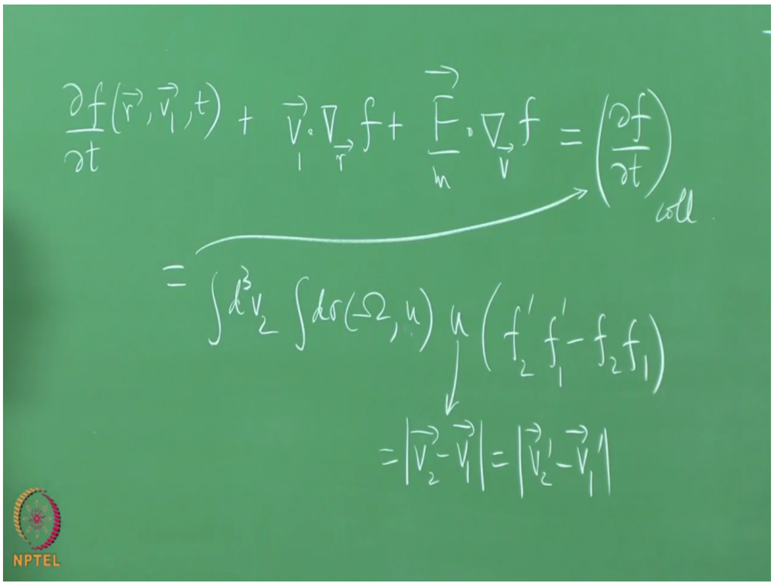
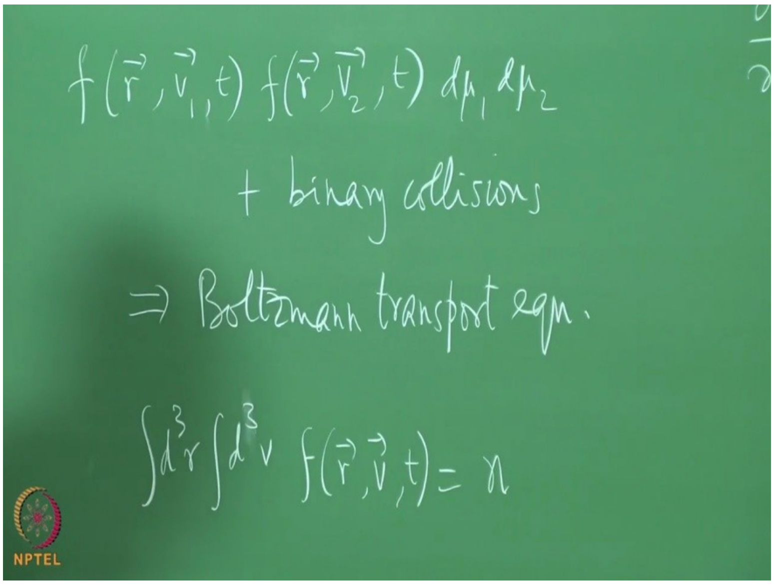


$$\frac{\partial f(\vec{r},\vec{v}_1,t)}{\partial t} + \vec{v}_1\cdot\nabla_{\vec{r}} f + \frac{\vec{F}}{m}\cdot\nabla_{\vec{v}} f = \left(\frac{\partial f}{\partial t}\right)_{coll}$$

$$= \int d^3v_2 \int d\sigma(\Omega,u)\, u \left(f_2' f_1' - f_2 f_1\right)$$

$$u \downarrow = |\vec{v}_2 - \vec{v}_1| = |\vec{v}_2' - \vec{v}_1'|$$



$$f(\vec{r},\vec{v}_1,t)\, f(\vec{r},\vec{v}_2,t)\, dp_1\, dp_2$$

+ binary collisions

⇒ Boltzmann transport eqn.

$$\int d^3r \int d^3v\, f(\vec{r},\vec{v},t) = N$$

And this collision integral under the assumption of binary collisions, that was equal to this equation, in here was equal to an integral d3 v2 for our scattering geometry, integral d Sigma v dependent on omega and now let me explicitly write its dependence also on the relative velocity, the magnitude of the relative velocity U times U times F, we have a strange notation F2 prime, F1 prime - F2 F1, the standard notation. Where the scattering geometry, just to recall once again was v1 and v2 coming in, the initial states and then going out with velocities v1 prime and v2 prime. And for instance F2 prime was identically F of r, v2 prime, t, etc. So the 4 velocities involved in the collision, v1, v2, v3 and oh yes, and of course this quantity U was mod v2 - v1 which is also equal to mod v2 prime - v1 prime.

this scattering cross-section is dependent on where you are looking with respect to the incident direction v1 and this quantity here U which is the relative velocity. Okay. So this equation as we can see here is a non-linear differential integral differential equation for this unknown function F, that is what creates a problem. Of course we know that certain quantities are conserved in this collision, we know that the total momentum of the system is conserved, we know that the total energy, kinetic energy is conserved and those will have to be put in. We also know that the total particle number is conserved, so no particles are going out anywhere in this gas, they are all in there.

So at any given time if you integrate this F, the normalisation was integral d3 r, integral d3 v which is what I call it d mu in mu space, this multiplied by F of r, v, t at any given time, this is equal to the number density. the number of particles per unit volume. Now this F has 2 interchangeable meanings under the assumptions we have made, large numbers and so on. the F multiplied by the volume element in free space is either the total number of particles in that per-unit volume in that space or it is also the probability that the given particle has the position r in the cell whatever is concerned, centred at the point r with a velocity v at time t, these 2 are interchangeably used.

Now this equation is satisfied by those Fs for which the assumption, for those states of the gas for which the assumption of molecular chaos is valid. It is not saying that this assumption is valid at all times, it is not saying that at all. It is just saying if the gas instantaneously happens to be a state, in a state in which this other option is valid, then the gas obeyed the Boltzmann equation, okay. And we found out that Boltzmann introduced and we proved this necessary and sufficient condition for equilibrium by various ingenious argument, we introduced the H function. And we discovered that dH over dt was not positive, it was either equal to 0 or less than 0 at any given time.

What it actually means is that if at some instant of time, the gas obeys the Boltzmann equation and obeys therefore the assumption of molecular chaos, then an infinitesimal instant later, so if at time t it obeys the equation, the molecular chaos, then at t + Epsilon, whether from goes to 0 from the positive side, you are guaranteed dH over dt is negative, the instantaneous slope will be negative at that point. that is what you are guaranteed, okay. you are not guaranteed that the gas by any means satisfies the assumption of molecular chaos at all times, the fluctuations of force. Now the h function also tells you something about those fluctuations, so we will discuss that.

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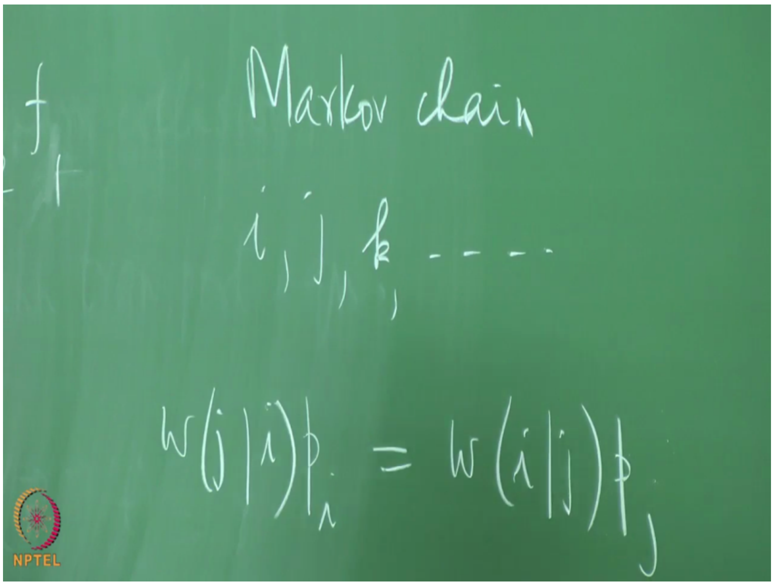

But before that let me remind you of what equilibrium meant, what was the meaning of equilibrium. We discovered that equilibrium distribution of course means that this quantity must be 0, there is no explicit time dependence. So that was achieved by saying that if F of r, v2 prime, so let me write this out. If this quantity v2 and it is in equilibrium distribution, so the time derivative is 0, that means this whole thing on the right-hand side, if it is independent of r or even if it is appended on it, even if there is an external force we do not care, we put a specific condition on when the gas is in equilibrium and the condition was that in the state of thermal equilibrium, the distribution was such that this bracket vanished identically.

So essentially we had the statement that F2 prime, F1 prime was equal to F1 F2, F2 F1, does not matter what order you write it. Where 1, 2, 1 prime, 2 prime are related by this scattering geometry in this fashion, okay. Now what does it remind you of? this thing if you like for the 2 particle system is like an initial state, just like the final state. So it is really saying that if you give me a v1, then all those v2s, such that this condition is satisfied on this side, then a Priory probability of v, having v1 is represented by F1, this is what causes the proposition put the final state because it comes and scatters. On the right-hand side this would have been a Priory probability and that is the scattering what causes the scattering.

So it is very reminiscent of what is happening in the theory of markov processes where you have the assumption of detailed balance in a markov chain. So if you recall when you write a massive equation for markov chain, if you have states which are labelled by i, j, k, etc., etc. in a markov chain and you say that piece of i is the a Priory probability, that the system is in the state i, then Pi multiplied by the transition probability for i to go to some j and that is equal to w, j, i, if this is equal to, on this side w, i j, p j, when you have the condition of detailed balance, when you have an equilibrium dissipation.

Of course this is a sufficient condition for you to have an equilibrium distribution because the markov master equation says dP over dt, if it write all this is a column vector, it is given the right-hand side by some W, some matrix, transition matrix time P and the elements of it are given by these transition rates. So since this appears inside the summation, if each term in the sum is 0, then of course the whole thing is, the sum is 0 and you have a stationary distribution. Okay. So this assumption is called detailed balance because each pair of states satisfies this condition here and there is no reason why that should be the case for equilibrium.

But if it is so, then the state, then you have an equilibrium distribution. So that is what is happening here, this is like a detailed balance condition, this condition. And now in this situation, in this physical system this dilutes gas, what the Boltzmann's equation, what Boltzmann's proof of necessary and sufficient condition for equilibrium being this says is that you automatically have detailed balance. that as t tends to infinity, any distribution you start with will tend to unique equilibrium distribution which satisfied detail, this detail balance condition.

No markov chain?

there is no, there is no markov chain here, there is no such thing But it is detailed balance all the same. So it says, so what it is trying, telling us of course that why this has come about, where this factor comes from, they came from this scattering process. So as long as the scattering is time reversal invariant, as long as the dynamics is time reversal invariant, you will not have this situation. So that is really what it is telling us, saying something about time reversal that it is a time reversal invariant system for these binary and of course the binary Collision assumption.

So if you recall how did Boltzmann do to prove a necessary and sufficient condition, well we went through this proof, we said 1st of all if this distribution in equilibrium is independent if there is no external force, this part is 0, it is uniform distribution, this part is 0 and then the fact that you have an equilibrium distribution means that this must vanish and that vanishes implies that this integral must vanish here. And a sufficient condition for the integral to vanish is for the integrand to vanish, that was the 1st part of the proof. the 2nd part of the proof show that it was also necessary by saying that what appears in the H function was the Boltzmann H function.

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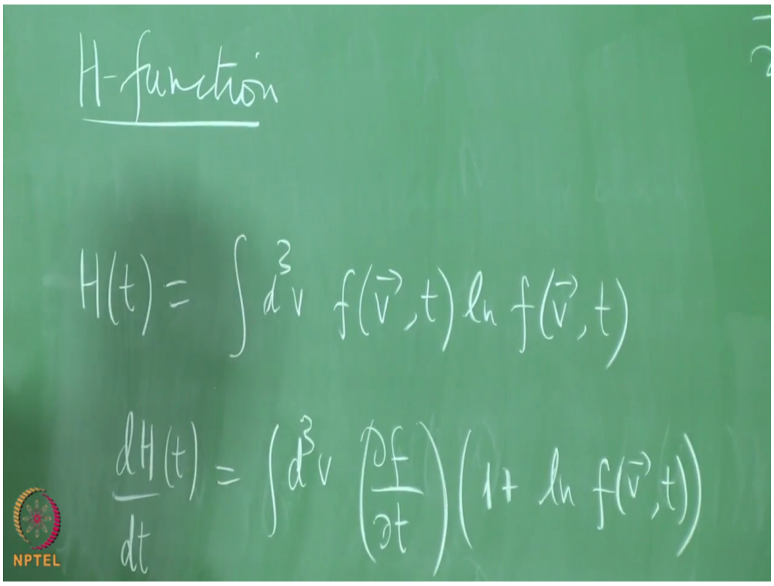

H-function

$$H(t) = \int d^3v \, f(\vec{v},t) \ln f(\vec{v},t)$$

$$\frac{dH(t)}{dt} = \int d^3v \left(\frac{\partial f}{\partial t}\right)\left(1 + \ln f(\vec{v},t)\right)$$

By the way the H is not the Hamiltonian, it is closely related to entropy, Boltzmann actually used the E and later on somebody else changed into H, so it should not be confused with the Hamiltonian. this function was for any distribution whatsoever, does not have to be distribution which satisfies the Boltzmann equation at all. you define an H of t to be equal to integral d3 v, F of v, t log F of v, t, that is the definition of H function. Pardon me?

(())(16:28) he did not put the - sign…

He did not put the - sign, now we have to see where the - sign comes in when we write the entropy of an ideal gas or entropy of a gas in equilibrium. So this immediately led to the conclusion that dH over trivially that this is equal to integral d3 v, delta F over delta t times 1+ log F of v, t. And now the statement was that if this vanishes, if you have equilibrium distribution, there is no time dependence then this vanishes. And the vanishing of this is a necessary condition because if this was not 0, there is no way this can be 0, it has to be finite clearly.

So necessary condition for an equilibrium distribution is that this should vanish. And we now put in for delta F over delta t, if you put in this whole thing here, then using the inequality about y - x, log x over y is always negative, non-positive, we discovered that you had to have this condition, the detailed balance condition for equilibrium, so it was necessary and sufficient in that case. So that much was the argument, okay. So the part that is ingenious was the fact that Boltzmann used the invariances of this object. that is to interchange v1 and v2 after you integrate over v1, if you interchange v1 and v2, nothing happens.

If you interchange, this does not change at all. If you interchange the initial and final states, nothing happens, except the change of sign here. So that gave us what was needed to show that the detail balance condition uniquely specifies the equilibrium distribution. Now given that much, we could ask what is this H function actually looks like. Is it always, if I plot this H function, what happens to it? After all the system, even if it is momentarily in equilibrium, in other words even if the distribution is, let me call this F equilibrium of v, the equilibrium time independent distribution which satisfies the detail balance condition.

If you say that this system is momentarily in this state, it does not remain so because collisions knock it out of that state instantaneously practically. So the system is always getting knocked out of equilibrium but it is being restored, the equilibrium is in some sense overall there is an equally be, right. So this is how statistical mechanics, the same molecular fluctuation or collisions that knock you out momentarily out of the state of equilibrium also have to restore this equilibrium, there are connection between fluctuation and dissipation which help you to restore this, okay.

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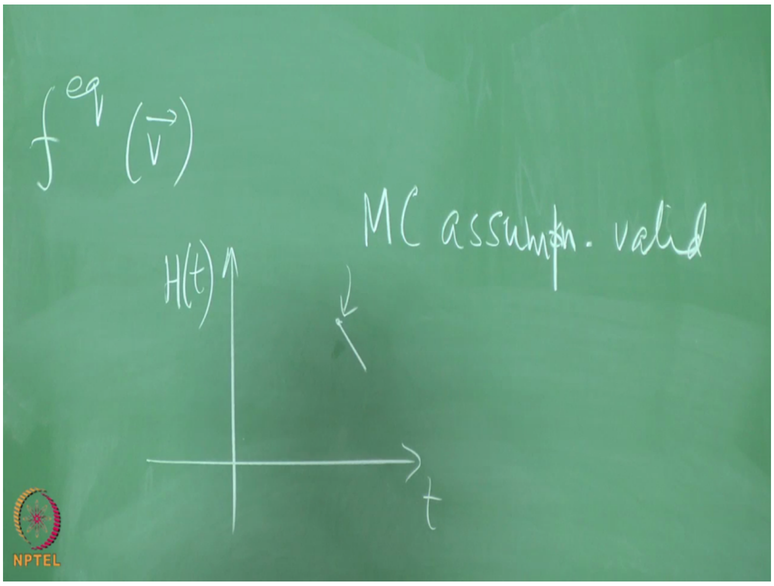

So there is a interesting proof which tells you that is momentarily this system is, if you plot this H of t versus t and momentarily insert a value here, then if the system obeys the assumption of molecular chaos, if F of v, t at this instant of time satisfies the assumption of molecular chaos. then you are guaranteed that the H function will decrease, the local slope will be negative, that is all you are guaranteed, okay. But there is an interesting argument due to (())(20:11) which also says that when the system is in, when this state, molecular chaos assumption is valid in this state, momentarily it comes down, right.

But the argument is, if you now reverse all the velocities in this state at this point, you are still in the state of molecular chaos, nothing changes. that has something to do with correlation between pairs of particles and that is independent of where the particles have positive velocities or negative or whatever, have velocities completely reversed or not, it does not matter. But then it also implies for that gas too with all the velocities reversed, you still have this negative slope. But the future of that reversed gas is the past of the present gas. So that in a few lines you can show that at this point is in the next instance you are going to have a molecular chaos, the slope would have to be positive.

there is a delicate argument here, it involves actually implementing the time reversal operation but the loose words what it means is as exactly what I said, that if you have in the state of molecular chaos at any instant of time, then at that instant of time the H function is at a local peak. But the function itself is fluctuating, so it is going ZigZag, ZigZag in this fashion. What you are guaranteed and there is nothing which says that this must, function must have a continuous slope, does not say that at all. So the slope can and it does jump discontinuously over the function is continuous.

All it says is these points at local peaks you have molecular chaos. It does not even say that all local peaks must be states of molecular chaos, it only says if you have a state of molecular chaos you have to be at a local peak, that is all it says, okay. But there are fluctuations about it and what statistical mechanics that is precisely to calculate what these fluctuations do at this point. Now the other party suppose you are in a rare state, you prepare the system in a rare state, it will relax back to the state and what it will do is.

If you are in a state like this which is far away and H is itself actually fluctuating like this but then it finds itself there, then it will come down, it will relax back and go back but it will do so in this fashion and it will fluctuate back into this little range in which the system approximately got the equilibrium distribution, differs from it infinitesimally, but there are always local fluctuations. So it is not that the system is always in a state of thermal equilibrium, even if you are in thermal equilibrium on the average. So the H function, the Boltzmann equation is that solid curve, is the continuous curve there, it is the smooth curve here.

So it is giving you something which is an average already, okay. And there are fluctuations about it and you need these fluctuations. For instance you know that the sky is blue because of molecular scattering. It is happening because of the local density fluctuations. If you did not have those density fluctuations you will not have the sky coloured blue at all. So the scattering is especially during and this becomes very very, these fluctuations become very pronounced near critical point which is what we are going to study next. that is where you get phenomena like critical scattering, critical (())(24:07). So the scattering can become enormous under those situations.

But even in the ordinary atmosphere, the fact that the sky is blue is due to the scattering, scattering of light due to density fluctuations. So they are always happening all around you and you can measure them, there are measurable consequences, etc. But the fact is that equilibrium is a statistical state, it is not as if the system is always in this state. Now the H theorem says something more also. When asked what is the connection between that and the thermal equilibrium, entropy in the thermal equilibrium state. Well, when F is equal to F equilibrium of v, in that state, then this H is equal to - the entropy divided by the units we have used B times K Boltzmann.

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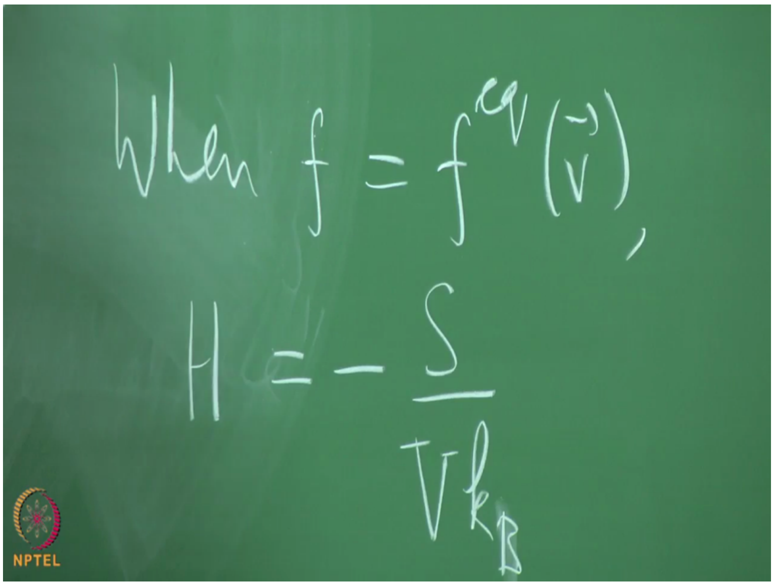

When $f = f^{eq}(\vec{v})$,

$$H = -\frac{S}{V k_B}$$

We define this F, F Log F without putting in the Boltzmann constant there but entropy is measured in units of Boltzmann constant. So this is the relation in the v because of the way we define this. Our normalisation is integral d3 vF is equal to N, the number density, number per unit volume. So this is what the connection is here. In the - sign as Suresh pointed out was because Boltzmann originally defined this H function with F log. So I hope this kind of tells you what is the role of this H function is, because now the going to go on and see what we can do with this Boltzmann equation. Whether we can try to solve it or whether we can try to extract something from it, etc.

1st let us look at how constants come out, how various conservation laws come out. We know that in mechanics, in dynamical systems, valid not just in particle mechanics but also in theory of fields as well as quantum mechanics and quantum fields etc. When you have continuous symmetry, you have associated with them certain invariance of the equation of motion and then you have conservation principles, which are like equations of continuity and when you integrate, then you get conservation laws. Okay. Now where does that kind of appear here?

Look at the following, keep that equation because we need it. Go back to the way we derived the sufficient and necessary condition from the Boltzmann equation further equilibrium distribution. the manoeuvring was that you integrate over v1 and then you interchange v1 and v2, Interchange initial state with a final state and interchange v 2 prime with v1 prime. And you get several averages and we argued that all these fellows are the same quantity and we took one 4th of the whole lot. Similar kind of thing goes on here because now let us suppose that you have the Boltzmann equation.

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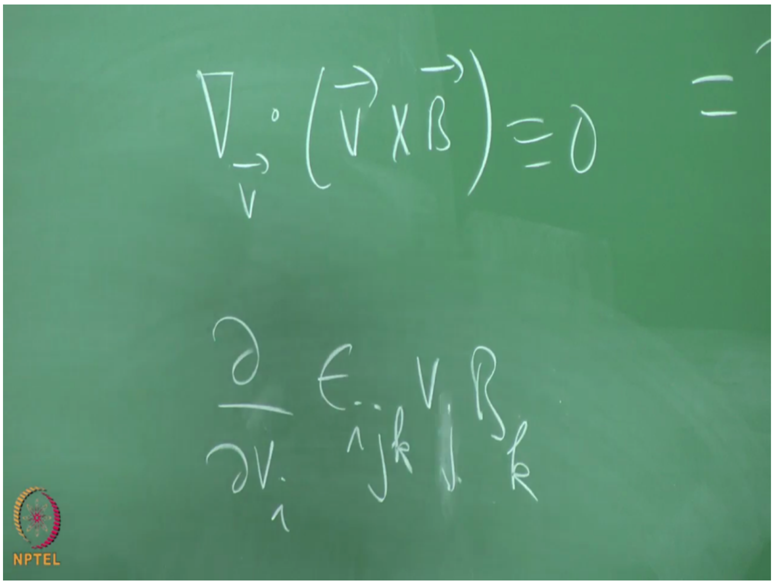

By the way one small side remark, this quantity here, when we have an external potential, when this F was - the gradient of some potential, I do not know what symbol used here, Phi, Phi of r when you have considered the force, then this was the term that appears in the Boltzmann equation, in the in the derivative term on the left-hand side, okay. Now you could ask what happens if I have a magnetic field. then you have a velocity dependent potential and can we now do this, can we extend this formalism to it. the answer is yes for the simple reason that if you compute dell v dot v cross B, this quantity is what appears ultimately is 0. Okay. It is identically 0.

If you differentiate it with respect to the components of v, this quantity here, it is immediately obvious that this quantity is 0 because we write this as dell i with respect to v. So dell over delta vi, Epsilon i jK, v, j, v, j, vK and of course the derivative of this fellow with respect to delta vi is chronic at delta i, j and that contracts with this and gives you 0. So this formalism trivially extends to the case of a magnetic field, even though there is a velocity dependent force, it does not matter, we still write this equation. Now what would like to do is to derive…

Half mv square (())(29:01).

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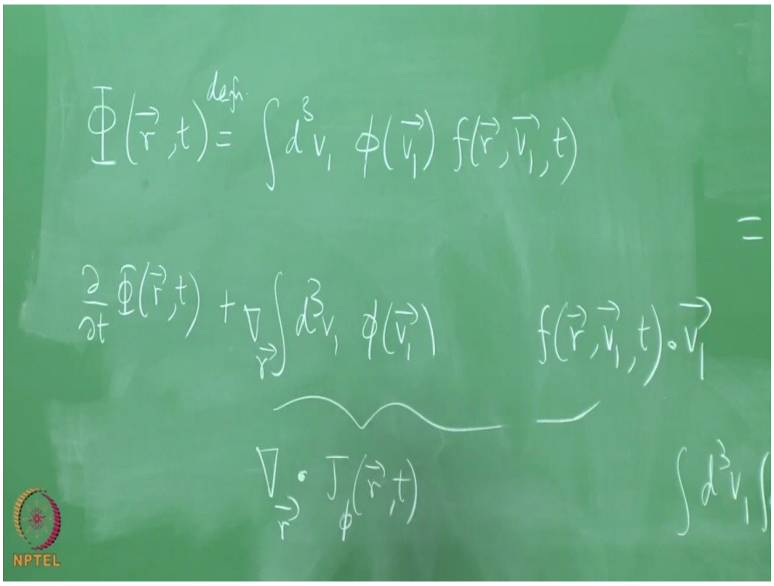

$$\Phi(\vec{r},t) \overset{\text{defn}}{=} \int d^3v_1\, \phi(\vec{v}_1)\, f(\vec{r},\vec{v}_1,t) \qquad =$$

$$\frac{\partial}{\partial t}\Phi(\vec{r},t) + \underbrace{\nabla_{\vec{r}}\int d^3v_1\, \phi(\vec{v}_1)\; f(\vec{r},\vec{v}_1,t)\cdot\vec{v}_1}_{\nabla_{\vec{r}}\cdot J_{\phi}(\vec{r},t)} \qquad \int d^3v_1 \int$$

yah, exactly, so what we are going to do is to write, there is no energy that is gained by, there is no work done by the magnetic field on a charge particle. Now what we are going to do is to see where the conservation Law comes from. So let us look at the following. Let us write, I need a symbol for this whole mess, what should I call it, let me call capital Phi of r and t equal to an integral over d3 and because I have always got v1 sitting here, this is v1, v1, etc., this is v1, okay. So let me call this v1, some function little Phi of v1 times F of r, v1, t, that we define. It is like the moment of this distribution with respect to the v variable. If I put Phi equal to1 or v or v square or v cube etc., I will get moments of this distribution, but it could be an arbitrary function of v in this fashion.

If I plug that in here, multiply both sides by little Phi of v1 and integrate over all values of v1, okay, what happens. So let us do d3 v1 Phi of v1, F of r, v1, t, this guy here. And the 1st, well we can write down the 1st term, it felt over delta t Phi of r, t, that is this term here, okay, + what should I write, think of what is notationally the simplest way of doing this without messing around. just write it down and see where it gets us.

We can pull the gradient outside.

I can certainly pull this fellow out, that is true, I want to be a little cautious. I want to be a little cautious, bear with me for a minute, I want to be a little cautious, I do not want to… So Phi v1 out here, and then F over m, assuming this to be a function of r alone, dot gradient v1 f. Is this fashion. Equal to, well, this part is okay but equal to on this side we have d3 v1, d3 v2 d Sigma, oh, let me just forget about this for a moment, d Sigma and then Phi of v1, U and all the rest of it on this site. Now let us come to terms with what is going on here. In the absence of an external force, this is 0, no external force, implies this is 0, there is no F, okay.

So you have this + this term here, what is this term looks like? It says Phi of v1 sitting here, and then there is v1, I want to simplify this term, so I want to be able to write, well, this term here is gradient r dot, well I am messing around with the notation, I want to write this as a current. Integral d3 v1 Phi of v1, v1 times f, this quantity here is a vector quantity. And v1 goes away, agree? So let me call this equal to j some current density related to this function, so let me put it as j Phi and it is a function of r and t. Okay. And am I right in calling this tilde dot j? yah.

I mean I can write this as grad, grad F dot v1, right and bring the gradient all the way out of the integral.

Is the point (())(31:17)

I put this as v1 and then take this grad outside because there is nothing to do with the integral and this quantity I call j whatever it is, okay. All right.

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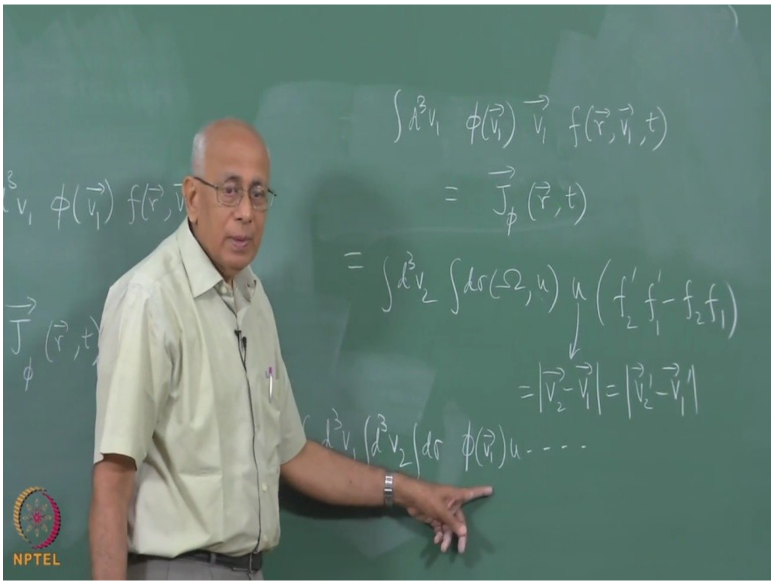

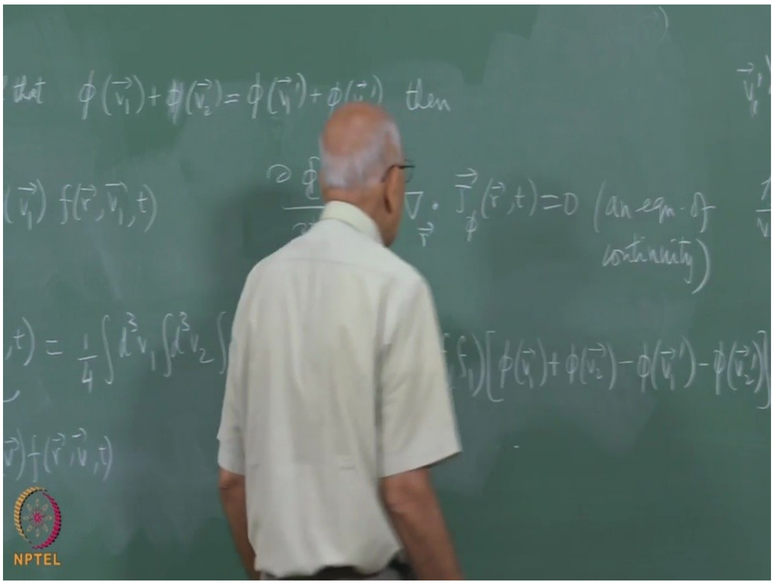

So I am getting an equation which says this (())(35:43), grad r dot j which is related to the function Phi r, t equal to something on the right-hand side, okay. Now look at what happens on the right-hand side. there is a Phi of it v1 here but I know that if I interchange v1 and v2, the rest of it does not change, I can interchange variables of integration, this becomes Phi of v2. But U does not change because it is mod v2 - v1. So let us keep that in our mind. this quantity on the right-hand side is the same, if you put Phi of v2 here. So I can add that to take the average, okay. then I will interchange the initial and final conditions, this fellow is going to become v1 prime and this is d3 v1 prime, v2 prime but in the process I can change variables of integration because d3 v1, d3 v2 is equal to d3 v1 prime d3 v2 prime, exactly as we did for the Boltzmann, for the H here.

But this becomes Phi of P1 prime and this comes here becomes F2 F1 - F2 prime x1 prime, so there is a - sign. And then I interchange v2 prime and v1 prime, so once again I get Phi of v2 prime out here with a - sign. So on the right hand side this quantity, now we are in a good shape, this quantity is equal to an integral one fourth because I took 4 of these fellows here, one fourth d3 v1 integral d3 v2 times integral d Sigma, d Sigma which is the function of this where our your r, and U times U, that is also sitting there times F2 prime F1 prime - F2 F1 times the following, Phi of v1 + Phi of v2 - pie of v1 prime - Phi of v2 prime. Okay.

Because all this remained invariant, including this and this fellow change sign when I interchange initial and final state, I have taken that into account here. And I have added all the 4 ways of writing the right-hand side and divided by 4. So this is an exact equation, okay, where this current was defined as d3 v or v1, we do not care, Phi of v, okay, times F of r, v, t. So this is identically is equal to that. this looks like the conservation Law if the right-hand side was 0. So now the argument is, if this function Phi of v is a function such that this quantity is identically 0 in the scattering process, then you have the conservation Law automatically.

So now you turn it around and say if Phi of v such that Phi of v1 + Phi of v2 is equal to Phi of v1 prime + Phi of v2 prime, then delta capital Phi of r, t over delta t + there divergence with respect to r, the usual divergence of j Phi of r, t equal to 0. that is an equation of continuity. Okay. Now we list all the quantities that are invariant in the scattering, okay. What is the simplest quantity that you can think of that is unchanged scattering?

(())(41:04).

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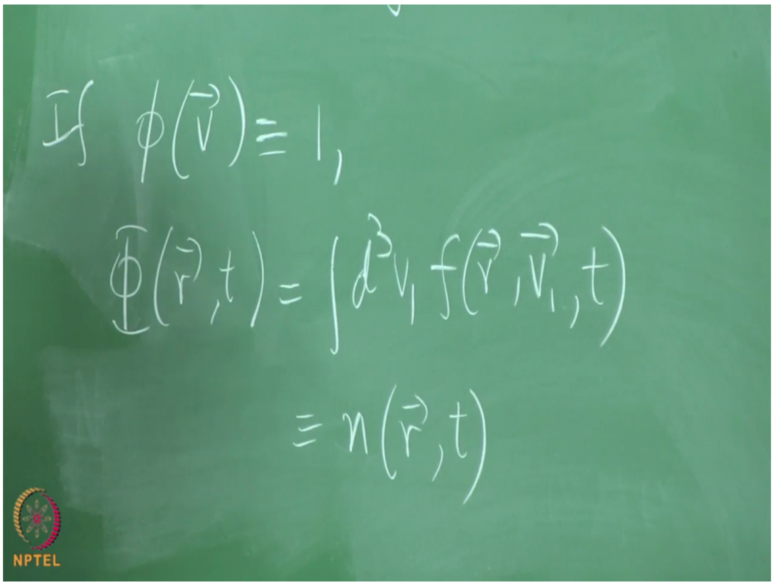

If $\phi(\vec{v}) = 1,$

$$\Phi(\vec{r},t) = \int d^3v_1 \, f(\vec{r},\vec{v}_1,t)$$

$$= n(\vec{r},t)$$

Phi equal to constant, if I put Phi equal to1, right, that is a constant, that is one +1 equal to1 +1, clear, right. So that is certainly a constant of the motion, trivial constant of the motion. What is it going to conserve, well if Phi is equal to 1, Phi of, if Phi of v is equal to 1, then the corresponding Phi of r, t equal to integral d3 v1 times F of r, v1, t and that is equal to the number density at the point r at time t. Okay. So our 1st conservation principle is the conservation of number of mass, if you like of matter. right. this is immediately going to tell us that delta N over delta t is a function of r and t. If there is a local density fluctuation, then then there has got to be flux of particles (())(42:34).

this is equal to + delta dot j of r, t equal to 0 where this j of r, t is equal to and now all we have got to do is to plug-in the formula for what this j was. If you recall this quantity was equal to integral d3 v, right, times what, what did we have? We had a v, definitely we had a v, and then what else? And then an f, we have…?

(())(43:24).

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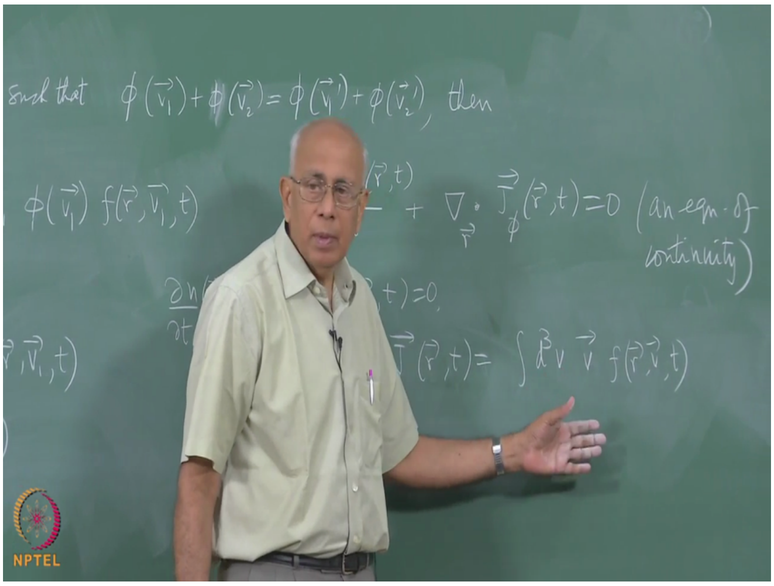

Phi is one in this case, Phi is one. So we have Phi of v and then f, right. And there was just a v and that is it. What is this representative of? We have been careless a little bit about putting in the dimensional quantities like mass and so on, we should be little careful. But it is okay as it stands. What is it, what does that represent? this is the momentum current, for a factor of m, this is the momentum current. So this is the usual continuity equation for matter. In the usual hydrodynamic language, the current is rho times v, the density times the velocity, okay, that is exactly what you have here. density, so there is a question of quantity for a number density, okay.

What happens if you put Phi equal to the energy? Incidentally there is one more thing that is conserved. I never said that Phi has to be a scalar, need not have to be a scalar, I have got to be a little more careful about how I manipulate these vector symbols but Phi could have been a set of 3 quantities, vector, components of a vector. v itself because I know that in the collision process, I know that when this process, if since it is equal masses, v1 + v2 equal to v1 prime + v2 prime, component for component, right. So certainly that is going to be again a conservation principle here, right.

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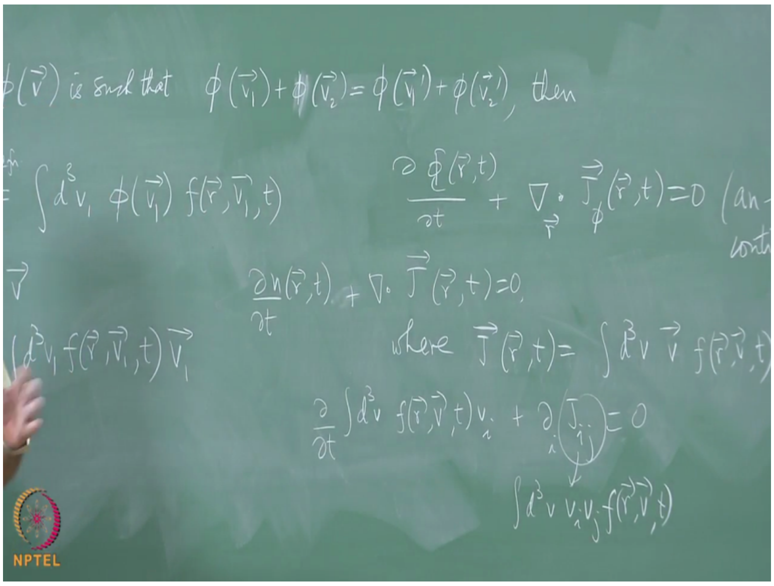

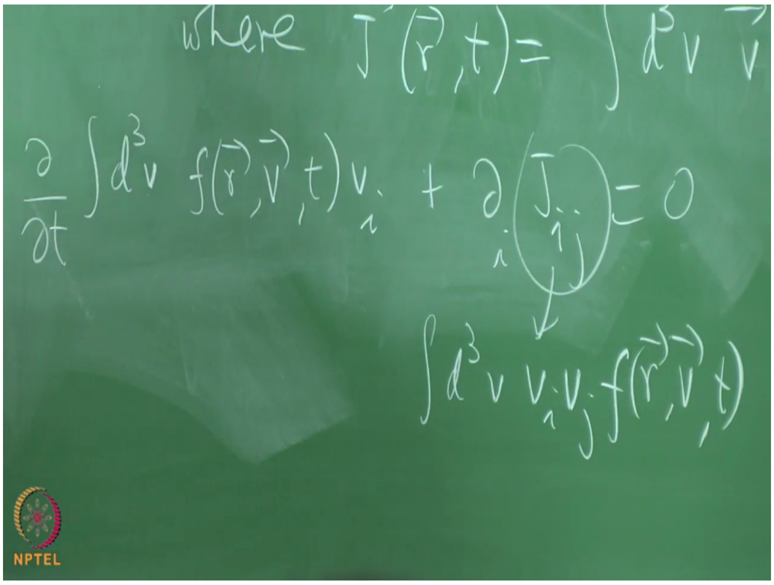

And what are we going to, so we put Phi of v equal to v itself. And in deference to notation, let us put a vector there, then this fellow is a vector out here and it is equal to v. So this is the momentum if you like and each component of it satisfies the continuity equation. On the right-hand side this j would now become a 2 index object, because for each component, each of the components of Phi, you are going to get, since you are going to integrate with Phi here, there is going to be vI, vj on the right-hand side. So you are roughly going to get something like delta over delta t, now fix these factors, I have not looked at it carefully enough.

So d3 v, F of r, v, t vi, any index i is going to +, this is going to look like some dell i, some current j i, j equal to 0, where this thing is going to involve an integral d3 v vi vj, F of r, v, t, apart from some factors. this is the kinetic part of the so-called stress tensor, okay. What happens if I put Phi of v equal to half m v square, that is the energy, right. Then there is going to be an energy flux density on this side. So I leave you to write down that equation. So there are basically 5 conservation principle rules that we can write down immediately. Phi of v equal to1 for the number, then Phi of v equal to v, that gives you 3 more for the moment of current and then you have for the kinetic energy one more.

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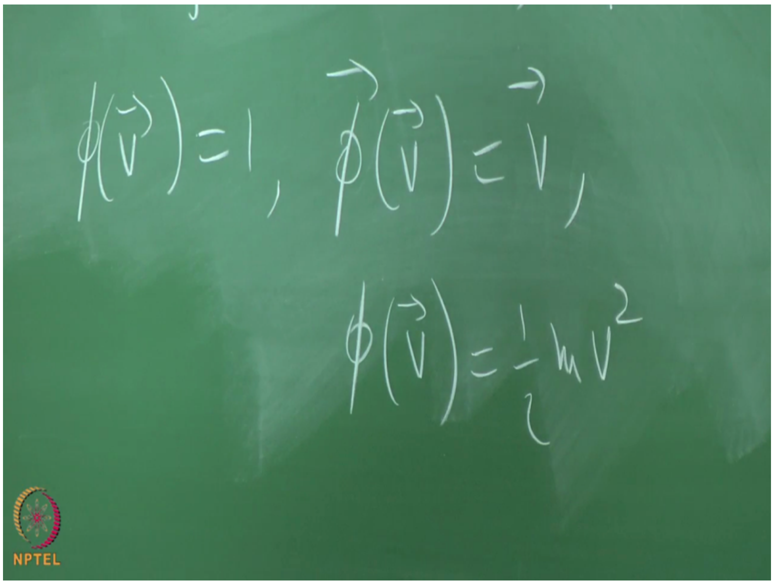
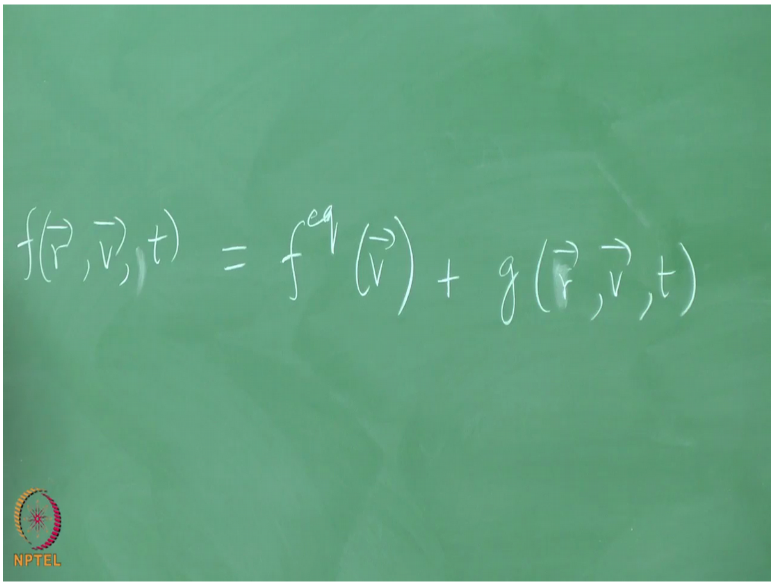


$$\phi(\vec{v}) = 1, \quad \vec{\phi}(\vec{v}) = \vec{v}, \quad \phi(\vec{v}) = \frac{1}{2} m v^2$$



$$f(\vec{r},\vec{v},t) = f^{eq}(\vec{v}) + g(\vec{r},\vec{v},t)$$

So the choices are Phi of v equal to1, Phi of v, vector equal to v itself and Phi of v equal to half mv square. Okay. And they will give you the corresponding current, continuity equations. From those continuity equations as you can extract the transport coefficients such other diffusion coefficient, viscosity, thermal conductivity, electrical conductivity, etc., If it is charged, then electrical conductivity and so on. And then you can put a magnetic field, you can construct a hall conductivity, etc. So the business of kinetic theory is to compute all these transport coefficients and they would follow from several conservation principles in this fashion. Now let us turn to what happens to the Boltzmann equation itself. And I would like to make, like to complete this, so I would like to make some comments on this, on the on the solution of the Boltzmann equation in the simplest case.

So let me schematically explain how this is done or maybe we will defer this till tomorrow since I have already run out of time. Basically the point is, let me just say it in words, the point is that if you have an equation for this quantity F of r, v, t, which is non-linear, an integral differential equation and you assume a small deviation from equilibrium. So you write this as basically equal to f equilibrium of v which satisfies detail balance, + the correction, a small correction which tells you how far away from the equilibrium you are in the distribution sense. So you write + g of r, v, t. Now put this into the Boltzmann equation.

this portion of course the delta over delta t does not act on it at all, the left-hand side is 0 for this portion. this portion will be acted on on the left-hand side and on the right-hand side you are going to have a product of 4 of these guys, 2 at a time and a difference. So you write this for v1, v2, v1 prime, v2 prime and then taking F2 prime F1 prime - F2 F1, the term which is 0 (())(50:16) will cancel out by detail balance because that is the definition of the equilibrium part. And what will be left will be combinations of one F equilibrium with the g, with the different index.

then you put delta functions and v to take out g common, with some given variable and then delta functions to fix what the actual velocity argument is for each of these and then you try to solve that linearised Boltzmann equation.

you throw away the terms like g square…

yes, you probably terms like g square. Now if you want a systematic approximation scheme, then you will have to keep schematically the second-order, higher-order, etc. But the linear term itself is hard enough and what I will do tomorrow is take the time to actually write this expression out and then we will see how to extract transport coefficient from it. In particular we would like to see if we can touch base with what we did earlier with the Langevin equation for diffusion for instance. We had a formula for the diffusion constant in terms of the friction constant, temperature, etc.

We would like to see how that comes out or if at all it does from the Boltzmann equation. We also had a Fokker-Planck equation for the velocity, conditional velocity. Now there we made assumptions about the mass of that particle, Brownian particle being much bigger than the rest of it, the molecular mass, we have made no such assumption here. In fact we are talking about elastic collisions here, so we would like to see how this differs from that and what happens. And what happens in the simplest approximation, we linearised the Boltzmann equation and made a lot of simplifying assumptions, essentially saying there is no positional dependence, everything is uniform distribution in space and only the velocity is distributed and let us see what it does, what it gives us, so we will go through that.