Physical Applications of Stochastic Processes
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Lecture - 16
Ito and Fokker-Planck Equations for Diffusion Processes

So today let us look at some properties of functions of process which is Brownian motion or a little more generally let us see what happens if we have a general Langevin type equation which I will now call by a slightly different name in a minute and the corresponding Fokker-Planck equation and from these two let us try and see if we can deduce the equal satisfied by functions of the random variables concerned.

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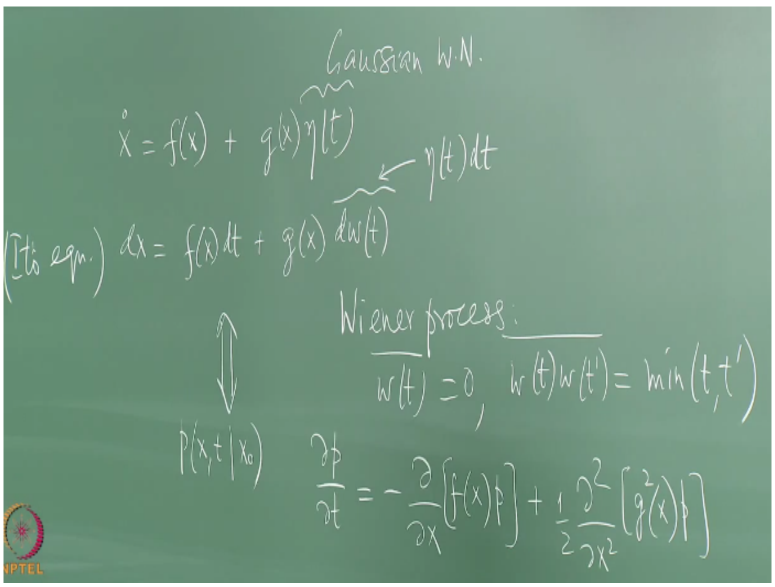

So to refresh your memory we have been talking about Langevin equation of the form x dot equal to some f of x + a g of x times white noise. This is Gaussian white noise, delta correlated in the usual fashion. But as I said the rigorous way to do this is not to deal in terms of a singular object like white noise but rather it is integral namely the Wiener process. So the right way, the mathematically better way of writing this is to write this as dx equal to f(x)dt + g(x) dw(t).

So this thing here stands for what is loosely written as eta of t dt. Because remember, the w of t was the integral of white noise. So eta of t dt is essentially the increment in the Wiener process okay. This process itself is Wiener process. This had a property w of t average was equal to 0 and w of t w of t prime, the average value is the lesser of t and t prime, a non-stationary process.

So this equation in general is called the Ito equation, Ito equation and it is the starting point of the so called Ito calculus about which I will make just a few preliminary remarks. It is a slightly different calculus than the one you are used to because this process here, this dw is a very weird one as we saw when I explained a little about Brownian trails and so on. So we will see what it implies.

What sort of rule one has when you are dealing with a Wiener process, thing as nonstandard as the Wiener process okay. Now corresponding to this we know there is a Fokker-Planck equation satisfied by the conditional density of this variable x of the random variable x. So that equation was and we wrote this down without proof, this quantity p(x, t) for some initial condition x not say this satisfies the Fokker-Planck equation.

It is the master equation and that equation was delta p over delta t equal to - delta over delta x f of x p + one half g square of x times p. So this is the correspondence between the stochastic differential equation and the conditional density for the corresponding conditional density of a Markov process okay. We are going to exploit this. We did not prove this but this can be rigorously established from Ito calculus that the moment you have this equation with this kind of noise you have this for the conditional density.

The result is Markov process x of t the driven process and its conditional density satisfies this equation Fokker-Planck equation. So we will exploit this fact back and forth okay. So the right equation, the right Ito equation or Langevin equation in various cases will be obtained by taking recourse to this Fokker-Planck equation in the cases we can actually solve for it for instance. We will be able to find out what related quantities what kind of equation related quantities satisfy okay.

Now the matter is quite trivial when g is a constant. Is then you have pure additive noise and then these niceties can be sort of slurred over but when you when it is a multiplicative noise one has to be cautious about the Ito equation. So the first example we look at is what happens to diffusion in d dimensions, d spatial dimensions.

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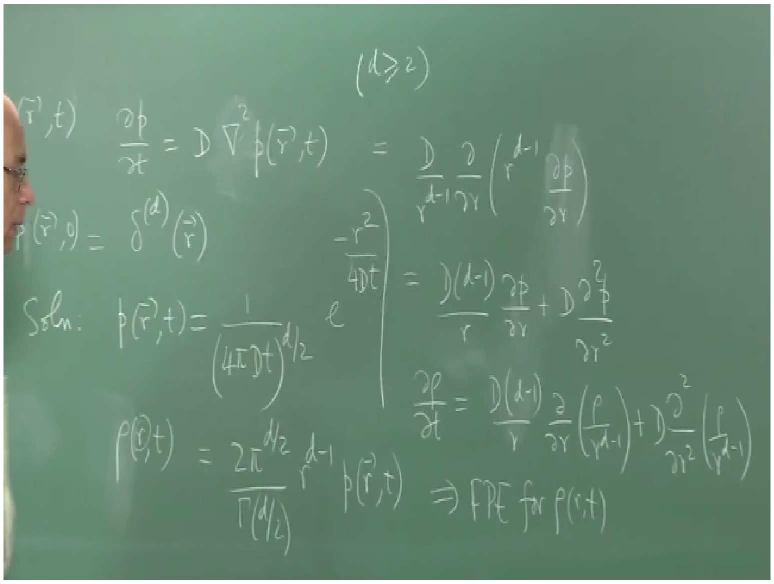

Remember that we have an equation for the conditional density of the position p(r, t) given that it starts at some point say the origin for instance. This quantity satisfies the equation delta p over delta t equal to D d2 p over del square sorry del square p of r and t okay. Where del square is a Laplacian in d Euclidean dimensions. Now we are interested in the solution which corresponds to p of r and 0 equal to delta d dimensional delta function at the origin.

So particle starts the diffusing diffusion starts from the origin and then it spreads out in d dimensional Euclidean space okay. Now this quantity is spherically symmetric and the equation is spherically symmetric as well and we are looking for a solution which is got which satisfies natural boundary conditions namely p of r and t dies to 0 as r tends to infinity along any direction in space okay.

So the boundary condition is spherically symmetric, the equation is spherically symmetric and the initial condition is spherically symmetric. Therefore, the solution is spherically symmetric right. So we are really looking for a solution that is spherically symmetrical and we know what the answer is. You can separate this in various dimensionalities. You can separate this for instance into Cartesian coordinates.

Solve each one dimensional equation etc. or you can do this in polar coordinates, spherical polar coordinates. But we know the solution and we will exploit this and work backwards from the solution. The solution is p(r, t) = 1 over 4 pi dt to the power d over 2 in d dimensions e to the - r square over 4 dt. This is the fundamental Gaussian solution. This r square stands for x 1 square + x 2 square up to xd square okay, square of the radial coordinate.

But what is the actual equation satisfied by this quantity here and just to remind ourselves that we are looking at a spherically symmetric solution. Let me in an abuse of notation write this p of r, vector t as p of radial coordinate r and t. Although we must remember that this is the probability density of the position of the displacement vector and not of the radial coordinate alone okay. So what is the equation satisfied by this.

This becomes equal to D times divided by r to the d - 1 delta over delta r r to the d - 1 p sorry okay. That is the Lapacian written out in spherical polar coordinates in d dimensions and retaining only the radial part. The angular parts are not the the p does not depend on the angular part and therefore those do not appear in the del square at all. Of course when you go to d = 3 you get the familiar one over r square d over dr r square d over dr okay.

And let us write that out. So this is equal to D times d - 1 r to the d - 2 over d - 1; so this over r delta p over delta r + d times. So you have a term which is not just the second derivative but also first derivative term sitting here. Now let us say we are interested in the probability density function of the radial distance from the origin okay. That is the modulus of r and we need to know what its equation is what its solution what equation its distribution satisfies and what is the actual corresponding stochastic differential equation.

So we are trying to work backwards and therefore let us call rho of r, t the radial distribution function of this fellow r. So r equal to mod r and we want the distribution function of this rho. Now what is the customary way of doing it? It is to say look I take this fellow, the actual solution for the PDF of the vector r and then I integrate over all angles and what is left is the radial distribution. In 3 dimensions it will be 4 pi r square e to the - r square over dt and so on.

What is it in d dimensions? You need the analog of this 4 pi in d dimensions in all solid angles if you like. The surface of a unit sphere in d dimensions is what we need. In two dimensions it is 2 pi because the radial the circumference is 2 pi and in 4 dimensions it is 4 in 3 dimensions it is 4 pi, in 4 it is 2 pi square and so on. The general formula is 2 pi to the d over 2 over gamma d over 2 multiplied of course by this factor r to the d - 1 and then a p of r and t.

So if I multiply this by this factor here and then put in this volume element factor in the volume element I get the radial distribution. You can check I mean when d = 2 for instance it is going to be equal to 2 pi. This is 1. 2 pi r times p and when d is equal to 3 then you get 2 pi to the 3 halves that is pi root pi over gamma 3 halves which is half root pi. The root pi cancels, the half goes up there and it gives you 4 pi r square okay. So this is what rho is.

Whatever it is this constant factor is actually irrelevant because this whole thing is homogenous in p and therefore it is going to cancel out. What is relevant is that you have an r to the d - 1 times p out there. So all you have to do is to remember that rho is r to the d - 1 times p apart from a constant and therefore p is rho divided by r to the d - 1. So this immediately says delta rho over delta t equal to d times d - 1 over r delta over delta r rho divided by r to the d - 1 + d.

You have guaranteed that rho is going to satisfy this differential equation right and now the task is to simplify it. Write out this second derivative term and so on and you get an equation which is going to tell you what. So this will lead to implies a Fokker-Planck equation for rho of r, t okay and that is trivial to write down. We will write down what this equation is and then you work backwards.

You work this correspondence backwards to find out what is the stochastic differential equation satisfied by r, the process r okay. So this is not a very trivial process in that sense because writing down the equation for each Cartesian component of this position vector is one matter. You have uncorrelated noise in the different Cartesian components. But now you are saying what is the stochastic equation satisfied by the square root of x 1 square + x 2 square up to xd square okay.

And to navigate between the Fokker-Planck equation and the stochastic differential equation you got to be very careful as you will see in a minute.

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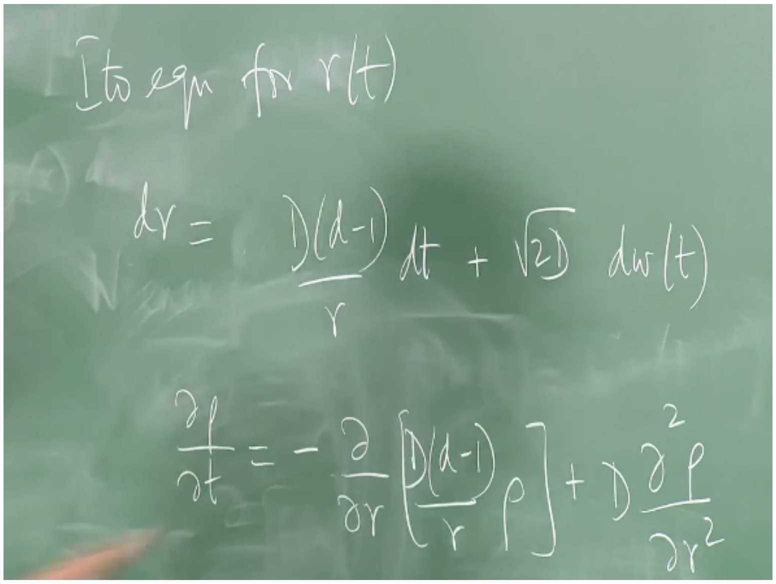

If you did that finally, the stochastic differential equation or the Ito equation for r of t, for this process itself turns out to be the following. It turns out to be r dot or dr equal to - d into d - 1 over r dt + root 2D times dw of t or if you like the equation satisfied by rho will be delta rho over delta t equal to D sorry it is plus here - D delta over delta r - delta over delta r D times d - 1 over r rho + D d 2 rho over d r square.

So I am skipping the intermediate steps. What you have to do is to write this out as an equation for rho by taking all these factors recombining them and so on and eventually you end up with a thing like this which of course will immediately imply a stochastic differential equation of this kind here. So notice that there is an extra term that is appeared here. What is happened is that this guy gives you a drift, this looks like the f of x, t.

This is like a drift here and the drift says that you are getting pushed away from the origin okay and as I explained very briefly last time a little earlier, this is a real effect because what is really happening is that the changes of any fluctuation increasing r are greater than the ones that decrease r and this tendency is enhanced as you go towards r = 0. Because if you are at the origin itself, the moment there is a fluctuation you have increased r definitely.

So this drift term is a real effect, it is really there okay. Even though the noise in different Cartesian components is completely uncorrelated the moment you combine these fellows together into the square root of x 1 square up to xd square you get this systematic drift term here even though there is no external potential there is no force of any kind you still get in the radial variable a drift term okay.

Now of course you already know the solution to the corresponding Fokker-Planck equation because we worked backwards, it is right here the normalized solution is, all you have to do is to substitute this Gaussian formed here and simplify it a little bit and that is the solution. But our interest here is in seeing what kind of Ito equation you get for this process. What about the square of this variable. What about r square?

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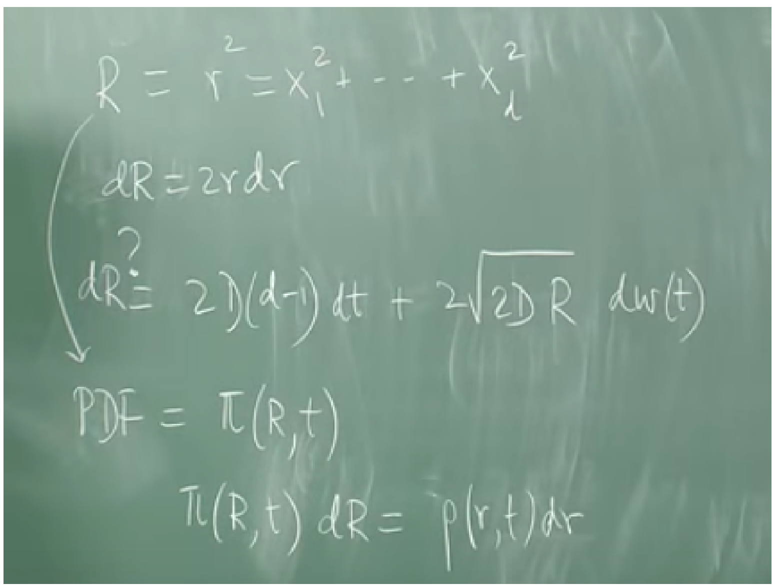

What happens if you deal with the random variable, let us say R square = x 1 square sorry R = r square which is x 1 square plus plus x d square. What sort of equation is that going to satisfy? What kind of Fokker-Planck equation or whatever do you get for that? Well, one possibility is to say the following. Is to say look this implies that 2 that dR equal to 2rdr, just differentiating.

And then I put that in here and multiply both sides by 2r right and then I end up with dR equal to twice D into d times - 1, the R cancels out and then there is a dt plus twice root 2D times little r but that is a square root of capital R so dw. So the question is, is this the Ito equation for dr? Looks very straightforward. All I have to do is to multiply this by 2r and that is the equation. That is wrong. This is wrong.

It is not possible to do it this way because the Ito calculus stops you from doing this. it is incorrect to do this. I will mention a little bit about the Ito calculus and then we will see what goes wrong in this because this quantity here in a sense the fact that w of t w of t prime is minimum t t prime shows that dw is of order square root of dt always. So that is got to be kept in mind when you play around with these differentials here. The right way to do this is to say alright.

One way to do this is to say the following. Is to say okay I have this R equal to r square. I have a solution for this little rho. In fact I have the Fokker-Planck equation satisfied by it. So what I should do is to see how the distribution of this capital R is related to the distribution of little r okay and use the relation capital R is little r square.

Then if I call the density function of this fellow, let us say it is PDF equal to some pi of R, t say for want of a better notation then I know that pi of R, t dR equal to and that is a monotonically increasing function of little r. So I do not have to worry about the sign of the Jacobian. This is equal to rho of r, t dr.

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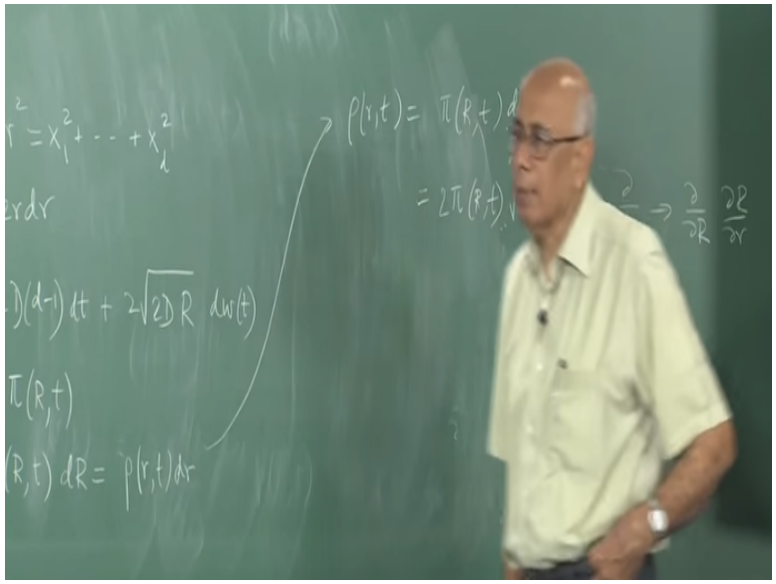

So this will tell me, that will immediately tell me that rho of little r and t equal to pi of R and t dR over dr that is equal to pi of R, t and this dR over dr of course is 2R that is certainly true. So this is equal to twice square root of R. But I have a Fokker-Planck equation for rho. So I put this into that equation for rho and convert all delta over delta little r's and delta over delta capital R and then delta R over delta little r okay which is twice square root of t okay.

So I should be careful to do that and once I do that I have a Fokker-Planck equation for this pi in which the independent variable, the random variable the the variable which I am interested in is capital R. So the derivatives will be with respect to capital R on the right hand side and from that Fokker-Planck equation I can go back to the stochastic differential equation satisfied by this capital R okay.

Now that is a completely correct method because we have not done anything playing around with the Ito equation directly. What we have done is to go to the Fokker-Planck equation and say that instead of little r I used capital R as a random variable and then it turns out the correct answer is almost this.

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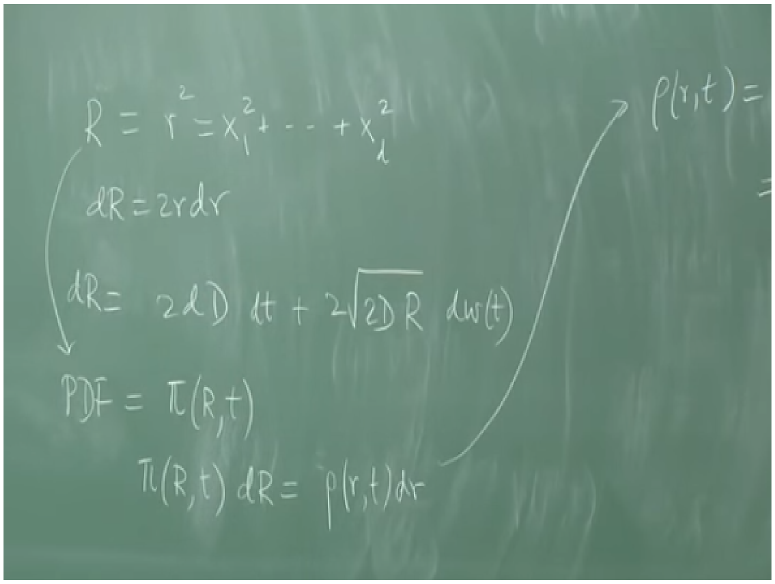

It turns out and I leave this as an exercise to you is 2dD dt. That is the correct equation. Not the one you naively get by multiplying this by 2R. So is the logic clear. We are doing taking a short cut to this whole business. I started with one solution to the diffusion equation, this spherically symmetric solution, that is the Gaussian solution in d dimensions right.

And then when I want functions of the various coordinates which are sufficiently symmetric and is spherically symmetric then what I do is to start with that original diffusion equation write it in radial coordinates and polar coordinates for the radial variable and then make changes of variables every time in the Fokker-Planck equation to go from one independent variable to another.

And that tells me what the correct Fokker-Planck equation is for the corresponding variable and density and from there I go back to the stochastic differential equation okay and that will give me the right answer always in every case. So notice that again for this capital R there is a drift here and so on okay. Now what does this thing actually look like, what does this distribution actually look like for the radial variable.

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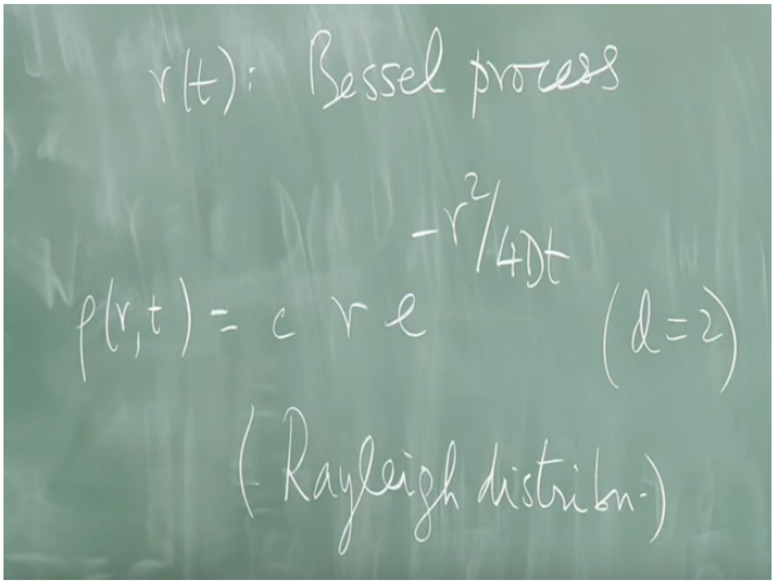

Remember that for the radial variable this rho, so by the way this process r of t it is called a Bessel process. This is square root of x 1 square up to xd square, square root of d squares of d Brownian motions. In the case of d equal to 2 the distribution is very nice and easy to write down particularly easy to write down. It was r to the d - 1 e to the - r square whatever it is and in 2 dimensions it is proportional to r e to the - r square over 4 Dt some constant times this guy. This is what rho of r, t equal to this in 2 d.

This is called Rayleigh distribution, special name for it, just need to check whether, yes it is called a Rayleigh distribution. What does it look like for small r? Linearly increases and then of course it dies down very fast exponentially in this form. It is particularly convenient. It is applied in various other places. We will not talk about that right now. But it is got a name because it is got some special significance.

This process here is useful elsewhere as well, not just in the context of Brownian motion okay. Now this formula here with this value little d if this were d - 1 it would vanish, this term would vanish in d = 1 right. But you can independently do the following thing. I can start with a one dimensional diffusion, Brownian motion and look at the square of that variable. Look at x square right and ask what is the probability density function of x square.

Of course x square will have a sample space running from 0 to infinity but you can ask what is the distribution of this x square and that is not very hard to write down and then from there from that solution you can now go back find out what is the Fokker-Planck equation corresponding to it; it is easy to write down and then go back to what the Ito equation is and it will turn out to be exactly this equation with little d set equal to 1.

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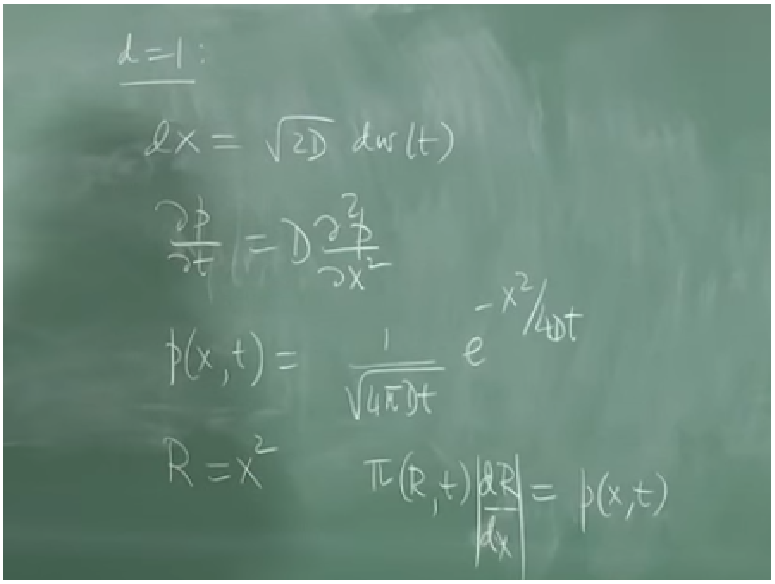

So if you look at ordinary Brownian motion in one dimension so d = 1. Now you have an equation, the stochastic equation is dx equal to square root of 2D dw of t, that is it. Because I had written this earlier as x dot as square root of 2D times eta of t. Now I have written it a little more rigorously if you like in terms of as a as an Ito equation. Now what is the Fokker-Planck equation corresponding to it?

It is delta p over delta t equal to D d 2 p over dx 2 of course. That is the usual diffusion equation right. And the solution, the fundamental solution p of x and t is 1 over root 4 pi Dt e to the - x square over 4Dt. And now I can ask what about the distribution of the random variable R equal to x square. It would be tempting to multiply this by 2 x but that is wrong, that is wrong because there is no drift term here at all. That is not correct.

What you have to do is to write this density. So let us call the density of this pi so you have pi of R and t dR equal to p of x and t dx. Actually what you have is dR over dx in this form okay and then compute this guy paying attention to the fact in this case what attention have I not paid. I have not paid attention to one simple fact. It looks very right; this thing looks absolutely right. What is what is missing here?

It is not monotonic, it is not monotonic because remember - x and + x give you the same capital R right. So what I have forgotten to do is the conversion which I did between the radial coordinate and it involved an angular integration, surface of the unit sphere in d dimensions. It was 2 pi in d equal to 2, it was 4 pi in d equal to 3, 2 pi square in d equal to 4. What is it in d equal to 1? 2 because how do you define unit sphere centered at the origin.

The locus of all points unit distance away from this point right. So in the case of one dimension there are 2 points. So there is 0 and then you have + 1 and - 1 those 2 points. So there is a factor 2, this surface. So you really got to multiply this by 2. This guy gets multiplied by 2. We did this earlier in the quiz we had you had to find the distribution of the probability that the modulus of the difference between 2 Poisson variables was equal to 3. So what you did?

Such a few has got the right answer. What you did was to say that this difference could be the + 3 or - 3. So you just add up the probabilities. In that case it was just not multiplying by 2 because their distribution was not symmetric but in this case it is because this Gaussian is symmetric and the x goes to - x. So this pi has a factor 2, extra factor 2 multiplying.

When you put that in you get a solution for this pi out here and then you can work backwards and ask what is the corresponding Fokker-Planck equation and then what is the actual diffusion what is the Ito equation for the variable x and it will turn out to be this with d equal to 1.

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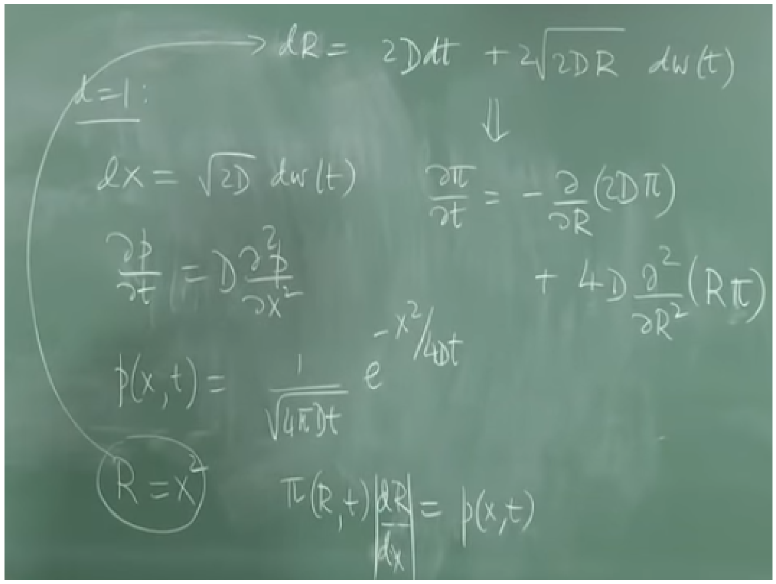

So it turns out in this case here this variable here obeys a dR equal to 2Ddt + 2 root 2DR dw of t okay. Notice that there is multiplicative noise. This implies multiplicative noise because if you write the Fokker-Planck equation down for it what would that be. This would delta pi over delta t equal to - delta over delta R 2D times pi out here + one half the square of this guy the 2 goes away so there is a 4D delta 2 over delta R 2 R square p ah sorry R times pi.

So this is the Fokker-Planck equation satisfied by the square of Brownian motion one-dimensional Brownian motion by by the density probability density for the square of one-dimensional Brownian motion okay. There is this term here, this extra term sitting here which you would not get if you did not take the correct Ito equation okay. What about the nth power of ordinary Brownian motion? What about this guy x to the power n.

So I will call this some other variable psi equal to x to the power n where x satisfies this Ito equation here. So what you would do is again say let me find first the density function of this guy here or at least the equation satisfied by the density of this variable by starting with the density of x itself paying attention to whether it is monotonic etc., etc. and then once I have that Fokker-Planck equation I will go backwards and write the Ito equation for psi okay.

So I will leave you to prove that this thing here also satisfies an Ito equation which has got a drift term and a diffusion term and there is multiplicative noise in this case. So the nth power of

Brownian motion or a diffusion process is also a diffusion process. What about the exponential of Brownian motion. What happens if I exponentiate it.

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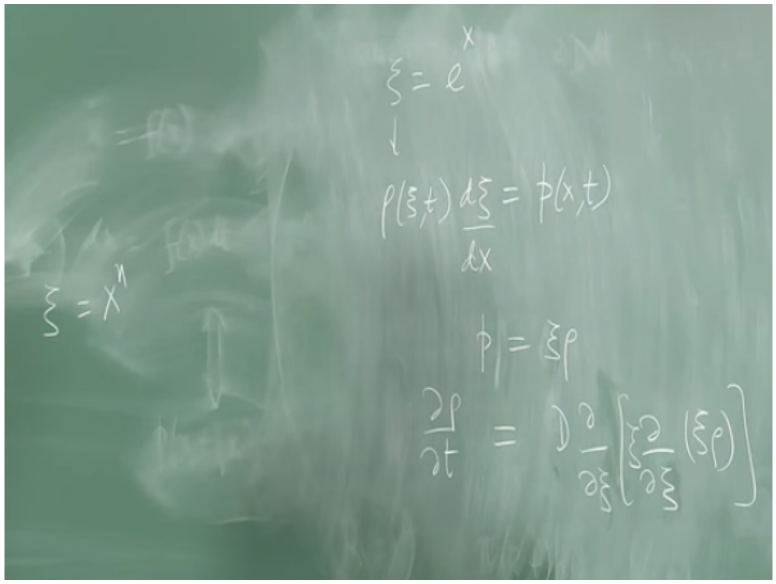

Let us see where that gets us. So let us put psi equal to e to the power x. Actually I should put e to the x. I should be careful about dimensions but let us assume I measure x in some units, appropriate units or put an l for a scale factor if you like but let us assume this is dimensionless now and then ask what is the equation satisfied by this exponential here okay. Now this is of course a monotonically increasing function of x. Starts at 0 when x is minus infinity.

Goes to 1 and goes off to plus infinity as x tends to infinity right. So there is no hassle about folding and doubling and so on and so forth. Then if I call the density of this rho of psi t d psi equal to p of x, t dx so d psi over dx; d psi over dx is e to the x which is psi itself right. So immediately I get p equal to x rho and then I go back and say delta over delta t oh sorry psi times rho okay. There is a rho here and there is a psi outside because that is not getting differentiated.

It is an independent variable. This is equal to, on the right hand side I have D times d 2 over d x 2 times p that is psi rho. So let us write this as delta over delta x psi rho. But this is delta over delta psi times delta psi over delta x which is psi itself. And this delta over delta x let us write it as delta over delta psi and multiply by a psi but that removes this out here okay and that is it. I simplify this guy and whatever I get is the stochastic is the Fokker-Planck equation for rho okay. From that I can get the equation satisfied by psi itself okay. So let us do that.

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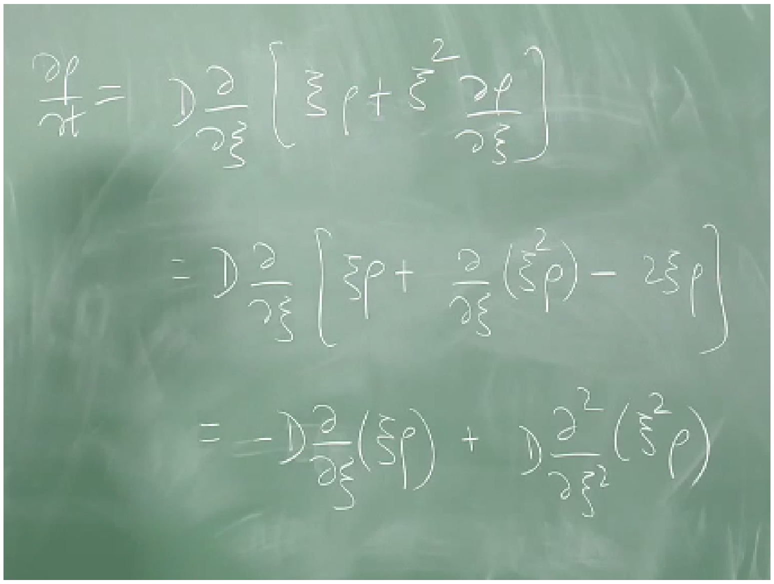

What does this imply? Delta rho over delta t equal to D times delta over delta psi times that is the first term when I differentiate this, well let us do it slowly delta over delta psi times psi rho + psi square delta rho over delta psi is equal to D times delta over delta psi is equal to rho + D psi delta rho over delta psi, I am not sure if I made a mistake somewhere. Okay let us see what happens. This + 2 psi D delta rho over delta psi + D psi square delta 2 rho delta psi 2.

This is not the way to write it. This is not the best way to write it. This is not the best way to write it because I need to write it as a drift term which is a delta over delta psi with a linear term rho on the right hand inside the bracket some function f of psi times rho + d2 over dx d psi 2 g square times rho. So I need to manipulate this a little bit. I am not thinking too clearly but anyway.

So you got to write this back as minus delta over delta psi f of psi rho plus one half d 2 over delta psi 2 g square of psi times rho. So you have to bring it to that form. So this term I write it as equal to D delta over delta psi, psi rho is already there plus delta over delta psi of psi square rho - 2 psi that is right. Thank you. Ya. Perfect. So this will give us equal to - D delta over delta psi of psi times rho. This is the minus sign + D d2 over d psi 2 psi square rho, very good. That is it okay. So that is the usual Fokker-Planck equation. So what will that imply now for the stochastic differential equation.

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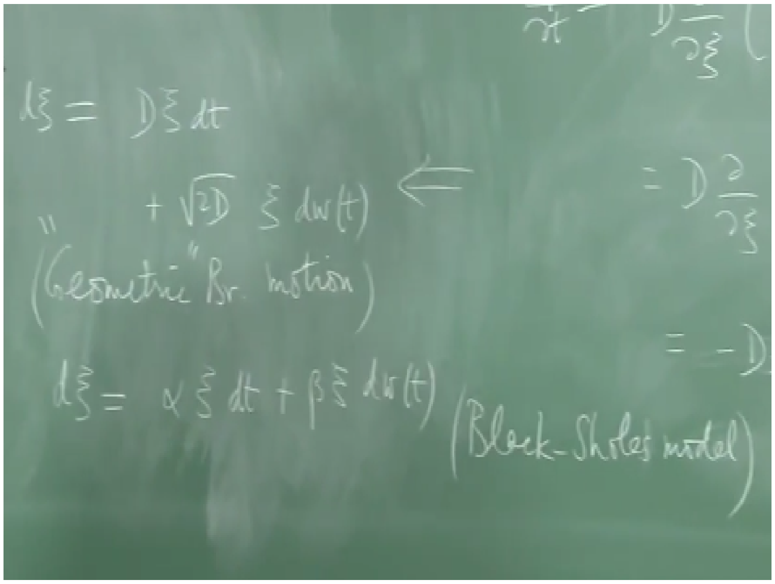

This will immediately imply that d psi is equal to D psi dt + square root of 2D times psi times dw of t. So when I take a half the square of it I get a D and then inside gets a psi square okay. So notice in this case that both the drift and the diffusion terms in the Ito equation are linear in psi okay. The standard one for x itself this is absent of course and this guy here is a constant but now you got a psi here. The Ornstein-Uhlenbeck had this as a constant and this linear but this is in between, a very strange combination here okay.

And this is called geometric Brownian motion, geometric in quotation marks okay. There is a small generalization of this and that is the model that is used in economics when they want to model stock market fluctuations and so on the model used is d psi equal to some alpha psi dt + beta psi dw of t in general okay and this is called the Black Scholes model okay. The process itself, the psi process itself if you use the Ito calculus turns out to be the functional of Wiener process.

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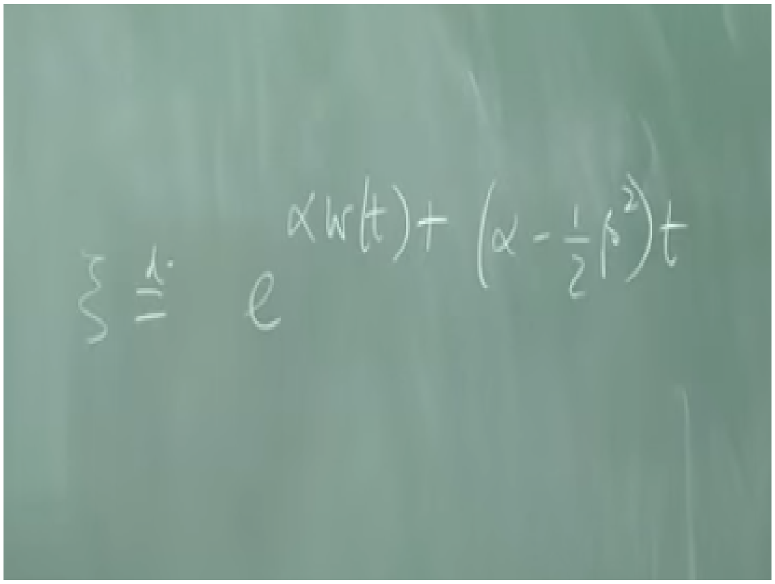

It turns out to be psi has the same distribution as e to the power alpha times Brownian motion + alpha - half beta square times t okay. Next time I will say a little more about the Ito calculus. We will write down the rules, the integration rule and the differentiation rule for the Ito calculus then this will become clear immediately. So the way to handle this strange object this dw is to change the rules of the calculus somewhat and we will do that in an intuitive way.

We will try to justify it in an intuitive way. It is after all a rule. There are other rules, other ways of arriving at stochastic differential equations. When you have multiplicative noise this is the problem. The correct stochastic differential equation to lead to the master equation we need to write the correct stochastic differential equation and I will explain this 2 different conventions, there is the Ito convention, there is the Stratonovich convention.

There are other conventions as well and we will see what the differences are between these. It has to do with how you write an integral as a sum over increments and depending on what convention you use you get different equations but the physics cannot be different finally. Finally, the probability distribution of the variables you are talking about are neatly specified in any case.

So the to put it very roughly you start with an equation for average values or moments quote unquote an engineering equation and then you make a model to a stochastic differential equation by adding noise on it and from there you go to a Fokker-Planck equation for the density function, probability density okay. The initial point and the final point have to be unique. They are measurable, they are testable in between.

But the point is what is the intermediate stochastic differential equation you are going to write down and what is the prescription to go from that stochastic differential equation to the master equation okay and you are going to have two different routes in each of these cases. So the equations could differ in between but the root from the equation to the Fokker-Planck equation, the prescription also differs in a manner to compensate for this and get the same equation no matter what interpretation you use okay. I will try to explain this with an example next time okay.