Select/Special Topics in 'Theory of Atomic Collisions and Spectroscopy'

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Lecture 12
Levinson's theorem

Greetings, we will have some more discussion on the Levinson's theorem which we already established in the previous class. We will also introduce the idea of effective range which was introduced by Fermi and later Bethe in nuclear physics and we will discuss some applications specially in the domain of ultra cold atoms specially Fermi mixtures and so I will just make a little mention of some of these things.

And we will conclude unit 1 and then come back to collision physics a little later in this course because we need to get into second quantization and other things which I would like to begin in unit 2. In fact unit 1 itself I wanted to wrap up a little earlier but because of our energy in slow energy collisions and the Levinson's theorem it got extended. But with today's class we will conclude unit 1.

So, essentially the Levinson's theorem we formulated for the S waves, the low energy limit of the phase shift of the scattering phase shift is either in pi or n + half pi. And it is npi typically but it can also be n + half pi if there is a resonant bound state solution which is in existence at that limit.

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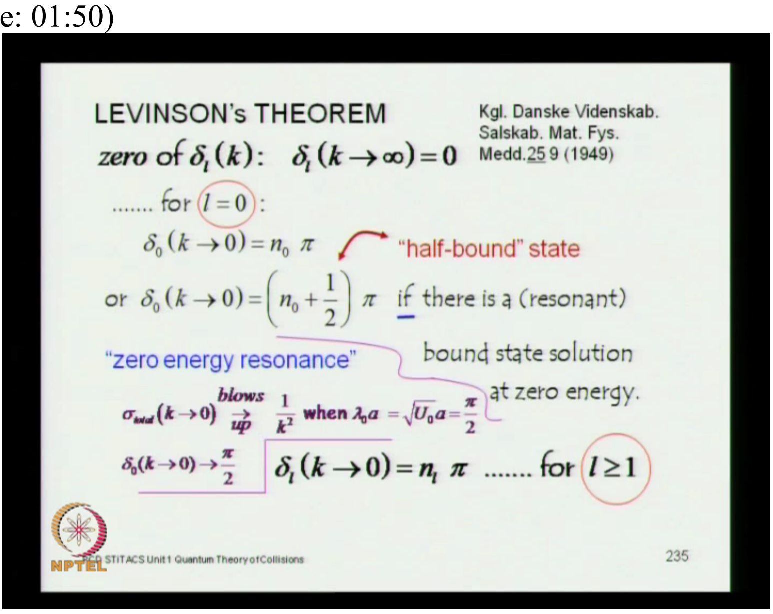

LEVINSON's THEOREM

Kgl. Danske Videnskab. Salskab. Mat. Fys. Medd. 25 9 (1949)

zero of $\delta_l(k)$: $\delta_l(k \to \infty) = 0$

.... for $l = 0$:

$\delta_0(k \to 0) = n_0\,\pi$ "half-bound" state

or $\delta_0(k \to 0) = \left(n_0 + \frac{1}{2}\right)\pi$ if there is a (resonant) bound state solution at zero energy.

"zero energy resonance"

$\sigma_{total}(k \to 0) \underset{up}{\overset{blows}{\to}} \frac{1}{k^2}$ when $\lambda_0 a = \sqrt{U_0}a = \frac{\pi}{2}$

$\delta_0(k \to 0) \to \frac{\pi}{2}$

$\delta_l(k \to 0) = n_l\,\pi$ .... for $l \geq 1$

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So, under those conditions you can have low energy bound state at zero energy but this will not be normalizable because this will be continuum state. Otherwise bound state energy discrete energies are essentially square integrable this is not. So, the boundary condition is different which is why it is called as a half bound state. And for l greater than 0, for either 1, 2 or any other value of l the phase shift is always npi for l greater than or equal to 1.

The resonant value of the cross section which we check for the square well spherical potential the cross section goes as 1 over k square and the resonances occur whenever the strength of the potential is given by this parameter root U0a or more condiment by its square which is U0a square. And when this parameter root of U0a is equal to pi by 2 or 3 pi by 2, 5 pi by 2 and so on that is when you get the resonances and the number of bound states increases.

So, the zero of the phase shift is set with at infinity that these phase shifts are measured with the phase shift at high energy limit, so this is the reference phase shift. And this reference is indicated over here that delta l my at zero energy minus delta l at high energy limit. This is the difference which is given by npi or n + half pi depending on whether l is 0 or 1, 2 and 3 and so on.

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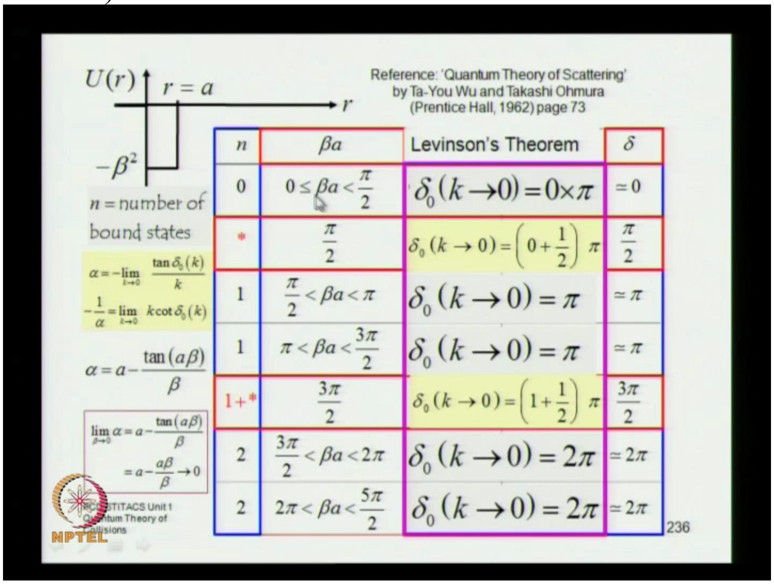

Reference: 'Quantum Theory of Scattering' by Ta-You Wu and Takashi Ohmura (Prentice Hall, 1962) page 73

n = number of bound states

$$\alpha = -\lim_{k\to 0} \frac{\tan\delta_0(k)}{k}$$
$$-\frac{1}{\alpha} = \lim_{k\to 0} k\cot\delta_0(k)$$
$$\alpha = a - \frac{\tan(a\beta)}{\beta}$$
$$\lim_{\beta\to 0}\alpha = a - \frac{\tan(a\beta)}{\beta} = a - \frac{a\beta}{\beta} \to 0$$

n	βa	Levinson's Theorem	δ
0	$0 \le \beta a < \frac{\pi}{2}$	$\delta_0(k\to 0) = 0\times\pi$	$\simeq 0$
*	$\frac{\pi}{2}$	$\delta_0(k\to 0) = \left(0+\frac{1}{2}\right)\pi$	$\frac{\pi}{2}$
1	$\frac{\pi}{2} < \beta a < \pi$	$\delta_0(k\to 0) = \pi$	$\simeq \pi$
1	$\pi < \beta a < \frac{3\pi}{2}$	$\delta_0(k\to 0) = \pi$	$\simeq \pi$
1+*	$\frac{3\pi}{2}$	$\delta_0(k\to 0) = \left(1+\frac{1}{2}\right)\pi$	$\frac{3\pi}{2}$
2	$\frac{3\pi}{2} < \beta a < 2\pi$	$\delta_0(k\to 0) = 2\pi$	$\simeq 2\pi$
2	$2\pi < \beta a < \frac{5\pi}{2}$	$\delta_0(k\to 0) = 2\pi$	$\simeq 2\pi$

So, we discuss this table in our previous class for a square well potential and notice that the potential can be made stronger either by making it deeper or by making it wider. And if the potential strength is indicated by this product beta times a where beta is a measure of the depth of the potential and a measure of the range of the potential.

Then in this range the potential is attractive but the number of bound state is 0. So, just the fact that the potential is attractive does not guarantee that a bound state exists. So, the potential has to be sufficiently strong to be able to bind a state bind an electron in that and as the strength parameter becomes pi by 2 that is when you get the first bound state.

So, that for all values of the strength which are greater than pi by 2 you have got one bound state and this number remains the same still this strength parameter becomes 3 pi by 2 which is when you get the next bound state. So, just the threshold of the bound state is what I have indicated by this asterisk. So, asterisk is just when the bound state is picked up okay. For all values of the strength lower than that you do not have the next bounce state.

And if you see the last column the phase shift it is either 0 or pi by 2 or pie or 3 pi by 2 or 2 pi completely in accordance with the Levinson’s theorem. So, what happens to the scattering lengths in this process? Now that is indicated in this table because the scattering length is given by this expression for the square well spherical potential and this scattering length it actually it is in this range, so it is negative in this range.

But the potential is not strong enough to bind an electron. So, the scattering length being negative is a necessary condition but not a sufficient condition it has to be negative. But it not only has to be negative it has to be as negative as it can get okay. How negative can it get, it really has to go toward minus infinity okay. So, you will see how that happens. So, here the scattering length is negative this is alpha this is the scattering length.

This is negative in this range and it becomes most negative when this beta a, becomes pi by 2, it becomes in fact minus infinity. And then across pi by 2 it reverses the sign from minus infinity to plus infinity it then becomes positive and it is almost like a repulsive potential because it cannot bind any more electron into the system or any more particle whatever be the particle.

So, the scattering length has to change sign at this threshold which happens at pi by 2 then it happens again at 3 pi by 2 when the scattering length again goes from minus infinity to plus infinity. So, I want you to pay attention to how the scattering length behaves and how it reverses the sign when it is about to, when the conditions of the potential are a property to pick up a bound state.

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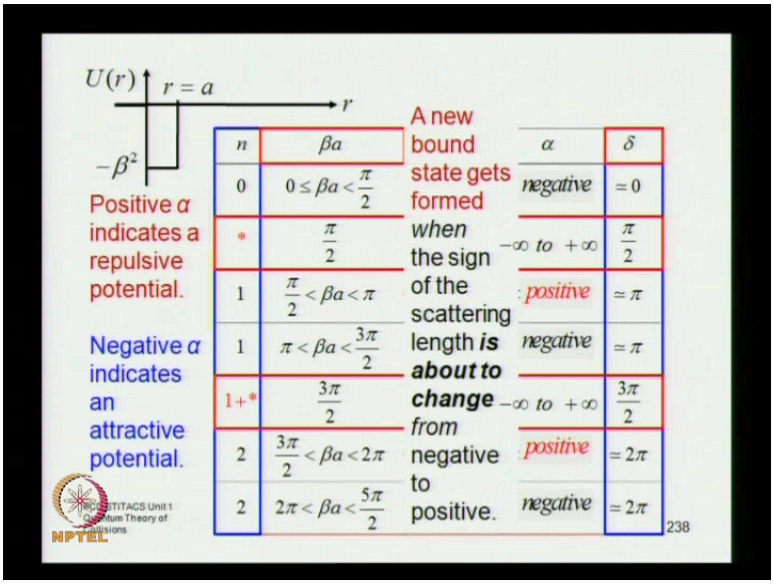

Positive α indicates a repulsive potential.

Negative α indicates an attractive potential.

n	βa	α	δ
0	$0 \le \beta a < \frac{\pi}{2}$	negative	≈ 0
*	$\frac{\pi}{2}$	$-\infty$ to $+\infty$	$\frac{\pi}{2}$
1	$\frac{\pi}{2} < \beta a < \pi$	positive	$\approx \pi$
1	$\pi < \beta a < \frac{3\pi}{2}$	negative	$\approx \pi$
1+*	$\frac{3\pi}{2}$	$-\infty$ to $+\infty$	$\frac{3\pi}{2}$
2	$\frac{3\pi}{2} < \beta a < 2\pi$	positive	$\approx 2\pi$
2	$2\pi < \beta a < \frac{5\pi}{2}$	negative	$\approx 2\pi$

A new bound state gets formed when the sign of the scattering length ***is about to change*** *from* negative to positive.

So, this I will discuss further, so a bound state gets formed when the sign of the scattering length is about to change it must change from negative to positive. But it cannot jump from negative you know 10 units to positive 10 units it has to get as negative as it can get, so is minus 20 more negative than minus 10 yes it is, is minus 100 more negative than minus 10 yes it is.

What is the most negative number you can think of? Minus infinity, so that is what has to happen okay. So, it has to be negative only then you have an attractive kind of potential which has the capacity to bind a particle. But it not only has the scattering like not only has to be negative it has to get as negative as it can.
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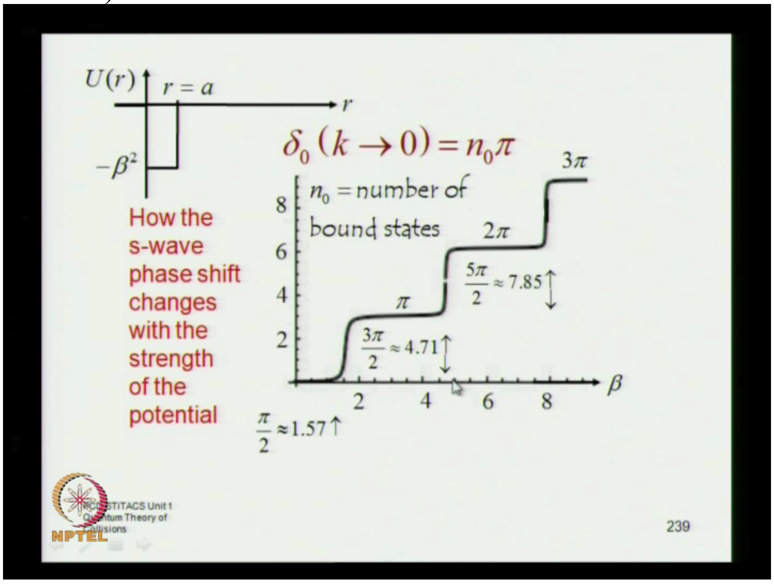

How the s-wave phase shift changes with the strength of the potential

So, this is what happens the phase shift is jumps through pi, so it picks the first bound state when this parameter is pi by 2 as I mentioned right. So, the value of pi by 2 which is 1.57 this is when the phase shift changes it picks up a one bound state and beyond this the phase shift is pi because the phase shift is for S wave this I am considering only the S wave over here.

So, the S wave phase shifted z energy is equal to pi times the number of bound states. So, now that there is a one bound state the phase shift is pi over here then beyond this value which is 3 pi by 2 you get a phase shift of 2 pi then it the phase shift is 3 pi and so on so it goes as a step function.

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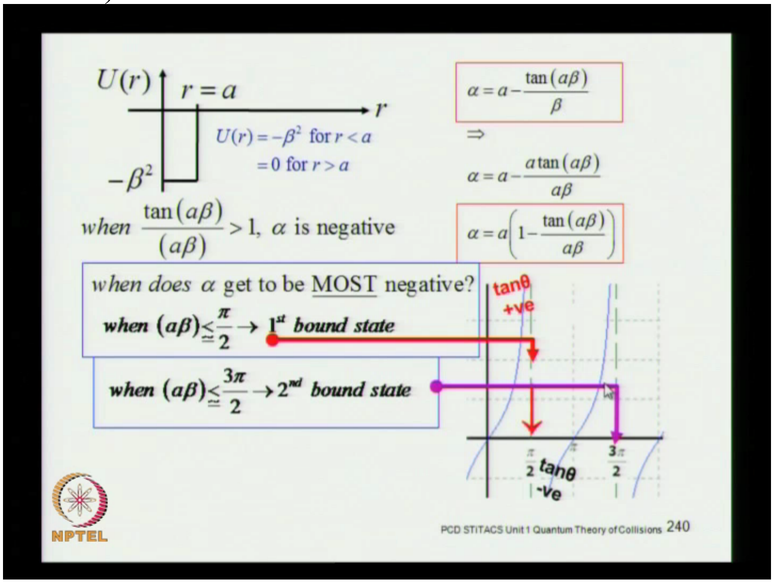

$U(r) = -\beta^2$ for $r < a$

$= 0$ for $r > a$

$$\alpha = a - \frac{\tan(a\beta)}{\beta}$$

$$\Rightarrow$$

$$\alpha = a - \frac{a\tan(a\beta)}{a\beta}$$

$$\alpha = a\left(1 - \frac{\tan(a\beta)}{a\beta}\right)$$

when $\frac{\tan(a\beta)}{(a\beta)} > 1$, α is negative

when does α get to be <u>MOST</u> negative?

when $(a\beta) \lessapprox \frac{\pi}{2} \rightarrow$ **1st *bound state***

when $(a\beta) \lessapprox \frac{3\pi}{2} \rightarrow$ **2nd *bound state***

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Now let us understand this idea, look at the expression for the scattering wavelength it is alpha is a, this is just a typical case which is the square well spherical potential which we are considering for the sake of simplicity. But the results are much more general than that now. For this the scattering length is given by a times this factor 1 - tan a beta by a beta this expression we have already obtained in our previous class.

Now the question is when does alpha get to be most negative, yes for alpha to be most negative it is 1 minus this right. So, this number that this ratio which is tan a beta by a beta this must be greater than 1 that is the first condition that is when alpha will go negative. But then when does it go most negative, when the tangent function becomes the largest and when does it happen, when the angle becomes pi by 2, 3 pi by 2 and so on right.

So, that is when the scattering phase shift is going to be such that we will be able to pick up one bound state at that stage. So, that is when alpha is most negative, so you get the first bound state when this angle a beta is pi by 2 but it means it should be infinitesimally less than pi by 2 because it must be negative okay. It should be negative but as negative so tending to minus infinity okay. So, this is when you get the first bound state okay.

Now likewise you get the second bound state again when this tangent function goes to plus infinity. The tangent function will go to plus infinity at 3 pi by 2 okay which is sine theta by cos theta. So, this will go to infinity when cos theta goes to 0 and this will again go to infinity and you will get the second bound state over here. Likewise you will get the third bound state when the angle becomes 5 pi by 2.

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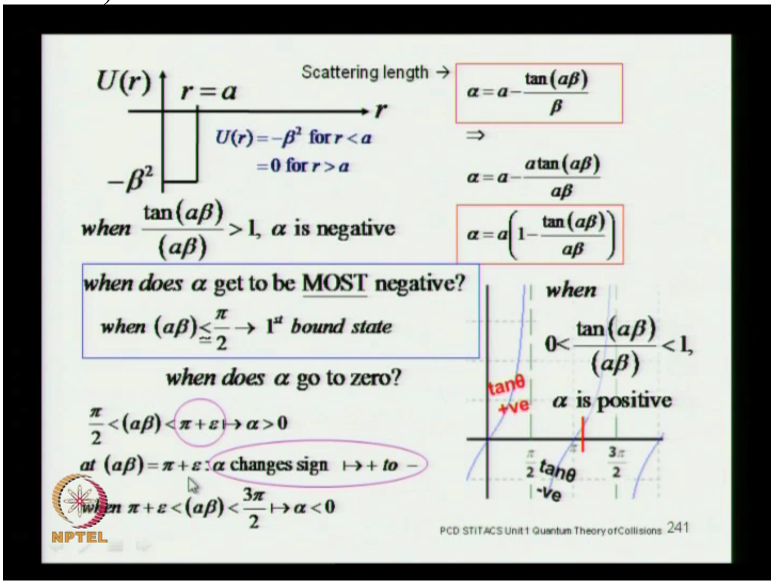

Scattering length → $\boxed{\alpha = a - \frac{\tan(a\beta)}{\beta}}$

$\Rightarrow$

$\alpha = a - \frac{a\tan(a\beta)}{a\beta}$

$\boxed{\alpha = a\left(1 - \frac{\tan(a\beta)}{a\beta}\right)}$

$U(r) = -\beta^2$ for $r < a$

$= 0$ for $r > a$

when $\frac{\tan(a\beta)}{(a\beta)} > 1$, α is negative

when does α get to be MOST negative?

when $(a\beta) \underset{\simeq}{\leq} \frac{\pi}{2} \rightarrow$ 1st *bound state*

when does α go to zero?

$\frac{\pi}{2} < (a\beta) < \pi + \varepsilon \mapsto \alpha > 0$

at $(a\beta) = \pi + \varepsilon$: α changes sign $\mapsto +$ *to* $-$

when $\pi + \varepsilon < (a\beta) < \frac{3\pi}{2} \mapsto \alpha < 0$

when $0 < \frac{\tan(a\beta)}{(a\beta)} < 1$, α is positive

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Now let us ask when does the scattering length become zero at pi by 2 and 3 pi by 2 it goes to minus infinity at pi by 2 it changes sign from minus infinity to plus infinity because the tangent function jumps from plus infinity to minus infinity right. When does alpha goes to zero, is it at the angle in between which is pi, it will be slightly more than pi right because this ratio tangent theta 2 theta must be equal to 1.

So, it will go to 0 at a angles which is slightly more than pi, so let us say that that angle is pi + epsilon and at this angle pi + epsilon the scattering length will change sign from plus to minus and then between this value which is plus pi + epsilon to 3 pi by 2 the scattering lengths will remain negative okay. So, that is how the scattering length changes its sign according to this prescription.

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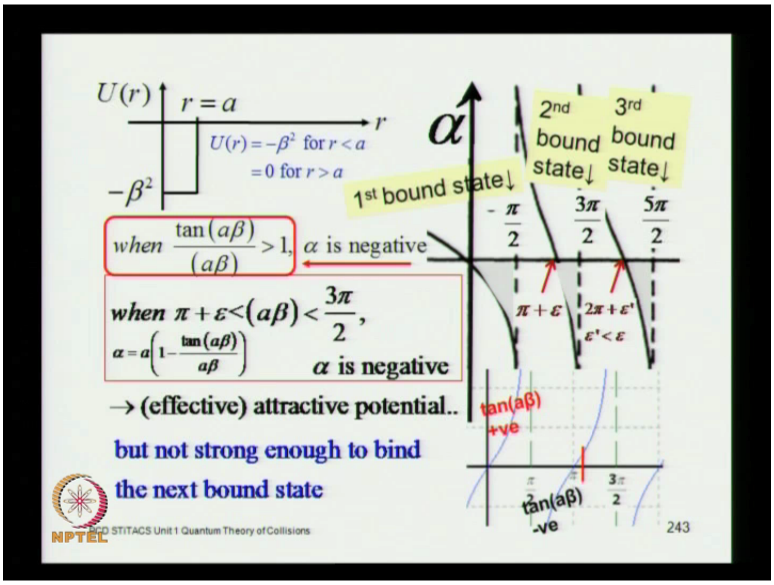

$U(r) = -\beta^2$ for $r < a$

$= 0$ for $r > a$

when $\frac{\tan(a\beta)}{(a\beta)} > 1$, α is negative

when $\pi + \varepsilon < (a\beta) < \frac{3\pi}{2}$,

$\alpha = a\left(1 - \frac{\tan(a\beta)}{a\beta}\right)$ **α is negative**

→ (effective) attractive potential..

but not strong enough to bind

the next bound state

And here it is a plot which shows you how the scattering length changes for an attractive potential as you increase the potential strength. So, the potential strength you have got this vertical line on the right of this the potential strength is attractive on the left of this it is repulsive okay. So, it is some measure of the strength of the potential.

And as you go toward right more and more toward right the potential becomes more and more strong. And it becomes capable of having more and more number of bound states as you go from left to right in this graph. Now if you see the scattering length goes to zero at pi +s epsilon over here. So, this is not the center point between pi by 2 and 3 pi by 2 it is slightly off the center toward the right.

The next time it will, the next zero will not be exactly between 3 pi by 2 and 5 pi by 2 it will again be off the center toward the right. But not exactly the same value because theta has increased okay, which is this angle a beta. So, the tangent will have to be a slightly different value. So, you will get that at a value of an angle which is slightly less than what it was in the previous case.

So, you will get 2 pi + epsilon prime when epsilon prime is less than epsilon, so that it is slightly shifted okay. So, these are not uniformly spaced zeros. So, this is when the scattering length changes its sign and this really has some very fascinating applications both in nuclear physics and atomic physics.

Because controlling the scattering length is a tool which is available to experimental is because they can have some external conditions with which they can fine tune the scattering length and have the potential either attractive or repulsive depending on how you tune it. So, it gives you, it gives the experimentalist certain tunability on the physics and the kind of the experiment that he is going to perform.

So, this is what we have and when you have an attractive potential you only have the necessary condition. So, if you look at the necessary condition alone you are not guaranteed that you are going to have a bound state or you are going to have the next bound state. Because if you look at this region over here this is the triangular region it is not of course exactly a triangle this is an arc.

But if you look at this segment this scattering length of course is negative over here. But it does not make the potential capable of having even a single bound state. Likewise it is negative in this region also which is in this shaded area okay. In this shaded area also the scattering length is negative as you can see. But it does not give the potential a capacity to get the second bound state.

To get the second bound state you have to hit 3 pi by 2 okay. And then you have another shaded area over here in which the scattering length is negative but you do not get the third bound state. To get the third bound state you have to get to 5 pi by 2 okay. And only then you pick up an additional bound state. So, you have the first bound state at pi by 2 the second at 3 pi by 2 under third at 5 pi by 2.

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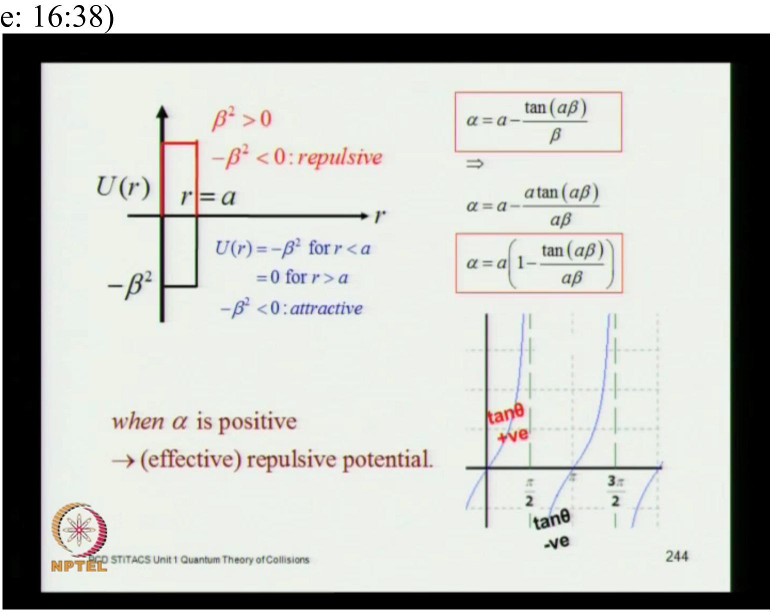

$\beta^2 > 0$

$-\beta^2 < 0$: *repulsive*

$U(r) = -\beta^2$ for $r < a$

$= 0$ for $r > a$

$-\beta^2 < 0$: *attractive*

$$\boxed{\alpha = a - \frac{\tan(a\beta)}{\beta}}$$

$$\Rightarrow$$

$$\alpha = a - \frac{a\tan(a\beta)}{a\beta}$$

$$\boxed{\alpha = a\left(1 - \frac{\tan(a\beta)}{a\beta}\right)}$$

when α is positive

→ (effective) repulsive potential.

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Now, a positive scattering length is not when you can have a bound state at all so it has some similarity with a repulsive potential because you just cannot have bound state under those conditions.

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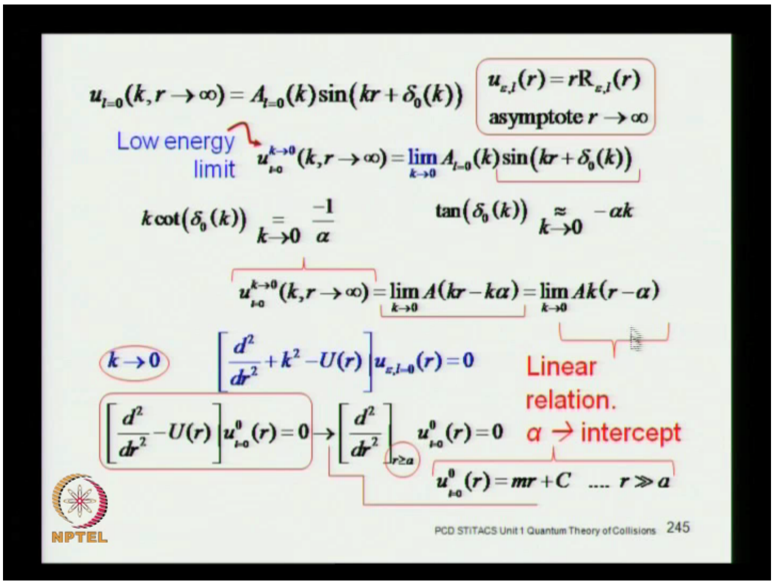

$$u_{l=0}(k,r\to\infty)=A_{l=0}(k)\sin(kr+\delta_0(k))$$

$$\boxed{\begin{array}{c}u_{\varepsilon,l}(r)=rR_{\varepsilon,l}(r)\\ \text{asymptote } r\to\infty\end{array}}$$

Low energy limit

$$u_{l=0}^{k\to 0}(k,r\to\infty)=\lim_{k\to 0}A_{l=0}(k)\underbrace{\sin(kr+\delta_0(k))}$$

$$k\cot(\delta_0(k))\underset{k\to 0}{=}\frac{-1}{\alpha}\qquad \tan(\delta_0(k))\underset{k\to 0}{\approx}-\alpha k$$

$$\overbrace{u_{l=0}^{k\to 0}(k,r\to\infty)}=\lim_{k\to 0}\underbrace{A(kr-k\alpha)}=\lim_{k\to 0}\underbrace{Ak(r-\alpha)}$$

$$k\to 0\qquad\left[\frac{d^2}{dr^2}+k^2-U(r)\right]u_{\varepsilon,l=0}(r)=0$$

$$\boxed{\left[\frac{d^2}{dr^2}-U(r)\right]u_{l=0}^{0}(r)=0}\to\left[\frac{d^2}{dr^2}\right]_{r\geq a}u_{l=0}^{0}(r)=0$$

Linear relation.

α → intercept

$$\overbrace{u_{l=0}^{0}(r)=mr+C}\quad\dots\quad r\gg a$$

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So, now let us look at the asymptotic solution for the S waves. Now as r tends to infinity your solution to the Schrodinger equation is this capital R this little u is a function which is defined as little r times the capital R which is the solution right. And the solution, this solution is a sine wave as we have seen for l greater than 0 you know you have got a sinusoidal solution which is phase shifted by l pi by 2 + a scattering phase shift which is delta.

But for the S waves l = 0, so the solution in the asymptotic region are tending to infinity is a sinusoidal function with argument kr + delta where delta 0 is the S wave scattering phase shift and we have seen how it behaves and how it is connected with the number of bound states through the Levinson theorem. Now the low energy limit of this is obtained by taking the limit k tending to zero.

Because h cross square k square by 2m is the energy. So, this is the low energy limit and I recapitulate the definition of the scattering length alpha which is given by these relations and this is the low energy limit of the scattering length right. The tangent of this function is minus alpha times k. So, now if you look at the low energy limit of this function u which is the radial function multiplied by the radial distance.

So, this function u in the asymptotic region when the scattering phase shifts are small this will become kr but this delta is the same as - alpha k, so you get almost a linear relationship between u and r, this is an equation to a straight line as you can see right. And you would expect it to be so, because if you look at the radial Schrodinger equation and write it in terms of the function u rather than the radial function r itself okay.

Then u must satisfy this differential equation in the limit k going to 0 this term k square drops out so you get only d2 by dr 2 – U. This is the potential operating on the function equal to 0 this is the form of the Schrodinger equation or the reduced Schrodinger equation. And then for the asymptotic region r greater than a, outside this range the potential vanishes.

So, the effective differential equation that you have to worry about is only the second derivative of u going to 0 okay. And if the first if the second derivative is 0, the first derivative is a constant okay and if the first derivative is a constant you have got a linear relationship which is u = mr + C, so you get a linear relationship.

So, that is what you have over here you get essentially a linear relationship and you recognize that the scattering length now you can assign a geometrical meaning to it. It is, if you compare it with the equation to the straight line you find that it is essentially the intercept right.

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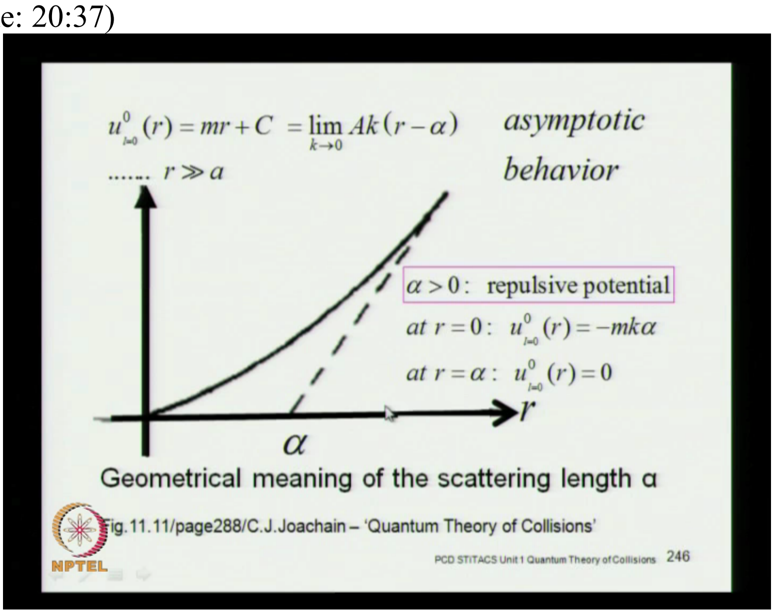

$u^0_{l=0}(r) = mr + C \; = \lim_{k\to 0} Ak(r-\alpha)$ *asymptotic behavior*

$\ldots\ldots \; r \gg a$

$\alpha > 0$: repulsive potential

at $r = 0$: $u^0_{l=0}(r) = -mk\alpha$

at $r = \alpha$: $u^0_{l=0}(r) = 0$

Geometrical meaning of the scattering length α

Fig.11.11/page288/C.J.Joachain – 'Quantum Theory of Collisions'

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So, this is the linear relationship that you get and this is the geometrical meaning of the scattering length. So, if you, here alpha is greater than 0, so it is a positive quantity which is what you have for a repulsive potential as I mentioned earlier. And at r = 0, which is here, this function will need to be somewhat negative at - mk alpha right.

Whereas at r = alpha which is here okay the function would go to 0. So, that is the geometrical behaviour of the scattering wavelength scattering length.

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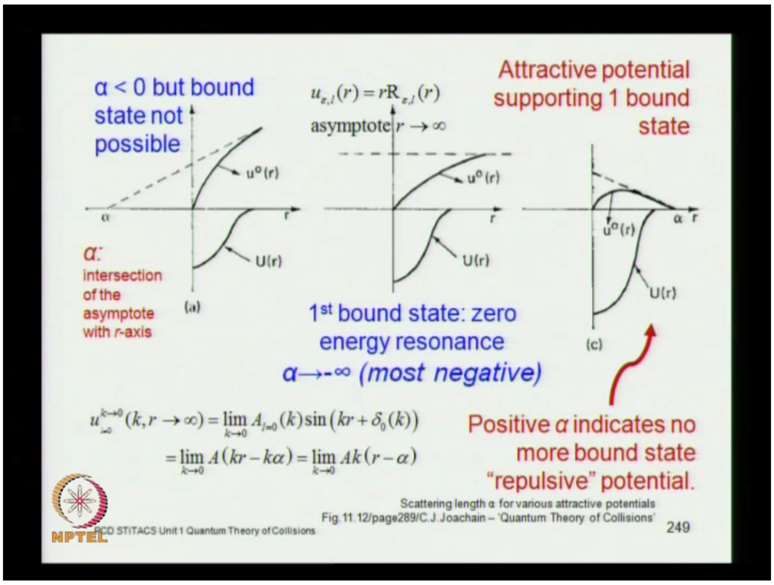

α < 0 but bound state not possible

α: intersection of the asymptote with *r*-axis

$u_{\varepsilon,l}(r) = r\mathrm{R}_{\varepsilon,l}(r)$

asymptote $r \to \infty$

1st bound state: zero energy resonance

α→-∞ *(most negative)*

Attractive potential supporting 1 bound state

Positive *α* indicates no more bound state "repulsive" potential.

$$u_{l=0}^{k\to 0}(k, r\to\infty) = \lim_{k\to 0} A_{l=0}(k)\sin\left(kr + \delta_0(k)\right)$$

$$= \lim_{k\to 0} A\left(kr - k\alpha\right) = \lim_{k\to 0} Ak\left(r - \alpha\right)$$

Scattering length α for various attractive potentials

Fig. 11.12/page289/C.J.Joachain – 'Quantum Theory of Collisions'

Now you have seen that the scattering length must be negative but it does not guarantee that the bound state exists. So here is the case of an attractive potential this is U, this is the zero of the potential this is below the zero. So, this is a negative potential which is an attractive potential.

So, you do have an attractive potential the scattering length is negative but not sufficiently negative to have a bound state. So, here the scattering length is negative this is the zero of the distance axis which runs from left to right. On the right of the zero you have got positive distances, on the left of the zero you have got negative distances.

So, the scattering length is negative in this picture as you can see right. But it is finite look at this intercept okay. You want the scattering lengths to become as negative as it can. So, this intercept has to go farther and farther and farther toward minus infinity right, only then you are going to have the bounce rate. So, that is what you get when it goes farther and farther this asymptotic line becomes parallel to this line.

Because parallel lines they told me meet at infinity I have never verified it. But I think they do okay. So, this parallel line this asymptote it will meet at infinity and that is the negative maximum scattering length that you can get and that is when you get the first bound state okay that is when you have the zero energy resonance okay that is the half bound state that we are referring to.

And then of course when the potential becomes a little bit stronger. So, here this is the potential you see that it has become a little stronger than what it was in the previous diagram. And in the next diagram we make it even more stronger so you see that it is a little deeper now. And now the scattering length just because it has become deeper does not mean that it will automatically pick the next bound state.

Because the scattering length has now become positive and in fact you will not get a bound state under these conditions you will get a bound state only when the scattering length again becomes negative and not merely negative but the most negative that it can get. So, those are the conditions that are necessary to pick an additional bound state.

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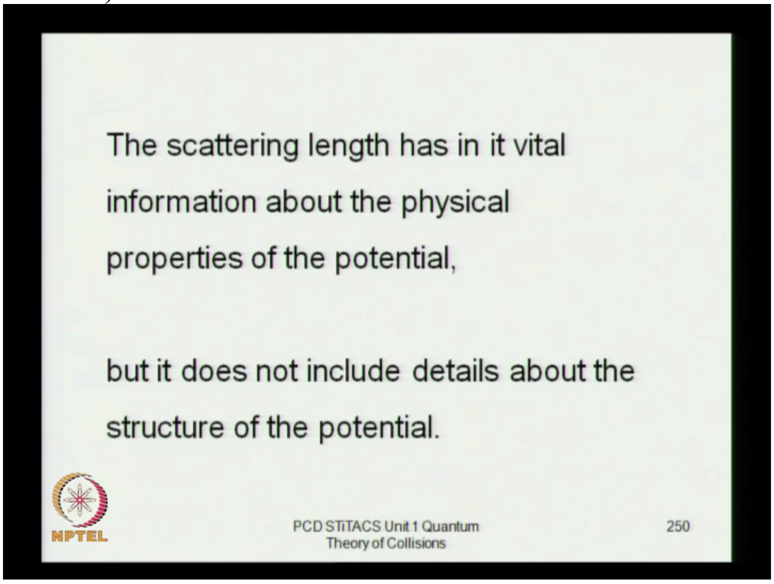

The scattering length has in it vital information about the physical properties of the potential,

but it does not include details about the structure of the potential.

And what where the scattering lengths find lots of applications is the fact that it gives you important idea about the physics of the collision process without having to look at the detailed structure of the potential. Like in the previous diagram we had this potential which is an attractive potential but the shape of the potential is not something like a square well.

But the general results of our discussion remain equally valid for this case as well. We have demonstrated it for a square well potential but it is valid for potentials having other forms also and the actual detailed structure of the potential is not necessary to be seen.

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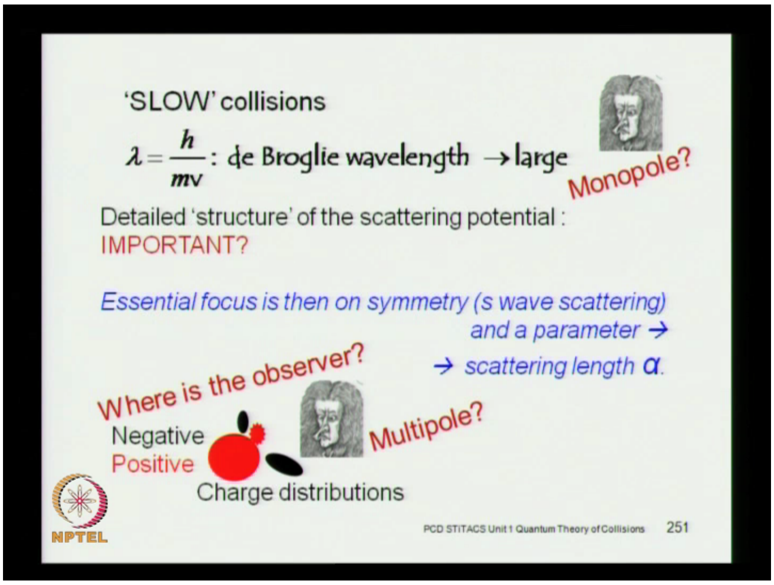

'SLOW' collisions

$\lambda = \frac{h}{mv}$: de Broglie wavelength → large

Monopole?

Detailed 'structure' of the scattering potential : IMPORTANT?

Essential focus is then on symmetry (s wave scattering) and a parameter →

→ scattering length α.

Where is the observer?

Negative

Positive

Multipole?

Charge distributions

And this is a case similar to how you will look at a charge distribution. So, if you are looking at a charge distribution there are some positive charges and some negative charges, so let us say the red positive charges then black are the negative charges they may have some random you know shapes okay. They may be diffuse they may be point charges it does not matter. So, the details of this picture are not really important.

But you may, let us say that you have got various charge distributions positive and negative. Now the question is where are you observing this charge distribution from are you observing it from a point that is close enough okay. There is only a coincidence this observer looks like Newton okay. And if he is close enough he will see a very complicated charge distribution okay. He see's mono pole, dipoles, quadruples, octa poles infinite right.

But if he goes far away, he we will see that okay. Everything is packed together and there is a single charge at a single point and the details of the structure or of the potential that we are talking about in the case of collisions. So, those details do not become relevant and they can be, the effect of that can be parametrized which is what the scattering length does for us. And how does it do it because when you are dealing with slow collisions.

If you look at the De Broglie wavelength because the energies are small the velocities are small and the velocity is being small the wavelength is large. And if a wavelength is really large it is not going to see the details of a small structure okay. Because to see something that is really tiny you need very short wavelength. So, this is a large wavelength and you are not going to see any details.

And the essential focus in S wave scattering at low energies which is the domain of slow illusions that we are discussing in this unit here, the all the essential properties are contained in the symmetry which is the S wave and then it is contained in a parameter which is the scattering length which is alpha.

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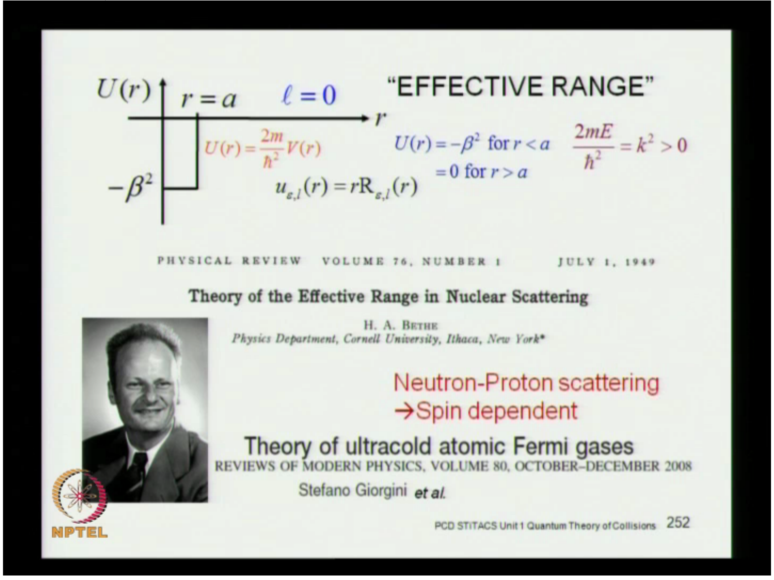

$\ell = 0$

"EFFECTIVE RANGE"

$U(r) = \frac{2m}{\hbar^2} V(r)$

$U(r) = -\beta^2$ for $r < a$
$= 0$ for $r > a$

$\frac{2mE}{\hbar^2} = k^2 > 0$

$u_{\varepsilon,l}(r) = r\mathrm{R}_{\varepsilon,l}(r)$

PHYSICAL REVIEW VOLUME 76, NUMBER 1 JULY 1, 1949

Theory of the Effective Range in Nuclear Scattering

H. A. Bethe
*Physics Department, Cornell University, Ithaca, New York**

Neutron-Proton scattering
→Spin dependent

Theory of ultracold atomic Fermi gases
REVIEWS OF MODERN PHYSICS, VOLUME 80, OCTOBER–DECEMBER 2008
Stefano Giorgini *et al.*

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So, we now introduce what is called as the effective range. So, we again go back to the square well potential because it is very easy to define it for a square well potential. And this formalism was developed by Fermi and Bethe and so on. So, the main formulation that we follow is due to Hans Bethe which came out in a paper in physical Review in 1949. And this was studied in the context of nuclear physics.

In the context of two nucleon scattering, neutron proton scattering and you know and it was found from these studies that these interactions actually depend on the spin state. So, that was a major outcome of these studies and in atomic physics it now has very fascinating applications in getting, you know very cold atoms and BCS or BEC kind of systems by controlling the scattering lengths in the low energy domains.

So, let me give you some brief introduction to this although the subject is quite vast.

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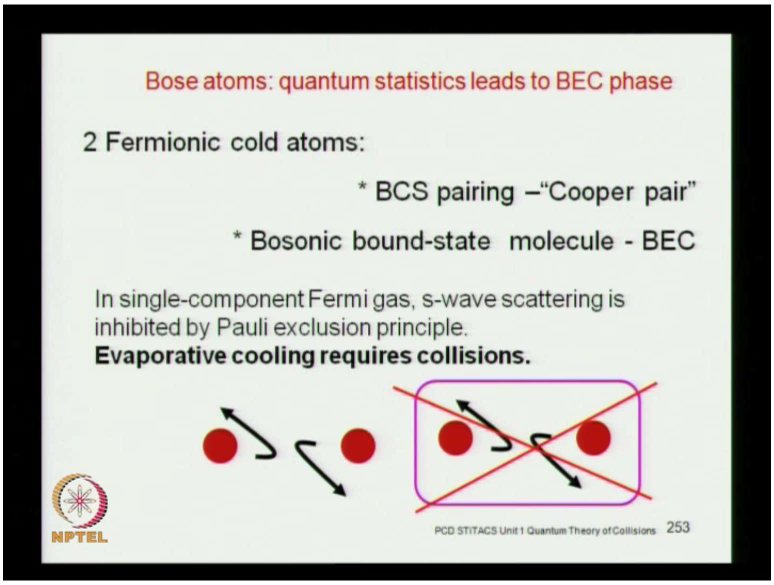

Bose atoms: quantum statistics leads to BEC phase

2 Fermionic cold atoms:

* BCS pairing –"Cooper pair"

* Bosonic bound-state molecule - BEC

In single-component Fermi gas, s-wave scattering is inhibited by Pauli exclusion principle.

Evaporative cooling requires collisions.

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So, if you have a Bose, if atoms are bosonic you can of course seek Bose Einstein condensation because that is the premise which Bose Einstein statistics gives you that there is no stopping there is nothing like a poly barrier which prevents identical particles from condensing it into one state which is what is achieved if you have particles with integral spins.

If you have bosons there is no such barrier the poly barrier or the Fermi Dirac barrier so there are Fermi Dirac particles for which you cannot do it. But for bosons you can get Bose Einstein condensation and this of course as all of you are well aware has been achieved and there is a lot of excitement in atomic physics because of the Bose Einstein condensates of atoms.

Now if you, if your atoms are not bosons but they are Fermion, you can have Fermi atom, you do have Fermi atom right. That depends on the total number of nucleons the atom has. So, now if you have Fermionic atoms you cool them and then you can have two kinds of; you know situations one is that you can have a pairing like the BCS pair as you have in a cooper pair or you can have a bound state molecule.

And whether or not you have a bound state depends on how you control the scattering length. So, you can have a BEC kind of pair or you can have a Cooper pair, so you have two possibilities over here. So, if you have a single component Fermi gas atom which are Fermi gases and you consider low energy scattering. So, the only scattering which is of important at low energies and why low energies that is what cold atoms are.
Because the speeds are very small okay, so the only scattering which is of importance in low energy, at low speeds, at low temperatures you have got low speeds and the only scattering

which is of importance is the S wave scattering which is what we have been discussing right. So, the case of S wave scattering because of the Pauli Exclusion Principle scattering is inhibited because the exchange interaction will tend to keep them apart okay.

And then evaporative cooling will not be achieved but you can always achieve it by having some mixtures of these Fermi atoms with other atoms like Bose atoms okay. So you may have you can have a Bose Fermi mixture and then you can achieve this is sometimes called as sympathetic cooling, what you do is you evaporate Bose atoms okay, because evaporative cooling requires this collision kind of thing. So, you can evaporate the Bose atoms.

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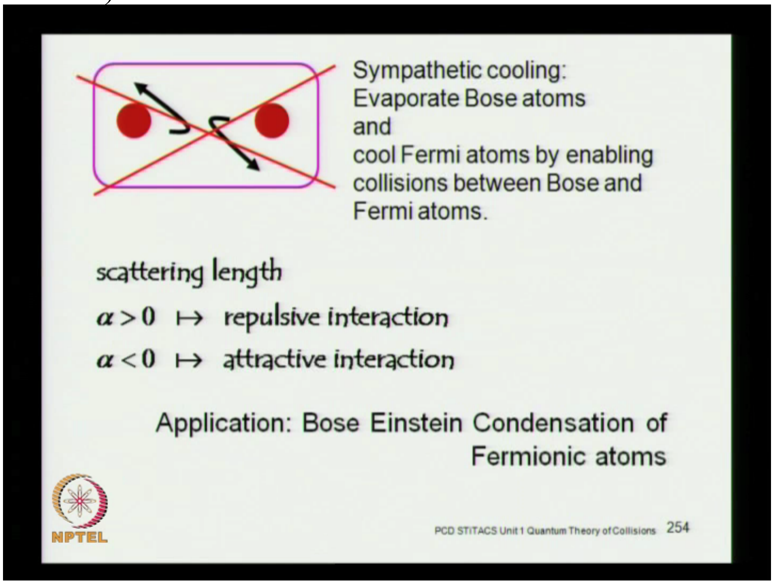

Sympathetic cooling:
Evaporate Bose atoms
and
cool Fermi atoms by enabling collisions between Bose and Fermi atoms.

scattering length

$\alpha > 0 \mapsto$ repulsive interaction

$\alpha < 0 \mapsto$ attractive interaction

Application: Bose Einstein Condensation of Fermionic atoms

And through the collisions between the Bose atoms and the Fermi atoms you can affect further cooling. So, that is one thing that can be done but then what is of importance in the context of the discussion we are having is that you can control the scattering length. Now we have already discussed that the scattering length is positive for repulsive interactions and negative for attractive interactions.

So, in the case of Fermi mixtures you know Fermi to get Bose Einstein condensation of Fermi atoms.

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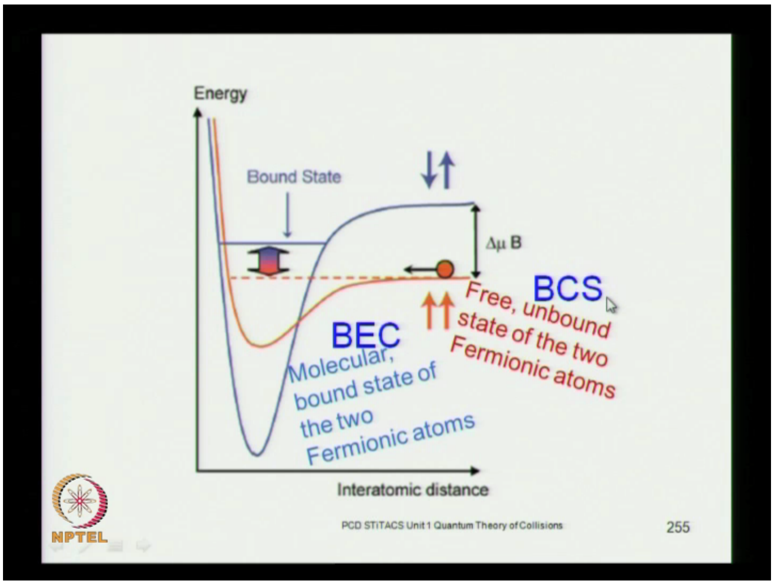

If you look at the energy diagram then you can have two atoms with parallel spins this is how they are in unbound state. So, you have got two atoms which are not bound to each other when they get bound to each other you get a molecule right, so that is a bound state. When they are not bound they are unbound.

And that is when they are free and you have parallel spins between them. But then you can get a bound state whose energy diagram is given by this blue line, so this is the inter-atomic distance okay. And the interim atomic distance if you look at this diagram you can see that as they get closer, if you can get the spin of one of these two flip because the molecular bound state requires S=0.

If you can get the spin to flip okay then you can get a bound state which will have a lower energy than this. So, you can affect a transition from one kind of pairing to another kind of pairing by controlling the spin orientation and how can you do that using a magnetic field okay. So, you can use the magnetic field as a switch and you can affect the transition from the unbound state to the bound state.

Now this is a resonance effect now this will remind you of at least those of you who have looped into the autoionization resonances or the Fano resonances as we sometimes call them. What do you have in it? You have got a transition; you have got two states which are degenerate. You have got a possible bound to bound transition and you also have a possible bound to continuum transition.

Now the bound to bound and the bound to continuum, these can be degenerate and when they are degenerate you have a resonance that is the Fano resonance or in the context of scattering theory it is called as a Fano Feshbach resonance. So, here you have a similar situation you have got a bound state of these two atoms or you can have an unbound state of these two atoms.

But in the unbound state you have got parallel spins in the bound state you have got anti-parallel spins. And in this case you can have when they are unbound you can have a cooper pair kind of pairing between those two atoms and you can still have a Cooper pair, so this is called as BEC, BCS pairing.

But then if you get bound molecular states of these two atoms. Then you actually have a boson and you can induce it by using some other you know cooling techniques into the into the Bose Einstein condensate phase. So that can be achieved by controlling these scattering lengths.

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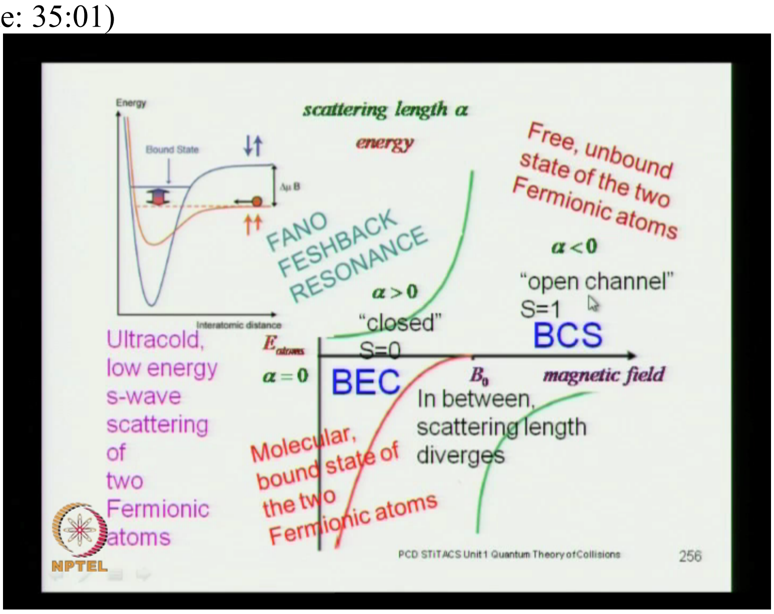

scattering length a

energy

Free, unbound state of the two Fermionic atoms

$a<0$

"open channel"
S=1

BCS

FANO FESHBACK RESONANCE

$a>0$

"closed"
S=0

Ultracold, low energy s-wave scattering of two Fermionic atoms

E_{atoms}

$a=0$

BEC

B_0

magnetic field

In between, scattering length diverges

Molecular, bound state of the two Fermionic atoms

So, these are the Fano Feshbach resonances as they are called and if you see what you are essentially doing is controlling the scattering length by this magnetic field. So, as you can change the magnetic field you change the scattering length it is greater than 0 on one side it is less than zero on the other.

And this is where you have the BCS and BEC kind of transition between the kind of pairs that the two Fermi atoms can make. So, this has got very fascinating applications in atomic physics in the physics of cold atoms and Bose Einstein condensation of Fermi gas atoms, so in between of course the scattering length diverges.

And this is where you have got a, this is what you call it a closed channel and this is an open channel. So, it is very similar to what you have in the Fano autoionization resonances where you have open channels and closed channels. And you can have a BEC molecular bound state of the two Fermion atoms but otherwise you can certainly have a cooper pair kind of BCS phase.

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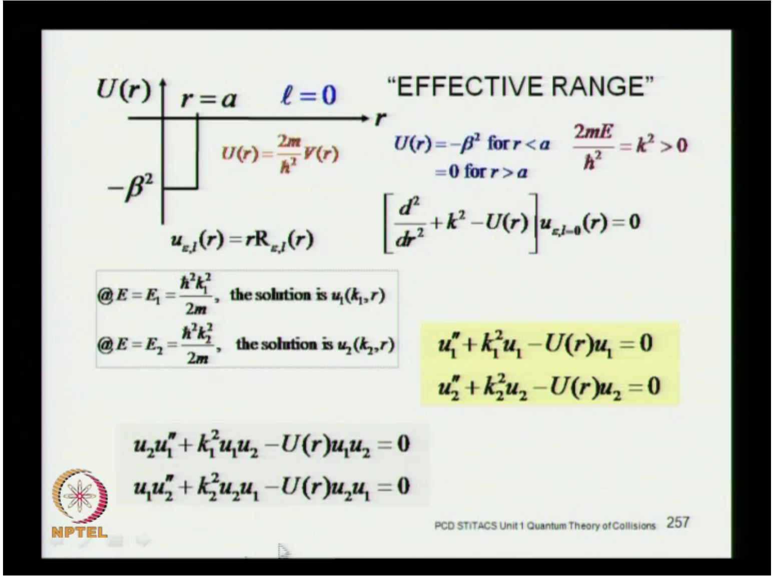

$\ell = 0$

“EFFECTIVE RANGE”

$U(r) = \frac{2m}{\hbar^2} V(r)$

$U(r) = -\beta^2$ for $r < a$
$= 0$ for $r > a$

$\frac{2mE}{\hbar^2} = k^2 > 0$

$u_{\varepsilon,l}(r) = r R_{\varepsilon,l}(r)$

$$\left[\frac{d^2}{dr^2} + k^2 - U(r)\right] u_{\varepsilon,l=0}(r) = 0$$

@ $E = E_1 = \frac{\hbar^2 k_1^2}{2m}$, the solution is $u_1(k_1, r)$
@ $E = E_2 = \frac{\hbar^2 k_2^2}{2m}$, the solution is $u_2(k_2, r)$

$$u_1'' + k_1^2 u_1 - U(r) u_1 = 0$$
$$u_2'' + k_2^2 u_2 - U(r) u_2 = 0$$

$$u_2 u_1'' + k_1^2 u_1 u_2 - U(r) u_1 u_2 = 0$$
$$u_1 u_2'' + k_2^2 u_2 u_1 - U(r) u_2 u_1 = 0$$

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So, that brings us to the effective range formulation of Bethe and I once again take the case of the square well potential because it is very easy to illustrate it for this particular case. And we consider solutions to the differential equation for the function u which is little r times the solution to the Schrodinger equation itself. And I consider solutions at two energies one at energy E1 which is h cross square k1 square by 2m.

And the other at E2 which is h cross square k2 square by 2m and so these are the two energies that I consider for this discussion. Now both u1 and u2 satisfy the differential equation. So, these are the two equations satisfied by u1 and u2 respectively. You can multiply the first equation by u2 and the second equation by u1 which is what gives you these two equations right.

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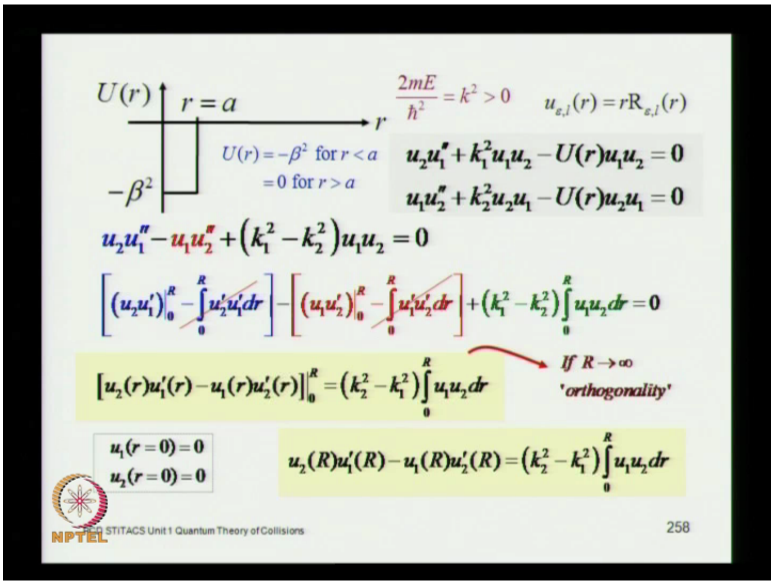

$\frac{2mE}{\hbar^2}=k^2>0$ $u_{\varepsilon,l}(r)=r\mathrm{R}_{\varepsilon,l}(r)$

$U(r)=-\beta^2$ for $r<a$
$=0$ for $r>a$

$$u_2u_1''+k_1^2u_1u_2-U(r)u_1u_2=0$$
$$u_1u_2''+k_2^2u_2u_1-U(r)u_2u_1=0$$

$$u_2u_1''-u_1u_2''+\left(k_1^2-k_2^2\right)u_1u_2=0$$

$$\left[\left(u_2u_1'\right)\Big|_0^R-\cancel{\int_0^R u_2'u_1'dr}\right]-\left[\left(u_1u_2'\right)\Big|_0^R-\cancel{\int_0^R u_1'u_2'dr}\right]+\left(k_1^2-k_2^2\right)\int_0^R u_1u_2dr=0$$

$$\left[u_2(r)u_1'(r)-u_1(r)u_2'(r)\right]\Big|_0^R=\left(k_2^2-k_1^2\right)\int_0^R u_1u_2dr$$

If $R\to\infty$
'orthogonality'

$u_1(r=0)=0$
$u_2(r=0)=0$

$$u_2(R)u_1'(R)-u_1(R)u_2'(R)=\left(k_2^2-k_1^2\right)\int_0^R u_1u_2dr$$

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And then you can subtract the second equation from the first so this is a very simple manipulation that I am doing. And having done this subtraction you integrate every term from 0 to R okay. What is capital R does not matter it is some value okay. It is of course not the radial function it is a certain distance up to which you are carrying out this integration. So, carry out this integration and you carry out term by term.

So, the integration of u, u double prime will give you these two terms. Integration of this will give you these two terms and you have to take the differences at the upper value little r equal to capital R and little r = 0 because you are carrying out this integration from 0 to R. Now you have this difference from 0 to R okay. So, we keep the value capital R as long as we need it and finally we will get rid of it.

So, this is the difference of this product u2 u1 prime at upper value of R and the lower value of R, so you must take the difference. And this for u2 u1 prime from the first term and here you get a similar term from the second term which is u1 u2 prime. And this difference must be equal to I take this term to the right hand side, so instead of k1 square - k2 square it becomes k2 square – k1 square.

So look at the integral on the right hand side if in that right hand side if the value of r happened to be infinity you would really have an orthogonal ready integral between u1 and u2 okay. So, now we already know that these functions both u1 and u2 these are solutions to the Schrodinger equation. Actually this times little r is the solution to the Schrodinger equation right.

So, at r = 0 these must be 0 right, so which means that the only value which is of importance is the value at uppercase R but the quantities that we are looking at will not depend on this so

that is yet to come. So, now we have gotten rid of the values of these factors at r = 0, so this is the expression that we get okay right.
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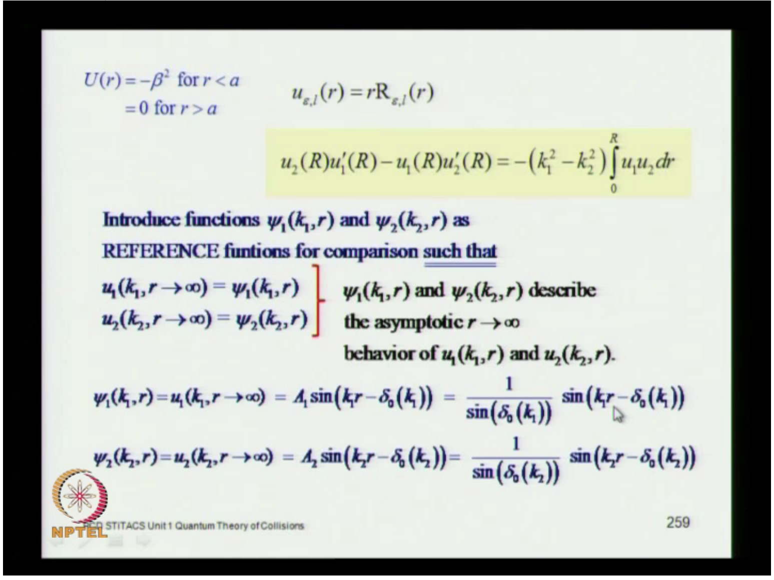

$$U(r) = -\beta^2 \text{ for } r < a$$
$$= 0 \text{ for } r > a$$

$$u_{\varepsilon,l}(r) = r\mathrm{R}_{\varepsilon,l}(r)$$

$$u_2(R)u_1'(R) - u_1(R)u_2'(R) = -\left(k_1^2 - k_2^2\right)\int_0^R u_1 u_2 dr$$

Introduce functions $\psi_1(k_1,r)$ and $\psi_2(k_2,r)$ as REFERENCE funtions for comparison such that

$$\left.\begin{array}{l} u_1(k_1, r\to\infty) = \psi_1(k_1,r) \\ u_2(k_2, r\to\infty) = \psi_2(k_2,r) \end{array}\right]$$

$\psi_1(k_1,r)$ and $\psi_2(k_2,r)$ describe the asymptotic $r\to\infty$ behavior of $u_1(k_1,r)$ and $u_2(k_2,r)$.

$$\psi_1(k_1,r) = u_1(k_1, r\to\infty) = A_1\sin\left(k_1r - \delta_0(k_1)\right) = \frac{1}{\sin\left(\delta_0(k_1)\right)}\sin\left(k_1r - \delta_0(k_1)\right)$$

$$\psi_2(k_2,r) = u_2(k_2, r\to\infty) = A_2\sin\left(k_2r - \delta_0(k_2)\right) = \frac{1}{\sin\left(\delta_0(k_2)\right)}\sin\left(k_2r - \delta_0(k_2)\right)$$

This is what we have, now we define two functions these are Psi1 and Psi 2, this is not the solution to the Schrodinger equation side that we started out with. These are two new functions that I have defined and how are they defined? They are defined to be such that they are exactly equal to the asymptotic behaviour of the functions u and that is known. Because the asymptotic behaviour of the function u is that it is a sinusoidal function.

The argument is kr + the scattering phase shift, so that is known. So, these describe the asymptotic functions so that is the definition of Psi1 and Psi2. So, these two functions are defined and I choose the normalization as 1 over sine delta the reason I choose this normalization is because if you look at this function Psi.

And find this value at r = 0 then put r = 0 over here this will become 0. And then you get sine delta over sine delta, so you will get essentially unit normalization at r = 0 okay. So, the normalization is chosen so that you get unit normalization at r =0, so the size is fixed. So, now the normalization is fixed.
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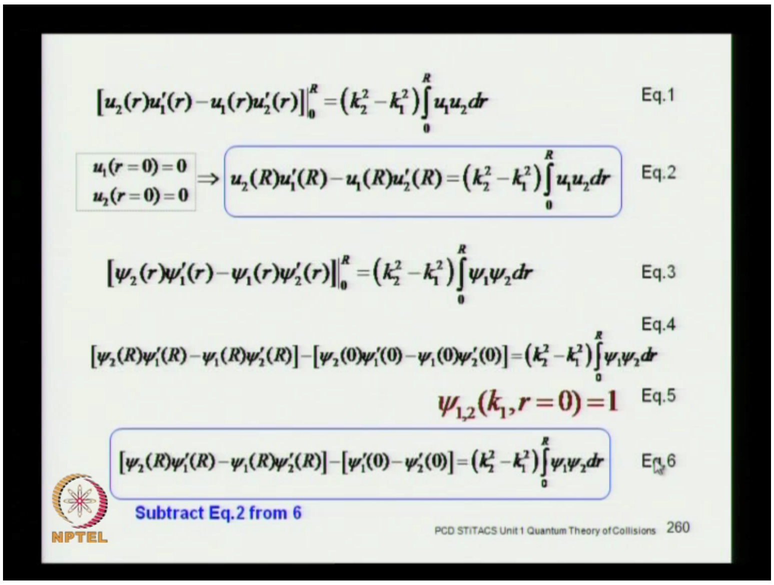

$$\left[u_2(r)u_1'(r)-u_1(r)u_2'(r)\right]\Big|_0^R=\left(k_2^2-k_1^2\right)\int_0^R u_1u_2\,dr$$ Eq.1

$$u_1(r=0)=0$$
$$u_2(r=0)=0$$
$$\Rightarrow \boxed{u_2(R)u_1'(R)-u_1(R)u_2'(R)=\left(k_2^2-k_1^2\right)\int_0^R u_1u_2\,dr}$$ Eq.2

$$\left[\psi_2(r)\psi_1'(r)-\psi_1(r)\psi_2'(r)\right]\Big|_0^R=\left(k_2^2-k_1^2\right)\int_0^R \psi_1\psi_2\,dr$$ Eq.3

Eq.4
$$\left[\psi_2(R)\psi_1'(R)-\psi_1(R)\psi_2'(R)\right]-\left[\psi_2(0)\psi_1'(0)-\psi_1(0)\psi_2'(0)\right]=\left(k_2^2-k_1^2\right)\int_0^R \psi_1\psi_2\,dr$$

$$\psi_{1,2}(k_1,r=0)=1$$ Eq.5

$$\boxed{\left[\psi_2(R)\psi_1'(R)-\psi_1(R)\psi_2'(R)\right]-\left[\psi_1'(0)-\psi_2'(0)\right]=\left(k_2^2-k_1^2\right)\int_0^R \psi_1\psi_2\,dr}$$ Eq.6

Subtract Eq.2 from 6

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Now this is our expression here with u1 and u2 both at r = 0 to be = 0. Now the only value which is of importance is the value at upper limit R and you have got an exactly similar expression for Psi. Because Psi is after all, by definition give you the asymptotic form for the same functions which already are solutions to the Schrodinger equation.

So, you will get an exactly similar relationship. So, you have got two relations now one is u the other is Psi. Notice the similarity between the two but notice also the differences between the two. These are coming from the solution to our equation right for our potential problem it includes a scattering phase shift. This is defined to be equal to the functions with respect to the asymptotic behaviour r tending to infinity.

So, that is the definition of Psi. Now having defined them this is a very smart way of defining the function Psi because now you can just carry out; expand this term so you take the difference between Psi to Psi1 prime at the upper limit R and the lower limits 0. So this is a limit at the lower limit, this is the value of this term at the upper limit and this is the value at the lower limit.

But then if the lower limit at R= 0, you have chosen the normalization, so you know what is Psi1 of 0 that is 1 right, that is how you chose the normalization. So, having chosen the normalization like that you get this factor Psi20 and Psi10 are both equal to 1, so you get this difference here.

So, instead of this product you get only one of them, the other is equal to 1. And now I have these equations which I have numbered because what I am going to do is to subtract from this equation 2, equation 6. So let me write equation 2 and 6 once again at the top of the next slide and subtract equation 6 from equation 2.

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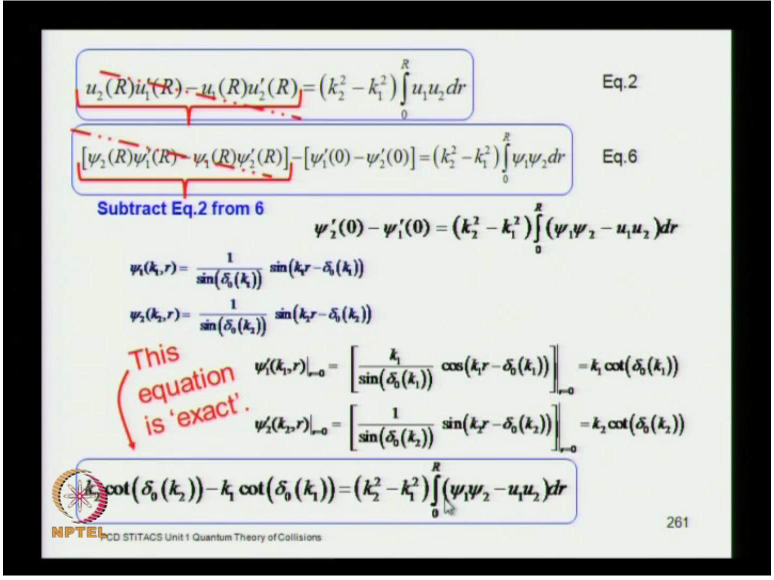

$$u_2(R)u_1'(R)-u_1(R)u_2'(R)=\left(k_2^2-k_1^2\right)\int_0^R u_1u_2\,dr$$ Eq.2

$$\left[\psi_2(R)\psi_1'(R)-\psi_1(R)\psi_2'(R)\right]-\left[\psi_1'(0)-\psi_2'(0)\right]=\left(k_2^2-k_1^2\right)\int_0^R \psi_1\psi_2\,dr$$ Eq.6

Subtract Eq.2 from 6

$$\psi_2'(0)-\psi_1'(0)=\left(k_2^2-k_1^2\right)\int_0^R\left(\psi_1\psi_2-u_1u_2\right)dr$$

$$\psi_1(k_1,r)=\frac{1}{\sin(\delta_0(k_1))}\sin(k_1r-\delta_0(k_1))$$

$$\psi_2(k_2,r)=\frac{1}{\sin(\delta_0(k_2))}\sin(k_2r-\delta_0(k_2))$$

$$\psi_1'(k_1,r)\big|_{r=0}=\left[\frac{k_1}{\sin(\delta_0(k_1))}\cos(k_1r-\delta_0(k_1))\right]\Bigg|_{r=0}=k_1\cot(\delta_0(k_1))$$

$$\psi_2'(k_2,r)\big|_{r=0}=\left[\frac{1}{\sin(\delta_0(k_2))}\sin(k_2r-\delta_0(k_2))\right]\Bigg|_{r=0}=k_2\cot(\delta_0(k_2))$$

This equation is 'exact'.

$$\boxed{k_2\cot(\delta_0(k_2))-k_1\cot(\delta_0(k_1))=\left(k_2^2-k_1^2\right)\int_0^R\left(\psi_1\psi_2-u_1u_2\right)dr}$$

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So, let us do that now let us carry out the subtraction. Now how was this defined, now this is some large value it up and does not matter what it is okay. Its actual value is not important but we know that it is a large value and because it is large, in this region, in the asymptotic region Psi and u are the same because that is how you define Psi. You chose Psi to be such that they agreed with the asymptotic forms of the functions u.

Which means that these two terms cancel each other okay when you subtract equation 6 from equation 2 this will cancel each other and now you have only this term equal to the difference of the right hand sides. The difference of the left hand side amounts only to this term and the difference of the right hand sides you must consider. So, this is the difference of the left hand side there is a minus sign over here.

So I have now written it as Psi2 prime – Psi1 prime at R = 0 and this is equal to k2 square - k1 square integral 0 to R of this difference Psi1 Psi2 - u1 u2 this is integrated from 0 to R . Now we already know what Psi1 and Psi2 are these functions have been strictly defined. So you can evaluate this, their derivatives automatically give you the cotangent function times the inverse wavelength k1 right.

Because this is just the original function being this, the derivative is k cos this theta divided by sine theta. So, you get these two terms and you can put these two on the left hand side and you get k2 cotangent of the phase shift at energy 2 - k1 times the cotangent of the phase shift at energy 1. On the left hand side and on the right hand side you get this difference times this integral from zero to R. Now there is an approximation so far.

Now take the limit R going to the infinity and if you do that you will convince yourself that what you chose your R to be really did not matter because we are interested in the limit R going to infinity okay. So, take the limit R going to infinity. So now the upper limit I have written as infinity over here.

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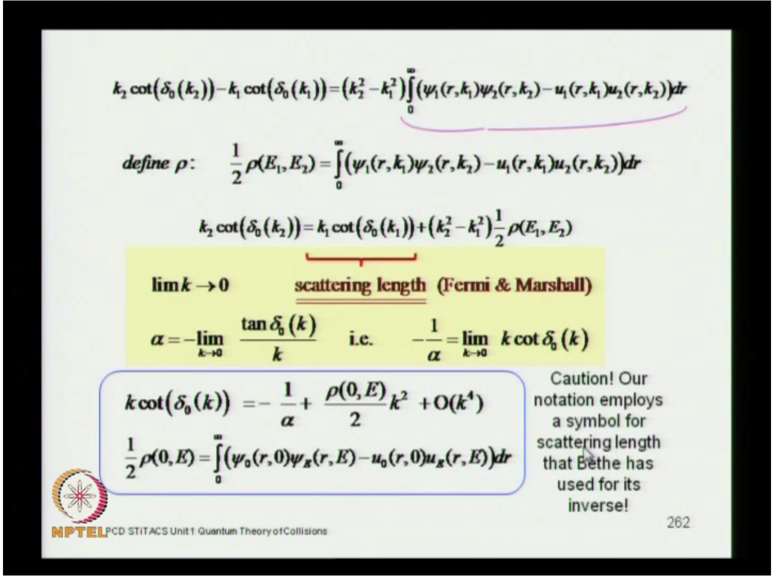

$$k_2\cot(\delta_0(k_2))-k_1\cot(\delta_0(k_1))=(k_2^2-k_1^2)\underbrace{\int_0^\infty(\psi_1(r,k_1)\psi_2(r,k_2)-u_1(r,k_1)u_2(r,k_2))dr}_{}$$

define ρ: $$\frac{1}{2}\rho(E_1,E_2)=\int_0^\infty(\psi_1(r,k_1)\psi_2(r,k_2)-u_1(r,k_1)u_2(r,k_2))dr$$

$$k_2\cot(\delta_0(k_2))=\underbrace{k_1\cot(\delta_0(k_1))}_{}+(k_2^2-k_1^2)\frac{1}{2}\rho(E_1,E_2)$$

$\lim k\to 0$ scattering length (Fermi & Marshall)

$$\alpha=-\lim_{k\to 0}\frac{\tan\delta_0(k)}{k}\quad \text{i.e.}\quad -\frac{1}{\alpha}=\lim_{k\to 0}k\cot\delta_0(k)$$

$$k\cot(\delta_0(k))=-\frac{1}{\alpha}+\frac{\rho(0,E)}{2}k^2+O(k^4)$$

$$\frac{1}{2}\rho(0,E)=\int_0^\infty(\psi_0(r,0)\psi_E(r,E)-u_0(r,0)u_E(r,E))dr$$

Caution! Our notation employs a symbol for scattering length that Bethe has used for its inverse!

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And it is the same expression as I had on the previous slide with the difference that the upper limit is now put at infinity or going to infinity. So, this integral which is from 0 to infinity this we introduce a compact notation for this, this was done by Bethe in the context of nuclear physics. And this was the definition of half Rho okay this is still not the range of the, not the definition of the effective range but it is getting there.

So, this is defined as half Rho it depends on two energies one is the energy k1 the other is the energy k2 okay. So, the corresponding energy is r h cross square k1 square by 2m and h cross square k2 square by 2m, so those are the two energies with respect to which this half Rho is defined as this integral. And in terms of this integral k2 cotangent phase shift is equal to this term will go to the right.

Which is k1 cotangent of this phase shift + k2 square - k1 square times half of Rho okay. Now if you take the low energy limit for E1, you let E1 go to 0 or k1 go to 0 then kcot delta at k = k1, will give you the scattering length right. It will give you the inverse of the scattering like -1 over alpha. So, you can put - 1 over alpha by letting k1 and go to 0. And now you do not have two k's to talk about because one of them has gone to zero.

You are taking the lower energy limit, so other need not be called as the second k or k2, we will call it as k okay. So kcot delta the left hand side which is now kcot delta becomes equal to k1 cot delta which is now -1 over alpha + k2 square - k1 square, k1 squared has gone to zero, so you are left with only k2 square.

But k2 is now what you are calling as k square, so you have got k square and then you have got Rho over 2 this depended on two energies E1 and E2 and E1is gone to 0 and the second is E. So, Rho now is defined in terms of two energy parameters 0 and E, one is the zero energy and second is E. Now you this is your expression for half Rho.

Now I should caution you that our notation is slightly different from what Bethe used in his paper our alpha is inverse of what he has used for alpha, he has used 1 over a for what we have used as alpha.

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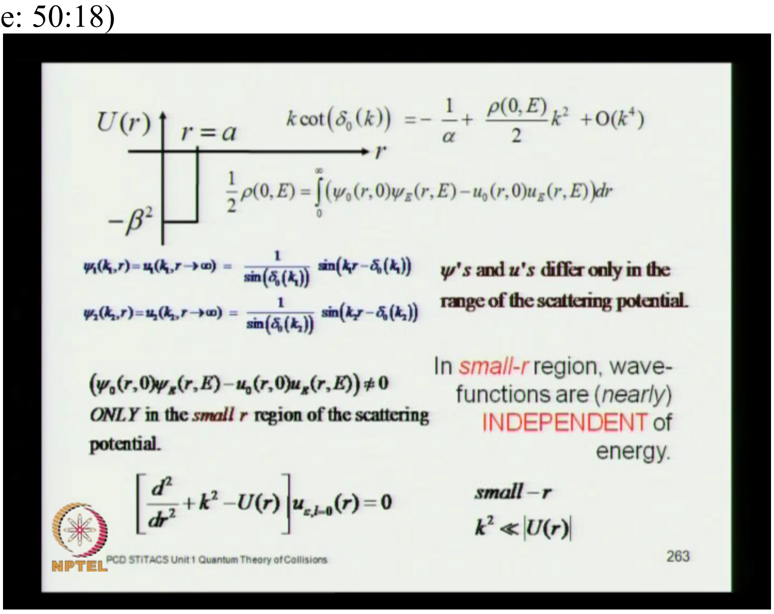

$$k\cot(\delta_0(k)) = -\frac{1}{\alpha} + \frac{\rho(0,E)}{2}k^2 + O(k^4)$$

$$\frac{1}{2}\rho(0,E) = \int_0^\infty (\psi_0(r,0)\psi_E(r,E) - u_0(r,0)u_E(r,E))dr$$

$$\psi_1(k_1,r) = u_1(k_1, r\to\infty) = \frac{1}{\sin(\delta_0(k_1))}\sin(k_1 r - \delta_0(k_1))$$

$$\psi_2(k_2,r) = u_2(k_2, r\to\infty) = \frac{1}{\sin(\delta_0(k_2))}\sin(k_2 r - \delta_0(k_2))$$

ψ's and u's differ only in the range of the scattering potential.

$$(\psi_0(r,0)\psi_E(r,E) - u_0(r,0)u_E(r,E)) \neq 0$$

ONLY **in the *small r* region of the scattering potential.**

In *small-r* region, wave-functions are (*nearly*) INDEPENDENT of energy.

$$\left[\frac{d^2}{dr^2} + k^2 - U(r)\right]u_{E,l=0}(r) = 0$$

small – r

$$k^2 \ll |U(r)|$$

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So, now let us ask ourselves what is the difference between Rho and u. The Psi and u okay u’s are the original functions that we started working with, Psi are the auxiliary functions that we defined such that they had the same asymptotic behaviour as u. So if Psi and u differ at all they differ from each other only where the potential is because the asymptotic forms are the same by definition by our choice okay.

So, they are different only in the range of the scattering potential and that is the small r behaviour that is the small r range. In the asymptotic region as R tends to infinity there is no difference between them. So, they are different only in the small r region and when you look at atomic properties in the small r region.

Whenever you look at any property in the small r region, all properties atomic properties which depend on small r they are independent of energy. To take an example you have seen that radial functions go as r to the power l as r tends to 0 right that is a small r part. Small r behaviour is r to the power l, is it the same for 2s, is it the same for 3s, is it the same for 10s, is it the same for 2p, 3p, 10p, 100p no matter what the energy is.

The energy of 2p, 10p, 100p are different but as r goes to zero they all go as r to the power 1 all the d wave solutions will go as r to the power 2 okay. So, the small r behaviour becomes independent of energy the reason it happens is that as you go to small r behaviour this term overtakes. And at low energy no matter how small you know the potential is. This term will be greater than k square, k square is energy. So k square becomes irrelevant okay.

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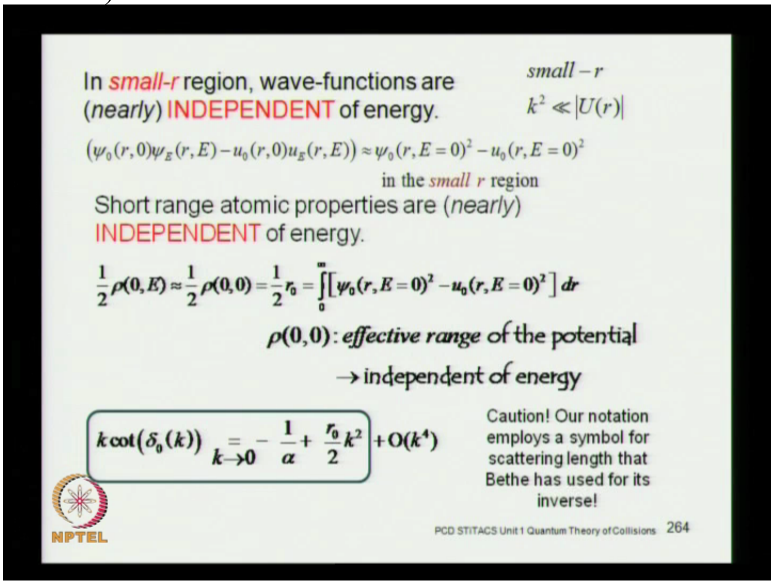

In *small-r* region, wave-functions are (*nearly*) INDEPENDENT of energy.

$small-r$

$k^2 \ll |U(r)|$

$$(\psi_0(r,0)\psi_E(r,E)-u_0(r,0)u_E(r,E)) \approx \psi_0(r,E=0)^2-u_0(r,E=0)^2$$

in the *small r* region

Short range atomic properties are (*nearly*) INDEPENDENT of energy.

$$\frac{1}{2}\rho(0,E) \approx \frac{1}{2}\rho(0,0) = \frac{1}{2}r_0 = \int_0^\infty \left[\psi_0(r,E=0)^2 - u_0(r,E=0)^2\right] dr$$

$\rho(0,0)$: *effective range of the potential*

→ independent of energy

$$\boxed{k\cot(\delta_0(k)) \underset{k\to 0}{=} -\frac{1}{\alpha}+\frac{r_0}{2}k^2} + O(k^4)$$

Caution! Our notation employs a symbol for scattering length that Bethe has used for its inverse!

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And you have the short range atomic properties become independent of energy and that means that this quantity that you have defined as an integral from 0 to infinity and this integral was the integrand had this factor it had the difference between Psi and u okay. But these two differ from each other only in the small r region not in the asymptotic region.

So, what is being integrated, the integrand is made up of such terms which become independent of energy. So, this energy parameter over here you can get rid of because and then this factor Psi0 at 0 energy times Psi at energy E not equal to 0 becomes nearly these two become nearly equal and this becomes Psi0 square.

Likewise the second term becomes u0 square for the same reason okay because they depend only on the small r behaviour. So, in the smaller behaviour now you have got a quantity which does not depend on energy and you now have the definition of the effective range of the potential. Because it depends neither on the energy nor on the capital R and because that is already taken care of you have taken the limit R going to infinity.

So, this is what is called as the effective range of the potential and in terms of this you can write this factor because you have got this half Rho, so the half Rho has come here and then you have got a k square behaviour. Then of course there will be some corrections the next order correction is of the order of k to the 4 okay.

So to the leading terms the scattering length -1 over alpha to this if this is your approximation to the k cot delta if you want to correct this with the next order in k, the next order in k is k square and this gets scaled by half the effective range of the potential. And then of course you have further corrections of the order of k to the 4.

So, this is the typical formula which you employ in effective range calculations and just remind you that Bethe's alpha is inverse of hours and that pretty much concludes our discussion on low energy collisions. Of course we want to discuss collisions at higher energies, the Barn approximation and so on.

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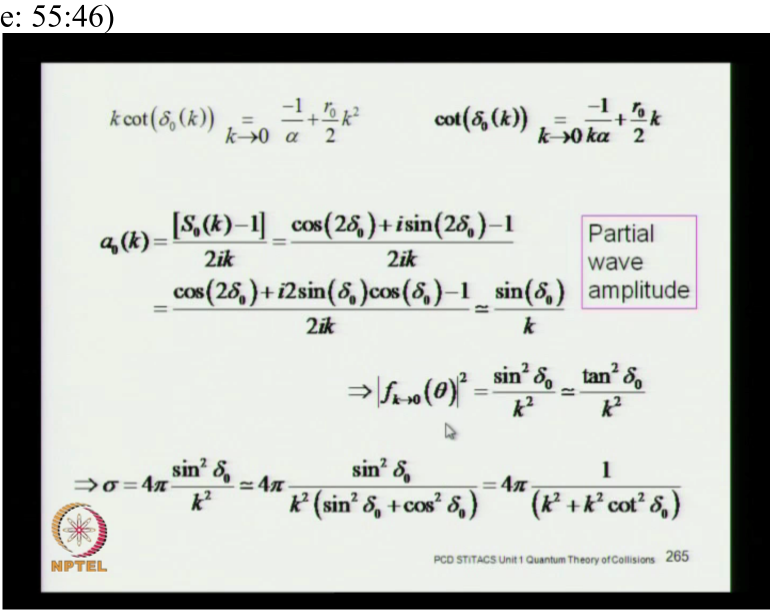

$$k\cot(\delta_0(k)) \underset{k\to 0}{=} \frac{-1}{\alpha}+\frac{r_0}{2}k^2 \qquad \cot(\delta_0(k)) \underset{k\to 0}{=} \frac{-1}{k\alpha}+\frac{r_0}{2}k$$

$$a_0(k)=\frac{[S_0(k)-1]}{2ik}=\frac{\cos(2\delta_0)+i\sin(2\delta_0)-1}{2ik}$$

$$=\frac{\cos(2\delta_0)+i2\sin(\delta_0)\cos(\delta_0)-1}{2ik}\simeq\frac{\sin(\delta_0)}{k}$$

Partial wave amplitude

$$\Rightarrow\left|f_{k\to 0}(\theta)\right|^2=\frac{\sin^2\delta_0}{k^2}\simeq\frac{\tan^2\delta_0}{k^2}$$

$$\Rightarrow\sigma=4\pi\frac{\sin^2\delta_0}{k^2}\simeq 4\pi\frac{\sin^2\delta_0}{k^2(\sin^2\delta_0+\cos^2\delta_0)}=4\pi\frac{1}{(k^2+k^2\cot^2\delta_0)}$$

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And that is something we will do in a later unit we will come back to collision physics after we do the next unit on second quantization. So, this is more or less the important piece of this result that you have got this k cot cotangent of this phase shift which is given by -1 over alpha times 1 over alpha + this effective range half of effective range times k square.

You can get the partial wave amplitude in terms of this because you have got the phase shift over here. And you can get the scattering amplitude which causes this okay. So, these things can be carried out worked out very easily. And from the scattering amplitude you can get the scattering cross section which goes 4 pi times 1 over k square + k square cot square delta zero.

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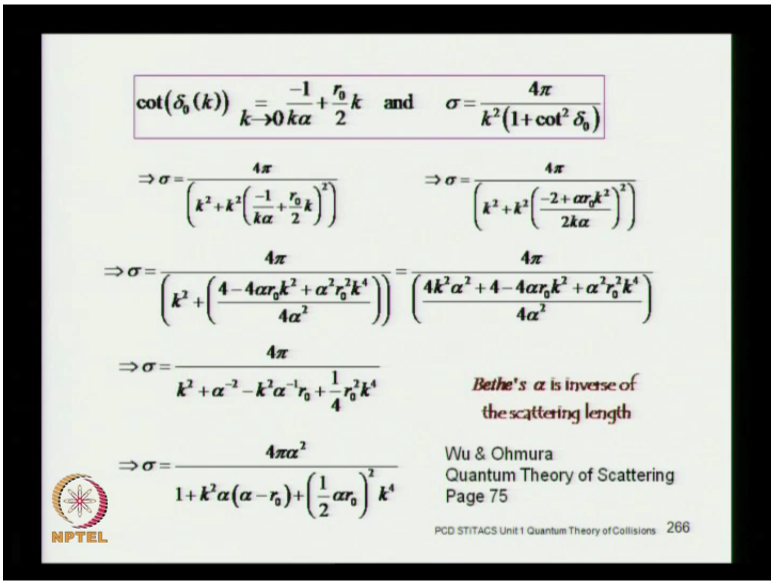

$$\boxed{\cot(\delta_0(k)) \underset{k\to 0}{=} \frac{-1}{k\alpha}+\frac{r_0}{2}k \quad \text{and} \quad \sigma=\frac{4\pi}{k^2\left(1+\cot^2\delta_0\right)}}$$

$$\Rightarrow \sigma=\frac{4\pi}{\left(k^2+k^2\left(\frac{-1}{k\alpha}+\frac{r_0}{2}k\right)^2\right)} \qquad \Rightarrow \sigma=\frac{4\pi}{\left(k^2+k^2\left(\frac{-2+\alpha r_0 k^2}{2k\alpha}\right)^2\right)}$$

$$\Rightarrow \sigma=\frac{4\pi}{\left(k^2+\left(\frac{4-4\alpha r_0 k^2+\alpha^2 r_0^2 k^4}{4\alpha^2}\right)\right)}=\frac{4\pi}{\left(\frac{4k^2\alpha^2+4-4\alpha r_0 k^2+\alpha^2 r_0^2 k^4}{4\alpha^2}\right)}$$

$$\Rightarrow \sigma=\frac{4\pi}{k^2+\alpha^{-2}-k^2\alpha^{-1}r_0+\frac{1}{4}r_0^2 k^4}$$

Bethe's α is inverse of the scattering length

$$\Rightarrow \sigma=\frac{4\pi\alpha^2}{1+k^2\alpha(\alpha-r_0)+\left(\frac{1}{2}\alpha r_0\right)^2 k^4}$$

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You can also manipulate these terms a little further because you can write these terms it write the cotangent function the kcot delta in terms of the scattering length and the effective range as we have done in the previous expression. And you can rearrange these terms you can see that you will get a 1 over alpha square okay and by cross multiplication you can bring that alpha square to the top.

So, that the expression for the scattering cross section will have 4pi alpha square and then it will be weighted by the remaining factors and this is just a rearrangement of these terms through cross multiplication and so on. You can work this out rather easily and you will find that the scattering cross section goes as 4 pi alpha square divided by these terms okay. So, this is a rather straightforward you know manipulation of these terms. So, I would not spend any time doing it.

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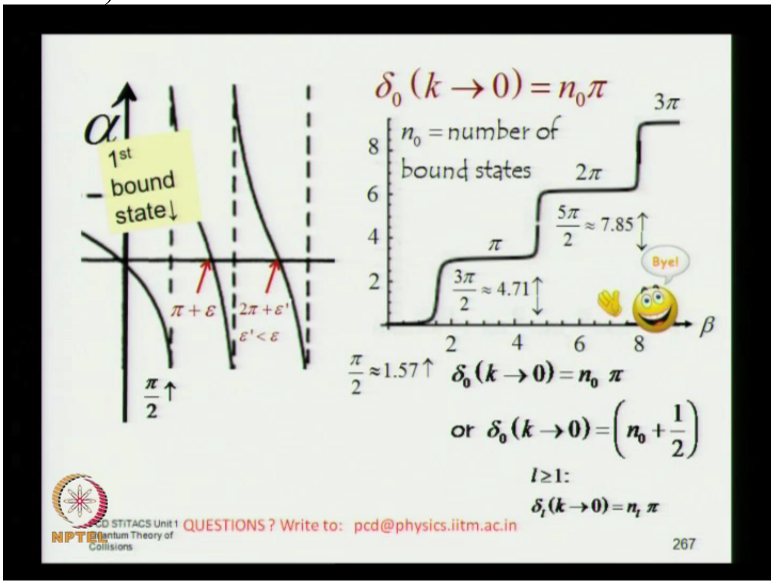

$\delta_0(k\to 0)=n_0\pi$

n_0 = number of bound states

$\delta_0(k\to 0)=n_0\,\pi$

or $\delta_0(k\to 0)=\left(n_0+\frac{1}{2}\right)$

$l\geq 1$:

$\delta_l(k\to 0)=n_l\,\pi$

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And that pretty much concludes our discussion for this unit. And essentially we have got the conditions under which you get the bound states the first bound state, the second bound state, the third bound state for the square well potential. It is very easy to demonstrate for other potentials it is a little more complex. But essentially the relationships go on similar lines.

You have seen that it has got major applications in nuclear physics in neutron, proton scattering in particular which was the genesis of all this formalism which was developed by Fermi, Bethe and so on. And now in the context of modern applications and atomic physics these are of great importance in getting BCS and BEC pairs of Fermi atoms. Thank You.