Quantum Mechanics - I
Prof. Dr. S. Lakshmi Bala
Department of Physics
Indian Institute of Technology, Madras

Lecture No - 35
Ehrenfest's Theorem

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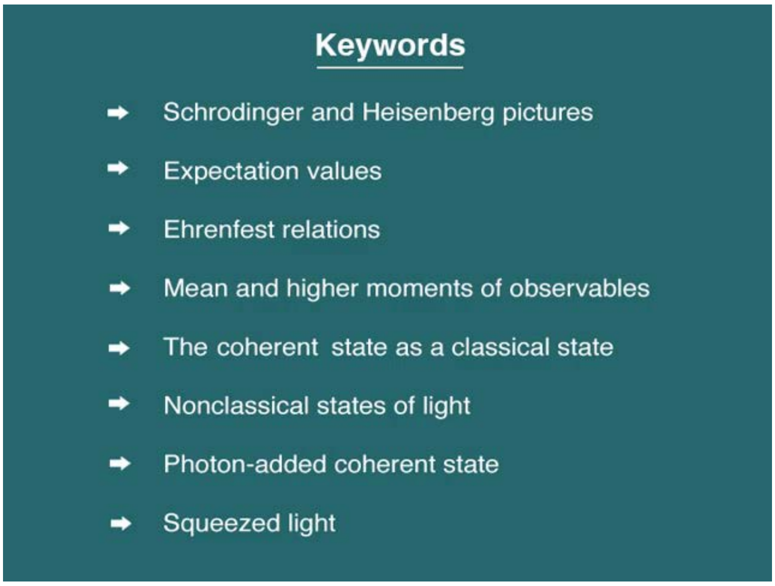

In the last lecture, we looked at the Heisenberg equation of motion.

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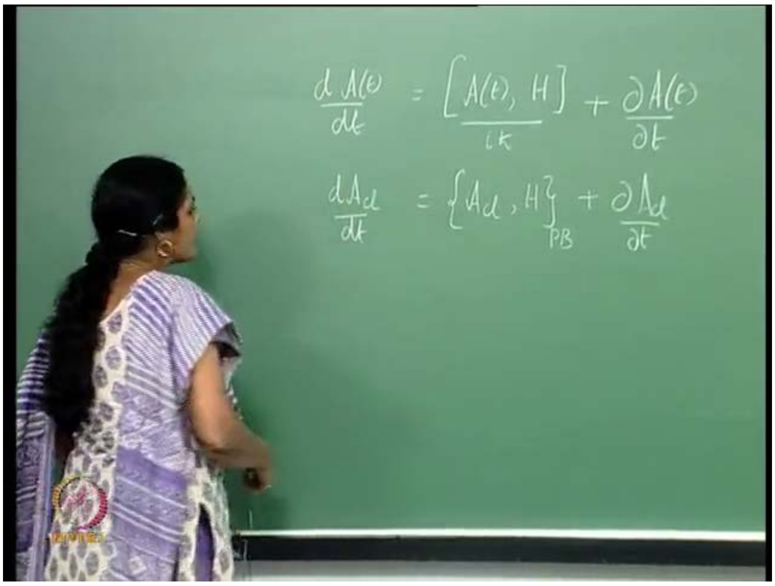

This was an equation for the operators and we had d A of t by d t, where a is an operator plus commutator of A of t with H by i H cross plus delta a by delta t. Where, this was a term which survived only if there was an explicit time dependence. So, this derivative means differentiation of the part of the operator which had explicit time dependence. So we also realized that this was very close to the classical physics equations, where you had d A classical by d t. Where, this is some dynamical variable it is the Poisson bracket of A classical with the Hamiltonian plus delta A classical by delta t.

And therefore, we realized that the Poisson bracket here is simply the commutator bracket by i h cross and you went from quantum physics to classical physics in the limit h cross going to 0. We explicitly saw using the commutator of x with p x what exactly happened and how you approached the Poisson bracket.

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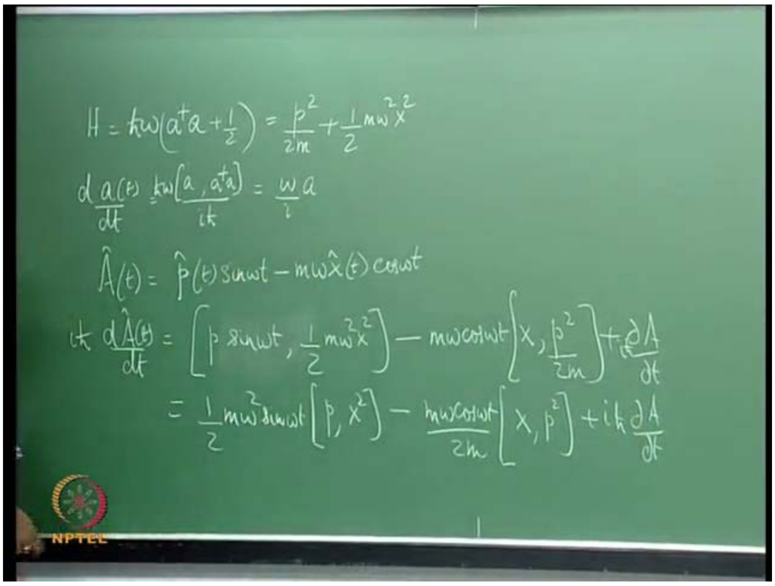

So, let us take a simple example; to look at the Heisenberg equation of motion. Let us take the Hamiltonian to be h cross omega a dagger a of course, plus half this is the harmonic oscillator Hamiltonian. And, so what is d a of t by d t? This would be the commutator of a with a dagger a plus delta a by delta t and that is 0 because, a does not have an explicit time dependence. And therefore, that term drops out so this is simply there is an h cross omega here; so it is simply omega by i a. Similarly, d a dagger by d t can be got. Let me take, a more non trivial example. Let me consider, the operator A of t to be p of t this, is a linear momentum operator sin omega t minus m omega x of t cos omega t. So in contrast to this case we have put in an explicit time dependence in the operator A.

Now, basically, this is simply a recapitulation of what we did towards the end of the last lecture. The A simply differs in sign from what we considered in the last class. But, I am repeating it so that I may reiterate an important fact. So, what is d a of t by d t? So first of all this Hamiltonian can well be written as p squared by 2 m plus half m omega squared x squared. So, the first part involves the commutator of p sin omega t, I will remove the argument t in the momentum with the Hamiltonian. And, that would just be half m omega squared x squared minus m omega cos omega t the commutator of x with p squared by 2 m. So, this is the explicit commutator term plus delta a by delta t and of course, there is an i h cross. So, let me start with i h cross this is this quantity plus i h cross delta a by delta t.

So, this is simply given by half m omega squared sin omega t commutator of p with x squared minus m omega cos omega t by 2 m commutator of x with p squared plus i h cross delta a by delta t.

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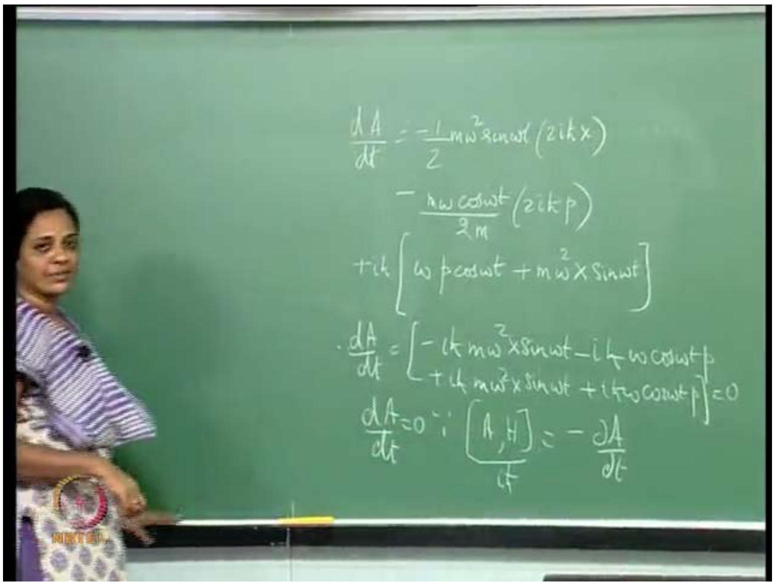

So, this quantity can easily be obtained and I have d A by d t, is half m omega squared sin omega t times 2 i h cross. That comes from this term with the minus sign because, it is a commutator of p with x that is involved and I (Refer Slide Time: 02:03) use the fact that commutator x p is i h cross. The next term is minus m omega cos omega t by 2 m times 2 i h cross p. I have used the a b c rule for commutators, so that x p is i h cross plus i h cross delta a by delta t. (Refer Slide Time: 02:03) And, since this is A the 1st term gives me omega p cos omega t and the 2nd term gives me plus m omega squared x sin omega t.

So therefore, d A by d t you can see that the 2 cancels here and I just have a minus i h cross m omega squared x sin omega t from the 1st term. The 2nd term gives me a minus i h cross omega cos omega t p and from the explicit time dependence I have a plus i h cross omega cos omega t p. And, then I have a plus i h cross m omega squared x sin omega t. So, plus i h cross m omega squared x sin omega t plus i h cross omega cos omega t p and therefore this is 0.

So, I have a situation where the explicit time dependence of a is 0 which means, that this is a constant of the motion having said that the important point to note is the following. It is not as if a commutes with the Hamiltonian this is a very (Refer Slide Time: 02:03) say tailored operator. I have chosen this to demonstrate the fact that, if A is a constant of the motion because d by d t of a is 0, the total time derivative is 0. It does not mean that a commutes with the Hamiltonian in general it means that the commutator of a with h apart from the i h cross suitably cancels the explicit time evolution of A.

So, here is an example where d A by d t is 0 because commutator of A H by i h cross is equal to minus delta A by delta t. So that is a thing worth remembering and you cannot naively imagine that always the explicit time dependence would not be there or that the commutator of a with h is 0 because, a is a constant of the motion. That need not necessarily be true.

Now, proceeding on these lines now that we know the Schrodinger picture and we also know the Heisenberg picture. We can choose our favourite picture depending on the context to work out the time evolution of quantum expectation values. As I have emphasized again and again expectation values are the experimentally measured quantities and quite independent of the picture you use answer should match for the expectation values and their dynamics. Whether, you work in the Heisenberg or in the Schrodinger picture.

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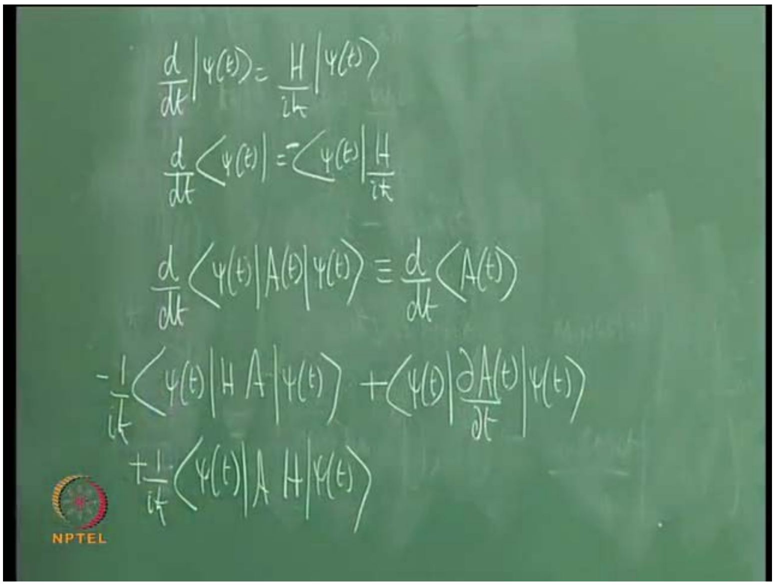

So, let me work with the Schrodinger equation, I have i h cross d by d t of psi of t is H psi of t. In general, the Hamiltonian could also have a time dependence and therefore, if I take the Hermitian conjugate equation minus i h cross d by d t bra psi of t is psi of t H. My aim is to see how expectation values evolve in time under appropriate Hamiltonians that I will select later on. So, I wish to find d by d t of expectation value of say an operator A which, could have an explicit time dependence that is allowed, this is what I want to find out. That means this is identical to d by d t of A of t, this is my notation. So, I can use this equation clearly this involves d by d t psi of t and d by d t of ket psi and d by d t of bra psi. So this will give me i h cross let me bring the i h cross to the other side.

So, the 1st term is d by d t Bra psi and that is out here, so I have psi of t H minus 1 by i h cross, A of t psi of t that is my 1st term. Then of course, plus psi of t, any explicit time dependence that differentiation has to be there and the 3rd part is plus psi of t a d by d t of ket psi of t is h by i h cross. So I can put the 1 by i h cross here, psi of t. So this is what this expectation value is about and you can see already that there is a commutator it is 1 by i h cross, commutator of A with H here, between these 2 terms.

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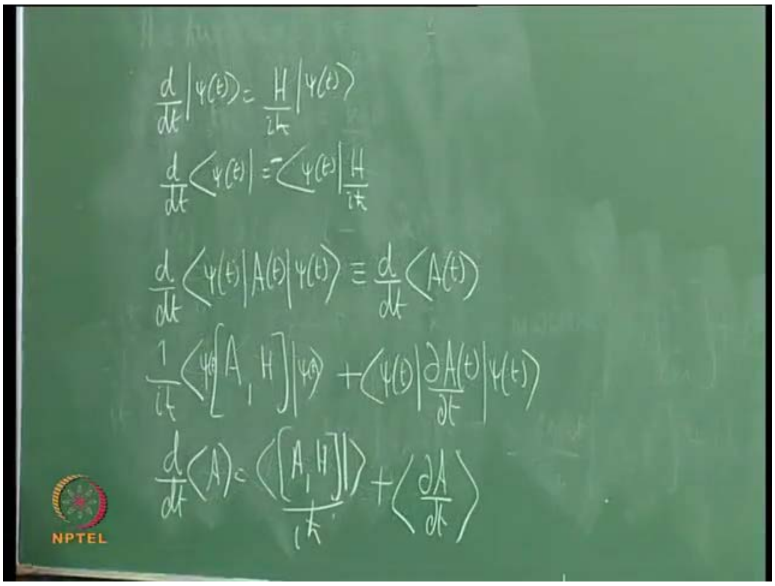

So, I can write this down as 1 by i h cross expectation value of the commutator of A with H so this is what I have. Of course, there is a psi of t here and this is my equation, so d by d t of expectation value of A of t, is expectation value of the commutator of A with H by h cross i h cross plus expectation value of delta A by delta t. So this is what I have, I have done this in the Schrodinger picture. Now, I could have done this in the Heisenberg picture as well.

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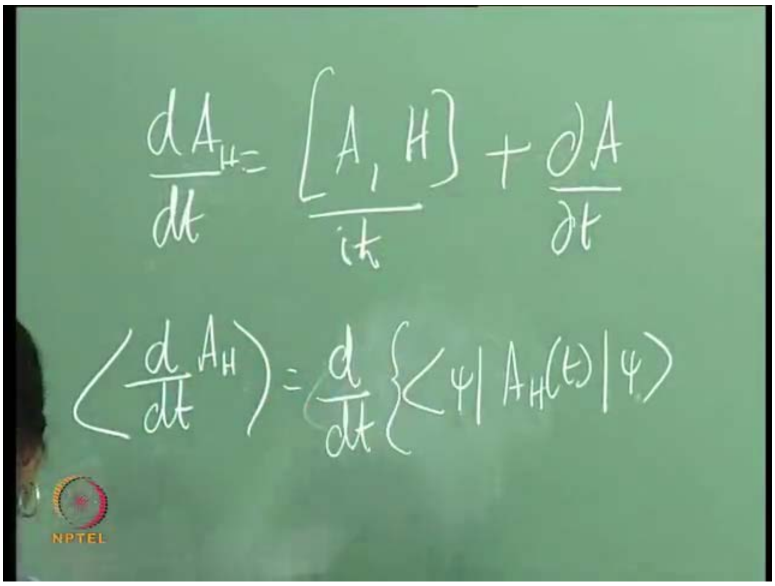

Now, in the Heisenberg picture I would have an equation that says, that d A by d t is commutator of A with H by i h cross plus delta A by delta t. So, if I take expectation values and I realize that in the Heisenberg picture expectation value of d by d t of A is expectation value of d by d t; could well be written as d by d t of expectation value of A. Because, this expectation value amounts to writing psi A H of t psi and since psi does not evolve in time in the Heisenberg picture, it amounts to simply differentiating A with respect to time. So, as you can see it gives me the same equation. (Refer Slide Time: 13:31) it gives me d by d t of expectation value of A is expectation value of the commutator of A with H by i h cross plus expectation value of the explicit time dependent term delta A by delta t.

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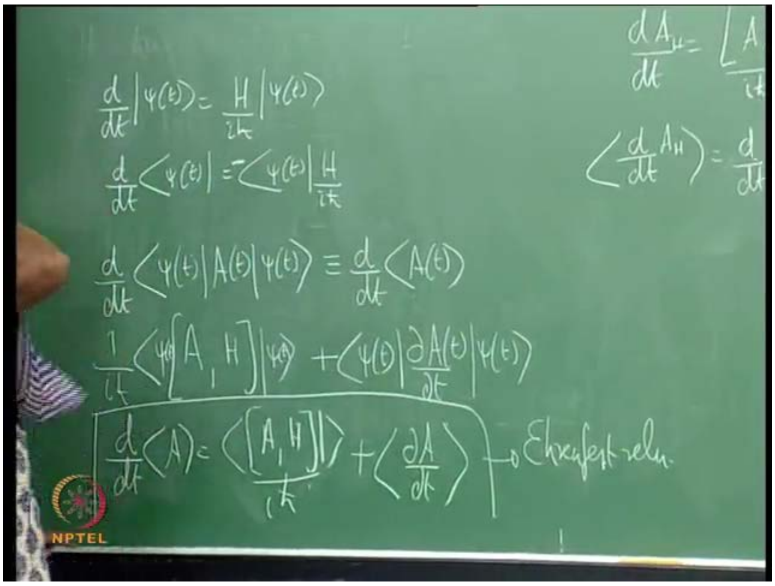

This is the Ehrenfest relation and this gains a lot of importance in the understanding of how to go from classical physics to quantum physics. The contribution of Ehrenfest becomes very significant, because in the early days of quantum mechanics there was this very grey area. Even now, there are some very formidable problems to be tackled in the area, when you pass from classical to quantum physics called semi classical physics. How do you go smoothly from the quantum to the classical world? This relation given here by Ehrenfest helps us significantly in this process, because of the following reason: Expectation values are in numbers and the idea is this if you want to go from quantum physics to classical physics.

You could perhaps, replace all operators by their expectation values because, this is precisely the kind of equation that you will have in classical physics, for a dynamical observable; for a dynamical variable. So, in classical physics you will have d by d t of any dynamical variable. It is the Poisson bracket of that variable with the Hamiltonian; the classical Hamiltonian plus delta by delta t of that dynamical variable. Where, this is the explicit time dependent part.

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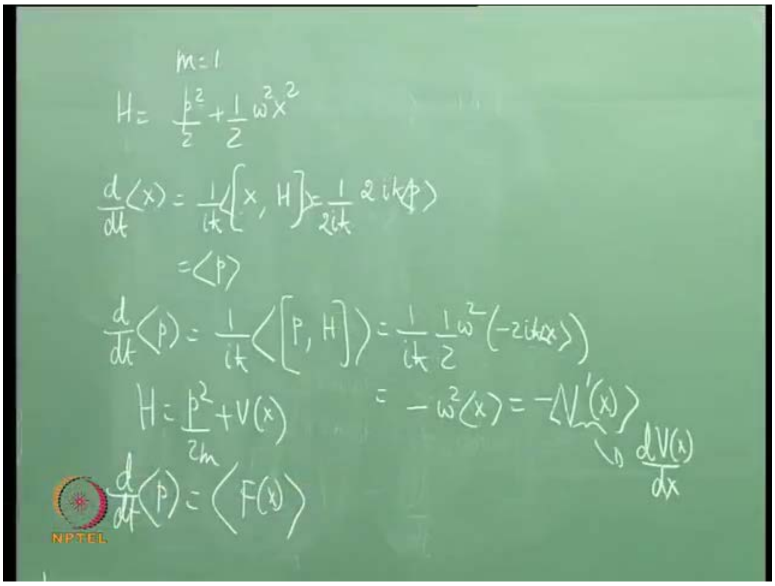

But, we need to understand this Ehrenfest relation in better detail its ramifications and its precise interpretation. Look at this example; a simple example let us again consider the harmonic oscillator of Hamiltonian that should do for our purpose. So you consider H is let us say the particle has unit mass m is 1 and so you have a Hamiltonian which is p squared by 2 plus half omega squared x squared. This is the Hamiltonian and I want to find d by d t of expectation value of X. The Ehrenfest relation there is of course, no explicit time dependence so we can forget that term.

So, this is simply the statement that d by d t of expectation value of x is 1 by i h cross commutator of x with H which is 1 by i h cross. So the commutator essentially is the commutator of x with p there is a 2 out here and I have 2 i h cross p. So, this is the same as expectation value of p, so I have d by d t of expectation X is expectation p. Let us look at d by d t expectation p once more, this is 1 by i h cross expectation value of the commutator of P with H and that is 1 by i h cross. The commutator of P with H just gives me a half omega squared minus 2 i h cross x expectation value and then that is just minus omega squared expectation x. But, this is the same as the derivative of the potential minus V prime of x its expectation value.

So, look at what I have I have d by d t expectation x is expectation p d by d t expectation p is minus omega squared expectation x. But, these equations are familiar these are precisely the classical equations, if you replace the operator by expectation values and if you identify these to be the analogues of the classical dynamical variables. These are precisely the equations that I have for a particle, which is executing simple harmonic oscillation. And therefore, it is in this sense that the Ehrenfest theorem is naively stated as quantum mechanical expectation values follow classical equations of motion.

But, you should understand that this is the general statement (Refer Slide Time: 15:50) of the ehrenfest relation this is merely an example. Because, if I did not have half omega squared x squared as my potential but, I had a Hamiltonian p squared by 2 m plus V of x an arbitrary potential. Then, d by d t expectation p is minus expectation V prime of x where V prime of x is simply the derivative of v with respect to x.

I would like to call this object therefore, as d by d t expectation p is expectation F of x. I use the notation F because at least, in classical physics for a conservative system I know that the force is the negative gradient of the potential and just by way of notation I use F here. It is just a notation as far as this is concerned. But, you see the harmonic oscillator example is a very interesting example because, it is quadratic in x. And therefore, I could have well replaced the operator by its expectation value.

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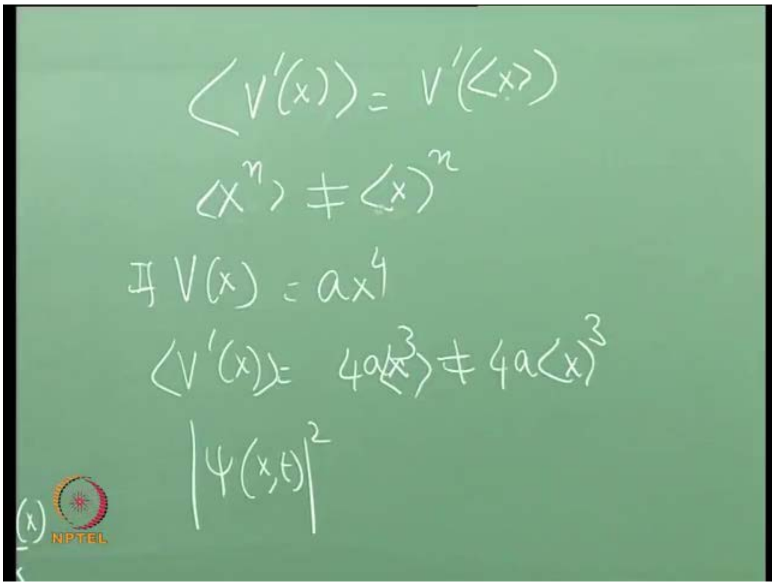

In other words, expectation V prime of x in this case can be just written as V prime of expectation x. Because, V had an x square in it and therefore, V prime was linear in x and therefore, expectation V prime is the same, as V prime expectation x. You cannot in general, replace all x to the power of n expectation by expectation x to the n. It worked very nicely in the harmonic oscillator case simply because, it was a quadratic potential. But, in general as you know this replacement is not possible because there are higher moments that contribute. It is not as if you can write the expectation value of x to any power, in terms of just the power of the mean there are higher moments that make a contribution. All of quantum physics is about understanding the wave function reconstructing the wave function from expectation values. And, that means that you need to know not just the mean but, all the higher moments; the infinite set of higher moments before you can reconstruct the full wave function.

Now, in practice in the laboratory, you would not be able to get an infinite set of moments, expectation values. So, you have a finite set of them but, surely it will involve the variance perhaps the higher moments, the skewness the kurtosis. And you try to reconstruct the wave function to some level of approximation. So, if V of x for instance is some a x to the 4 V prime of x is 4 a x cubed and expectation V prime of x is 4 a expectation x cubed, which is not the same as 4 a expectation x the whole cubed.

So, when can you simply replace? When can you do this replacement? Well, you can do this certainly it is an approximation in general. It can be done if the position probability density is very close to expectation x of t for all times. Then this can be done, but, if that is not true obviously the higher moments will contribute and you will not be able to make such a replacement. And that is why, if you want to find for instance d by d t of expectation x squared. It will not simply involve expectation p squared.

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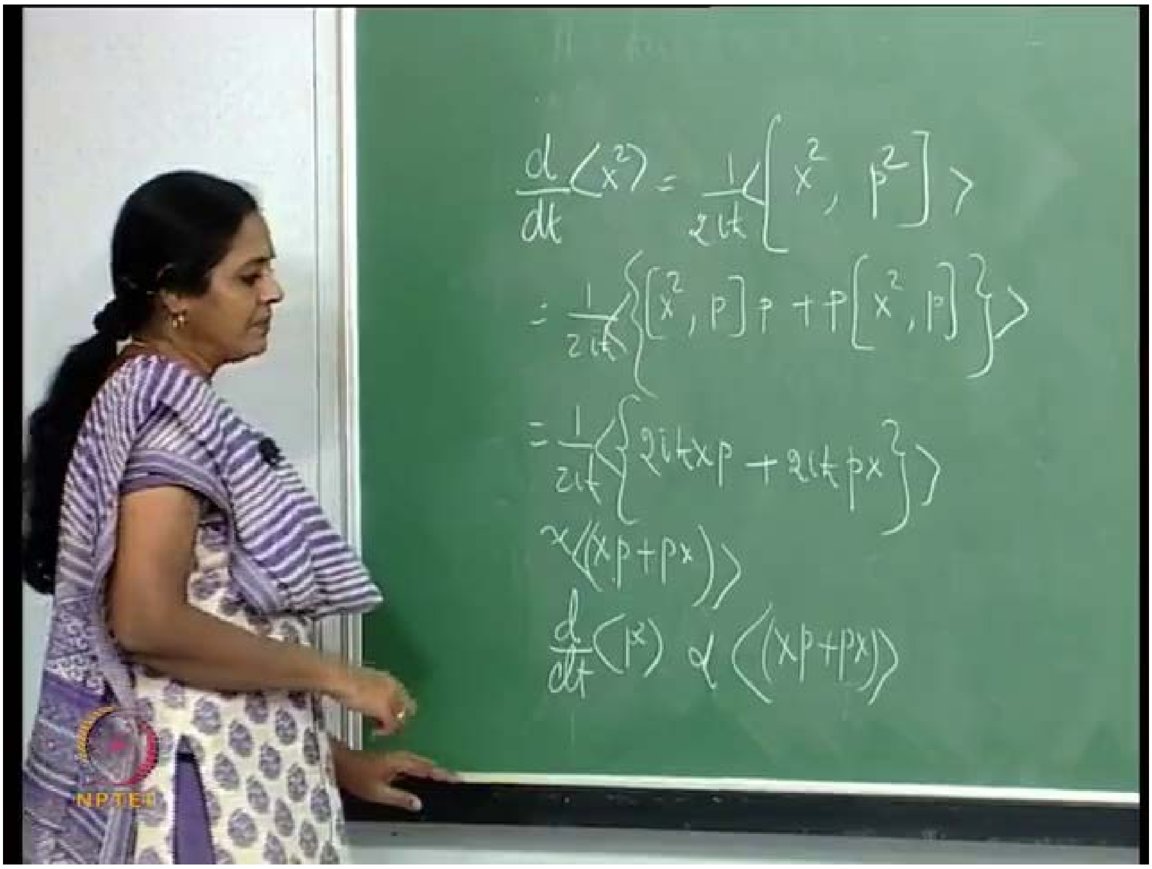

Let’s calculate d by d t expectation x squared. Now, this object is 1 by i h cross commutator of x squared with h, I am using the harmonic oscillator Hamiltonian. So this involves commutator of x squared with p squared by 2 and therefore, that is 1 by 2 i h cross x squared with p p plus p commutator of x squared with p. And this is 1 by 2 i h cross i h cross x p twice and this is plus 2 i h cross p x therefore, this is essentially x p plus p x.

So, d by d t of expectation x squared involves x p plus p x and it is not as if the algebra simply closes between x squared and p squared, other moments are also involved.

Similarly, if I find d by d t of p squared that too will involve the expectation value of x p plus p x. So this 2 is proportional to the expectation value of x p plus p x, can be checked out. So, inevitably, the algebra does not simply close between x squared and p squared or x to the m and p to the m it involves many other moments. So this is the other important point about the Ehrenfest relation.

Now, 1 thing is clear clearly, this relation and what it predicts depends crucially upon the state of the system that we consider. Of course, it depends crucially upon the Hamiltonian of the system I have used just a Hamiltonian of the oscillator form. It is evident that, if I use something like a dagger squared a squared, it is going to create more complications. But, for the moment I restrict myself to the oscillator Hamiltonian and then I find that, depending upon the state; depending upon the Hamiltonian the dynamics of the expectation value is going to change tremendously.

So, the way in which I would like to demonstrate the Ehrenfest relation in the context of expectation values of course, is to look at states of the radiation field. Since there are very many interesting states of the radiation field, some of which you already know like: coherent state that is laser light, the photon added coherent state and the squeezed state of light. It would be interesting to use these 3 states of the radiation field and see what happens.

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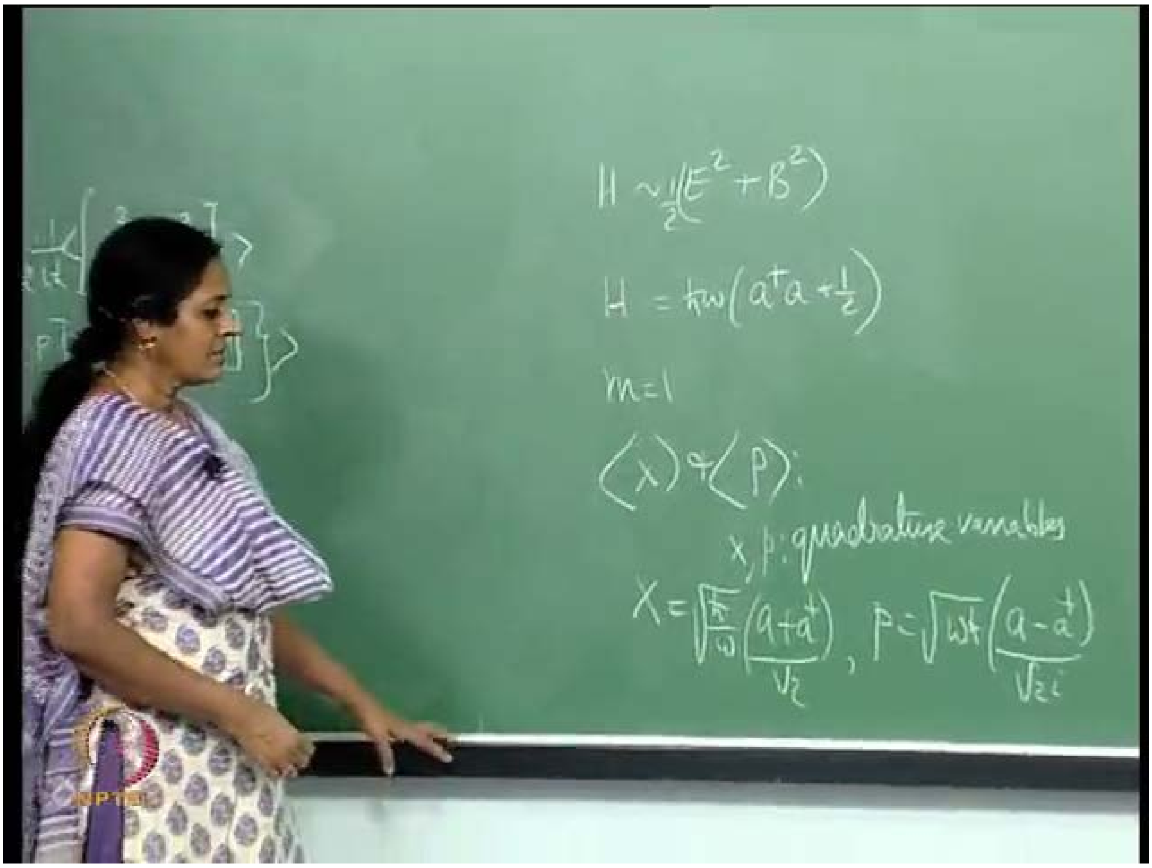

Let us recall, that the Hamiltonian was essentially, this is the Hamiltonian in quantum optics was essentially E squared plus B squared. It is clear that I have set I have ignored mu naught and epsilon naught for the moment and we did appropriately right E and B in terms of a and a dagger. The photon destruction and creation operators and we could get the Hamiltonian in the form of the oscillator Hamiltonian. This is just a formal mapping this is the Hamiltonian for the Hamiltonian density for a free electromagnetic field. So, if there is a radiation field propagating in free space, then, instead of looking at the behaviour of the radiation field propagating in free space. I could well, look at a particle of unit mass subject to an oscillator potential, so that is all that this needs.

So, if I look at various states of the radiation field propagating. I could mimic that or I could have an equivalent formalism where, I take a particle of unit mass subjected to a parabolic potential omega squared x squared by 2. And, look at its expectation values expectation x expectation p and so on and that should tell me the behaviour of expectation x and expectation P in the case of the radiation field. I call these the x and p are the quadrature variables. Clearly, in the case of optics these expectation values would be related to the intensity of the electric field. And, I would just formally define x as root of h cross by omega a plus a dagger by root 2 and p as root of omega h cross a minus a dagger by root 2 i.

So while a and dagger have physical meaning that they are the photon destruction and creation operators. x and p are the hermitian counterparts and would have in principle relations with would be related in principle to the intensity of the electric and the magnetic fields.

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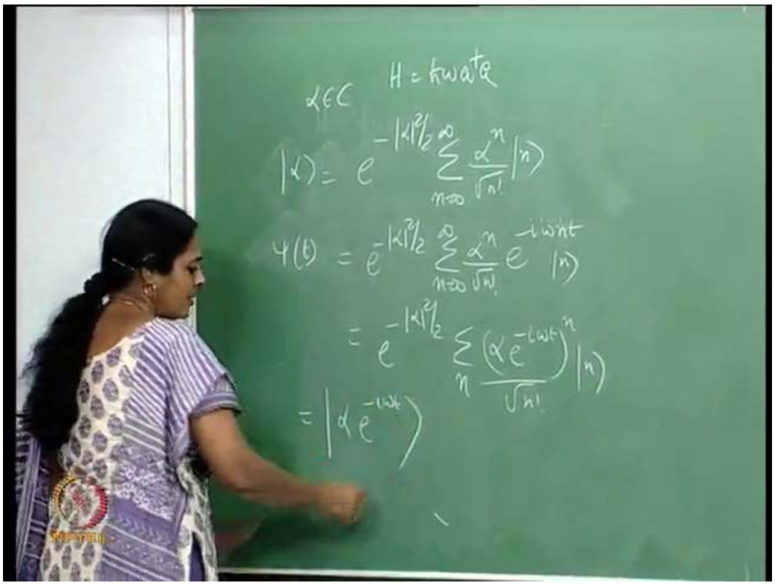

So given that, I would now look at 3 types of states of the radiation field the 1st step that I want to look at it is the coherent state; the standard coherent state ket alpha. And as you know in terms of the photon number states, this is e to the minus mod alpha squared by 2 summation n equals 0 to infinity alpha to the n by root n factorial ket n. Yesterday, we looked at this state in the position representation it is a gaussian. So, now I am interested in finding out what happens at time t. So what is psi of t if the initial state is alpha?

Yesterday, we showed that a gaussian continues in the harmonic oscillator a Hamiltonian when subject to the harmonic oscillator Hamiltonian the gaussian continues to be a gaussian merely, oscillating. The cheap way of seeing it is by saying that in this context psi of t is e to the minus mod alpha squared by 2 summation n alpha to the n by root n factorial e to the minus i H t by h cross ket n. And, this is simply e to the minus i omega n t ket n, I have used the Hamiltonian h cross omega a dagger a. Otherwise, you would have to put an n plus half there.

So, this can be written as e to the minus mod alpha squared by 2 summation over n alpha e to the minus i omega t to the power of n by root n factorial ket n. And therefore, psi of t is simply alpha e to the minus i omega t, executes harmonic motion simple harmonic motion and continues to be a coherent state. Therefore, in the position representation it is a gaussian, so this is the way in which the state changes in time.

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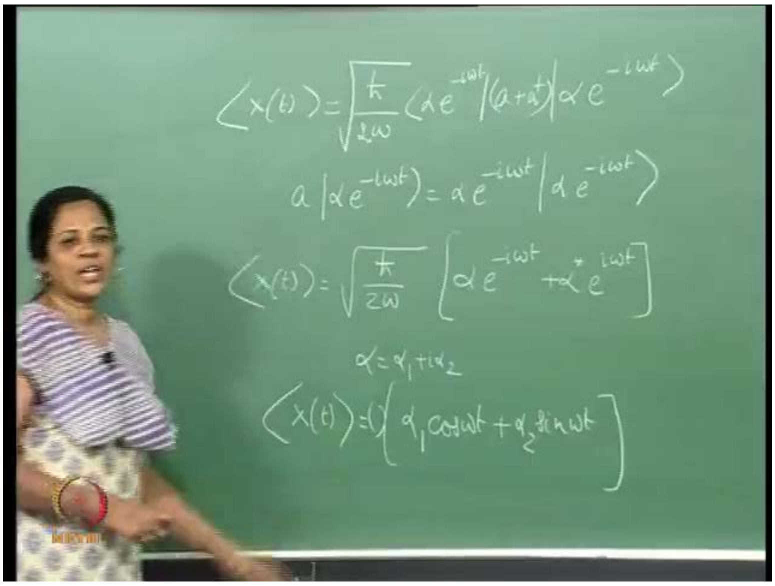

Now, let us look at expectation value of X. Let us now calculate expectation value of x of t in the initial state alpha. So alpha evolves and at any instant of time alpha is just alpha e to the minus i omega t so that is the state of the system at time t. And therefore, this expectation we’ve set n equals 1. So, there is a root of h cross by omega and then there is an a plus a dagger sandwiched between the ket and the Bra. So there’s an a plus a dagger here and of course, there is a root 2. So this is the object that we need to calculate this is certainly true, this is alpha e to the minus i omega t times the ket itself; it is an Eigen value equation.

Similarly, I can act a dagger on the Bra and pull out alpha star e to the minus e to the plus i omega t as an eigen value when a dagger acts on this bra the eigen value is alpha start e to the plus i omega t. Therefore, x of t is root of h cross by 2 omega alpha e to the minus i omega t plus alpha star e to the i omega t. I need to take the complex conjugate of this number which is what I have put down there and that is all, that is the expectation value of X. Of course, I can write alpha as alpha 1 plus i alpha 2, write e to the minus i omega t in terms of sins and cos and then expectation x of t, is apart from an overall constant which you put down here. That involves h cross omega the number and so on it is just alpha 1 cos omega t plus alpha 2 sin omega t. So, this is what we have for a expectation value of X.

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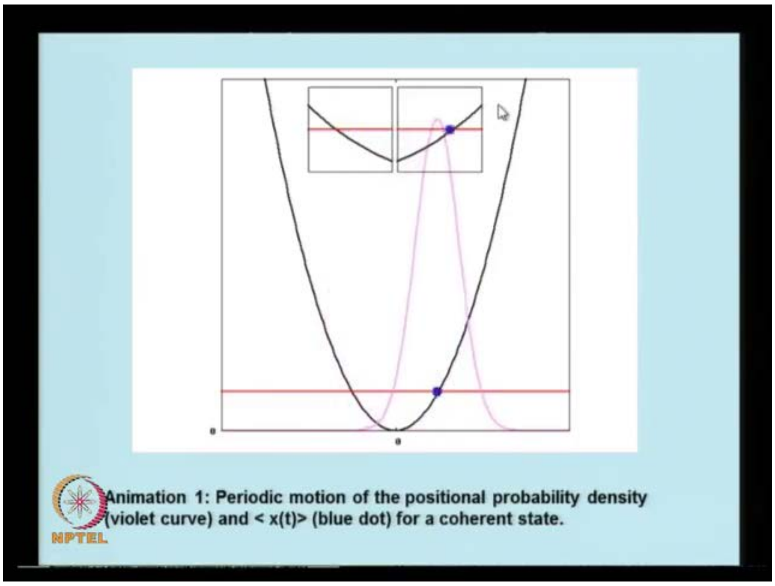

So, here I have it in animation, this is the oscillator potential the black parabola and what we have is an initial state which is the coherent state. And, what we have here in pink is the gaussian form as I have just now told you. We demonstrated it explicitly the gaussian continues to remain a gaussian as time proceeds.

So you can see that once it comes to the extreme right or left it is simply a gaussian, so is it at the centre no change in amplitude, no change in height, no change in width, no change in height. Execute simple harmonic motion in the following sense that as we have seen expectation X apart from constants is alpha 1 cos omega t plus alpha 2 sin omega t and expectation P you can easily verify is alpha 2 cos omega t minus alpha 1 sin omega t. The blue dot here which keeps moving is the expectation value; the red horizontal line is merely the energy value that I have selected. So, there is a constant energy which is given and that is a constant of the motion. And you can see that expectation x of t the blue dot for an initial coherent state, simply moves like a particle of unit mass subject to the parabolic potential, it simply executes simple harmonic motion.

Now, look at the inset and that will tell you how far, the blue ball moves. Moves all the way till it touches the potential the parabolic potential that is what you see from the inset. So those would be the classical turning points, so you see what we have shown you is this the expectation value expectation X and P behave exactly like the position and momentum of a classical linear harmonic oscillator. The correspondence is actually even more exact because, the amplitude of oscillation you can easily check in position space is root of 2 h cross by omega times mod alpha. But, the mean energy in this state is h cross omega mod alpha squared because it is simply the expectation value of a dagger a in the state ket alpha. Notice that this is a conservative system so energy does not change in time. So the position of the classical turning point is root of 2 e divided by omega.

So, the quantum mechanical expectation values in the coherent state behave exactly, like the classical position and momentum of a linear harmonic oscillator. And the state remains a minimum uncertainly state at all times. So you can see as it goes from 1 end to the other that this really behaves like a classical harmonic oscillator. The dynamics of the higher powers of expectation X and expectation P and their combinations can be established similarly, with the help of the Ehrenfest theorem.

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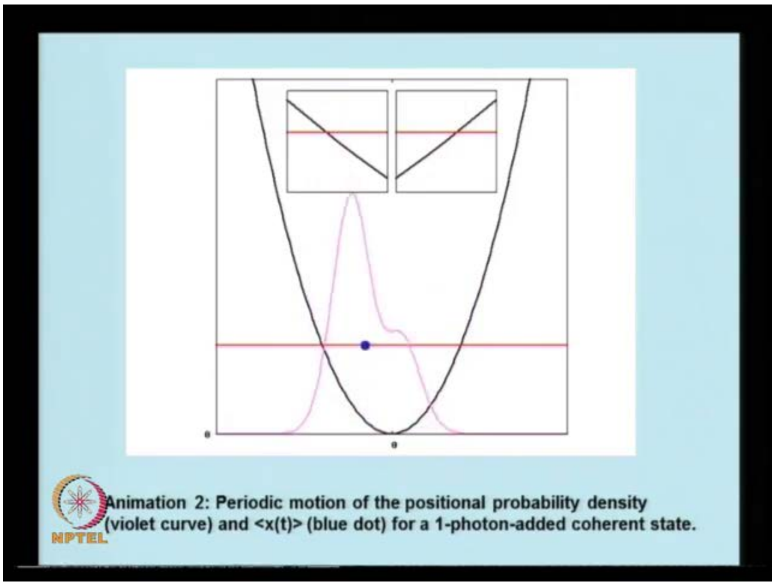

So, now, let us consider the case when the initial state is a photon added coherent state. You will recall, that a photon added coherent state; the n photon added coherent state for instance adds m photons to the standard coherent state. That means, you repeatedly apply a dagger m times 2 ket alpha and then you normalize it to get the normalized photon added coherent state, alpha comma m. So, this is the initial state that we will now consider because, we would now, look like to look at a situation where the state does not have coherence.

So, here in this figure, I show the periodic motion of the position probability density that, is the pink curve as you can see that is the pink curve. And of course, the blue dot is expectation x the initial state is a 1 photon added coherent state. You will recall that the I have already mentioned that this was produced in the laboratory a few years ago. And therefore, understanding the nature and behaviour the dynamics of the single photon added coherent state becomes that much more interesting and important. As you can see this is in sharp contrast to the coherent state; the state is not a gaussian function of x. It is not a minimum uncertainty state at any time and moreover the violet curve; the pink curve which is the position probability density, it does not retain its shape as it evolves under the influence of the oscillator potential.

So, the dotted red line again corresponds to the mean energy in the photon added coherent state. That means the expectation value of h cross omega a dagger a. But, remember a dagger a is no longer equal to mod alpha squared, which was its value in the case of a coherent state. Now, if you look at the probability density in a position space it has 2 maxima rather than just 1. Moreover, the curve changes shape in a rather drastic but, regular fashion as it oscillates back and forth. The expectation values x of t and p of t continue to vary sinusoidally, as in simple harmonic motion.

Now, this feature is guaranteed by the Ehrenfest theorem and it is really implicit in the equations that we wrote down that is: d by d t of expectation x is expectation p and d by d t expectation p is minus omega squared expectation x. These equations remain valid for all states and all times since, the Hamiltonian is quadratic in x and p. However, there is an important difference between the present case and that of the coherent state. Notice that the amplitude of oscillation falls short of the classical turning point the classical turning point was, root of 2 e by omega. But, then if you look at the inset you will see that the blue dot does not touch the edge of the potential at all; in fact does not come up to the classical turning point.

So, this is a new feature that we have seen and in a sense there is a simple way of understanding why the amplitude of oscillation of expectation value of x falls below the classical turning point. In the case of the fock state you will recall that expectation x is 0 and expectation x remains equal to 0 for all time. In the case of the coherent state it periodically reaches the classical turning point that is what we saw. So, the photon added coherent state interpolates between the fock state and the coherent state and therefore, it is possible when the amplitude of oscillation of expectation x for the photon added coherent state lies somewhere between 0 and the classical turning point given by root of 2 e by omega.

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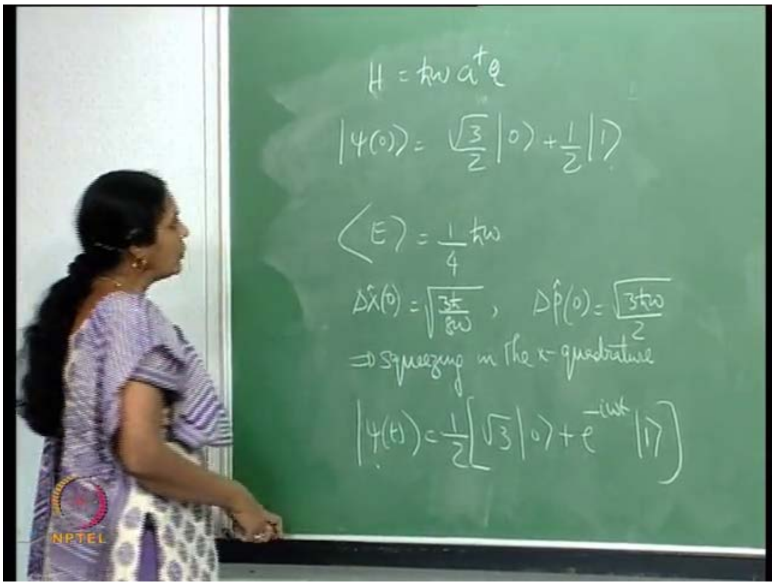

Finally, I want to consider the squeezed state we have examined this state in 1 of the earlier lectures. So the state is given as a super position of the 0 photon state and the single photon state and this is an example of a squeezed state; it is not the squeezed vacuum it is a super position, which certainly shows squeezing it is normalized as you can see, the mean value of energy in the state is quarter h cross omega. Because, E is h cross omega; the Hamiltonian is h cross omega a dagger a and then you can easily check that this term contributes whereas, that gives me a 0. And this is the expectation value of E. Now, if you calculate for this state the variance if you calculate delta x which is given as root of 3 h cross by 8 omega and delta p is root of 3 h cross omega by 2. It is clear, that there is squeezing in the x quadrature because, if you set h cross and if you set omega equals 1. Then this object is certainly less than what you would have for the ground state of the oscillator or the standard coherent state. So, there is squeezing in the x quadrature.

Now, if you let this state evolve subject to this Hamiltonian you can trivially check that this, is the state at a later time. Apart from a factor half it is root 3 ket 0 plus e to the minus i omega t ket 1.

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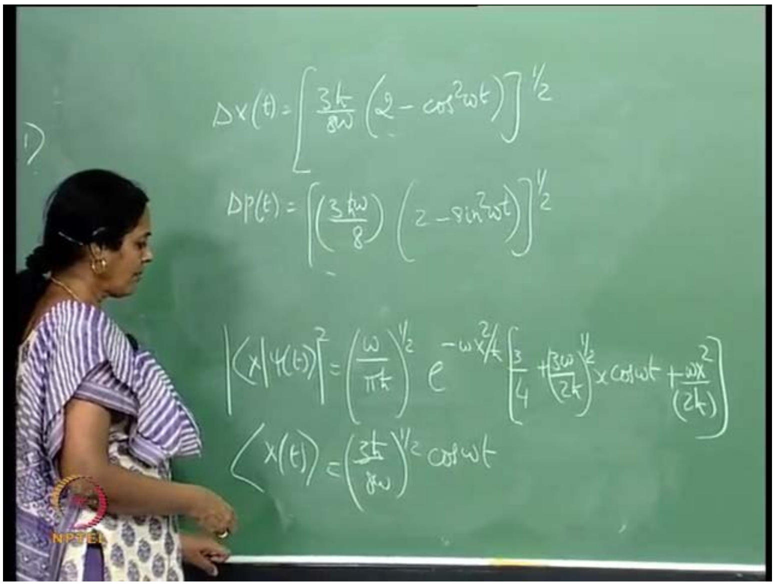

I will leave it to you as an exercise to calculate delta x at time t as a function of time and delta p as a function of time. These are the expressions, so these would be exercises for you to do. Apart from a coefficient 3 h cross by eight omega delta x is essentially, 2 minus cos squared omega t. So, the variance would be remove the half and get the rest of it. Similarly, delta p whole squared which is the variance is 3 h cross omega by eight times 2 minus sin squared omega t. Just by way of information and to complete the picture, if you look at the manner in which this initial state evolves in time when subject to the Hamiltonian h cross omega a dagger a. If you find the probability density this is interesting because, apart from constants it is e to the minus omega x squared by h cross. Times there is a function of x squared here, there is x squared there; a constant and then there is a cos omega t as well x cos omega t.

If you look at expectation value x of t as you know, (Refer Slide Time: 45:10) ket zero is expectation value at all times is 0. Ket 1 has an expectation value the power of quantum super position is seemed here because, despite the fact that the expectation value is 0 in this state; fock state. You find that expectation x of t turns out to be non-zero and in fact varies as cos omega t.

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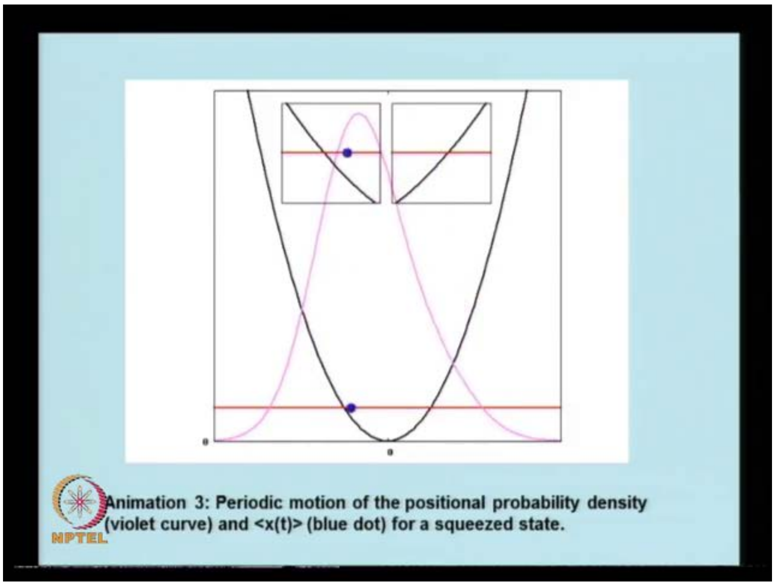

So, I have an animation here for you to show the behaviour of this initial squeezed state. So I have for you the periodic motion of the position probability density as before, this is the pink curve and then of course, expectation x of t is the blue dot. And, this is for the initial squeezed state that I was talking about, the horizontal red line here is simply the mean value of the energy; that is a constant. The black is the parabolic potential the oscillator potential. And then, you can see various features you can see for instance that in the inset expectation x of t very clearly, does not get anywhere near the classical turning point; it does not touch the classical turning point. It stops there and it stops a little short of the classical turning point you can see that.

The squeezed state changes its shape and this is the position probability density and that keeps changing its shape as time elapses. And of course, in all the 3 cases that I have shown you it is pretty clear that there is a tunnelling, because, as you can see, the position probability density is non-zero even outside the parabolic wall. So there is a nonzero thing out here and here, so there is certainly this quantum mechanical the leakage, leakage beyond the containing potential.

So, there is a leakage into the classically forbidden regions, which we can see very clearly here. So, these are the 3 cases; the 3 cases of non classical states of 2 of them are certainly non classical states. The coherent state is considered to be a classical state for reasons that I have explained to you. And once again in the case of the squeezed state you can find out the time evolution of expectation x squared and expectation p squared by just using the ehrenfest's relation in its general content.

So, in any faux state n the expectation value of x is 0 but, looking at this it is clear that quantum super position produces various changes. There are more stringent quantitative measures of non classicality of quantum mechanical states in general. You see even among states of radiation there are many other interesting non classical states for instance, there is a so called cat state ket alpha plus ket minus alpha or the Yurke Stoler state which is ket alpha plus i ket minus alpha; the squeezed vacuum state and so on. In all these cases, the Ehrenfest theorem provides a convenient way to analyze the dynamics and to understand why are the behaviour of the higher moments of the quadrature concerned, the role played by quantum fluctuations.

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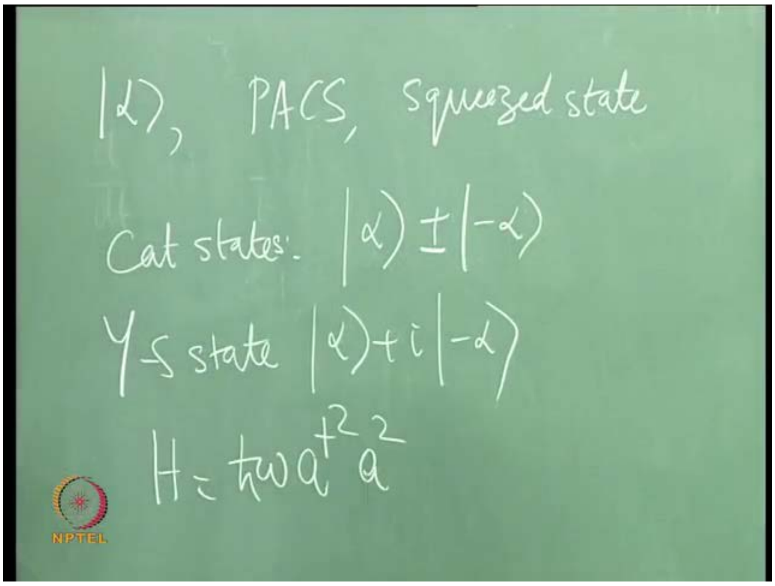

So, you see we have looked at 3 states of the radiation field ket alpha and then, the so called non classical states the P A C S which was the photon added coherent state and the squeezed state. And, we have demonstrated how expectation values behave in time, there are other interesting non classical states for instance, there is the Cat state: this is ket alpha plus ket minus alpha or ket alpha minus ket minus alpha. Then the Yurke stoler state, this would be: ket alpha plus i times ket minus alpha and so on In all these cases, you can look at the Ehrenfest's relations and see how exactly expectation values evolve in time. Now, what we have used is the Hamiltonian h cross omega a dagger a and then you find that for all these states considered, there is just periodic motion and nothing more.

Now, you look at a Hamiltonian like the Kerr Hamiltonian a dagger squared a squared, very many interesting things happen for instance even a coherent state; an initial coherent state does not continue to be a coherent state for all times. Almost immediately, since this is a wave packet; that means it is a super position of plane wave states for instance. And therefore, it does not have a sharp momentum or a sharp position but, it is more or less there it is a gaussian. Approximate momenta and approximate positions can be given, it is somewhat localized that is what I mean. You will find that almost immediately after being subject to this Hamiltonian, the coherent state puts out ((Refer Time: 54:37)) it is no longer a gaussian. Changes in all sorts of fashions very very interesting fashion.

But, then because of some very interesting peculiar properties, it could revive that means at periodical instance of time it could exhibit this feature that, it comes back to its initial state apart from a phase factor. And then, expectation values return to themselves to their initial values at those instance of time and so on. These are called wave packet revivals and we will not be considering that here, it is a more advanced topic. But, I hope I have given you a flavour for what kind of complicated dynamics can be there, what sort of interesting dynamics can be there in the behaviour of expectation values depending upon the states initial state and depending upon the Hamiltonian.

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