Quantum Mechanics - I

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Lecture - 34
Illustrative Exercises – II

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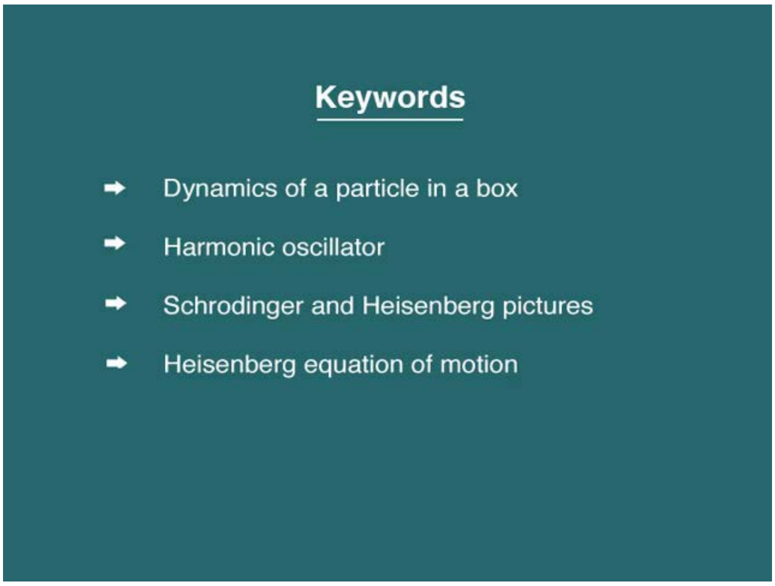

In the last lecture, I considered dynamics, in the sense that I looked at the time dependent Schrodinger equation. And, we will apply it to certain systems; we will continue to apply it to certain systems today.

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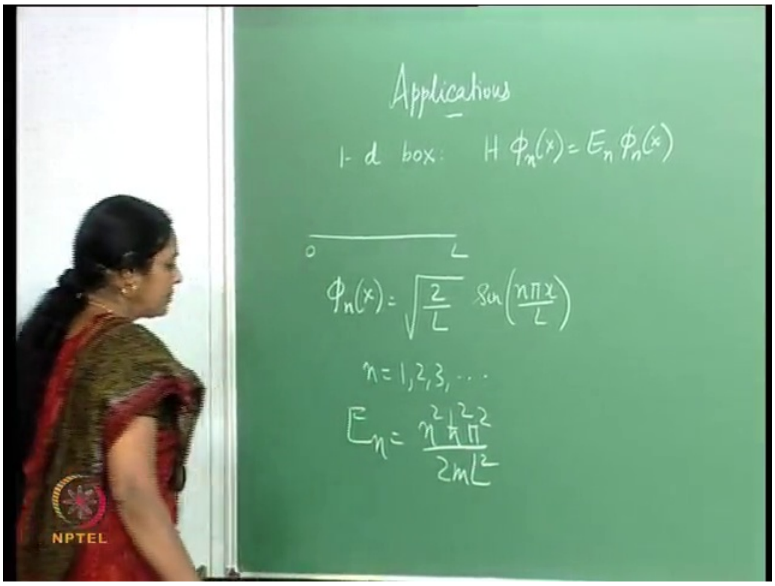

So, I had called this lecture also title it applications, would be discussing the dynamics of various initial states, in a box; various initial states of a particle in a box. Also look at a certain superposition of a harmonic oscillator energy Eigen states and the way they behave, under the quadratic potential. The manner in which they change, with time, I also said in the last lecture that, we will look at an alternative formalism of quantum mechanics, where the operators evolve in time and the state vectors do not. The Heisenberg picture as it is called. I will get to that towards end of this lecture perhaps, after working out these applications that I was talking about. The dynamics of a particle in a box, different initial states given and also the dynamics of the harmonic oscillator states.

So, the first thing I will consider is a box. A 1 dimensional box, size is L 0 to L and the Eigen states of this 1 dimensional box. The energy Eigen states, are given by phi n of x and since, we have already done this problem. You know what the phi n's are, the normalized energy Eigen states, are given by root of 2 by L, sin n pi x by L. Where, n takes values, 1, 2, 3 and so on. The energy E n itself is known to us, n squared h cross squared, pi squared by 2 m l squared.

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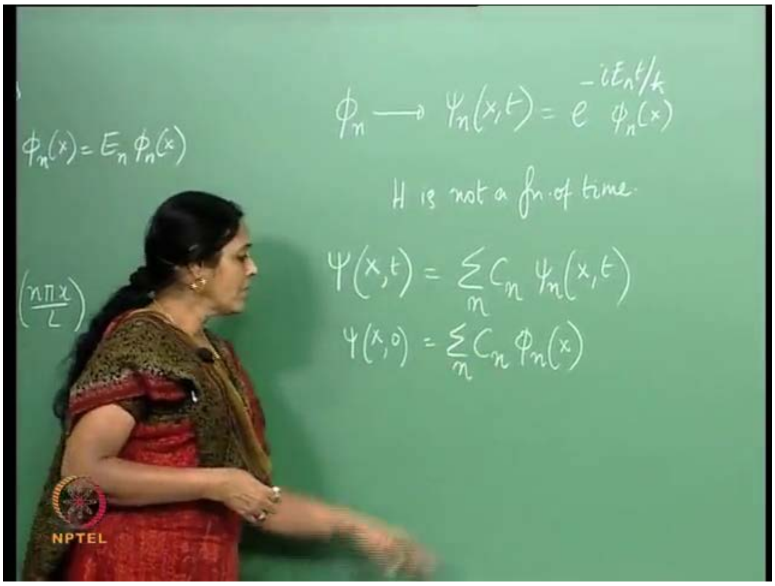

So, these are the energy Eigen values and since these are Eigen states, the phi n of x. The manner, in which they evolve, with time, is very simple. Phi n goes to some psi n of x t, with time. Which, is e to the minus i E n t by h cross, phi n of x. These are stationery states and this is the time evolution. Of course, were member that H is not a function of time. So, consider to the system, the energy does not change in time. Hamiltonian is not a function of time. So, now we ask the question. What about an arbitrary state, psi of x t? How do I write it in terms of the basis states? Well psi of x t can always be written, as summation over n C sub n, psi n of x t.

So, at any instant of time, this is a linear super position of the Eigen states, the evolved Eigen states of the Hamiltonian. And, it is clear therefore, that at time t equals 0, psi of x is simply C n, psi n of x 0, which is phi n of x. So, now, let us start with arbitrary initial states, psi of x comma 0 and see how exactly it evolves in time. In other words, one wants to find out what the C n's are.

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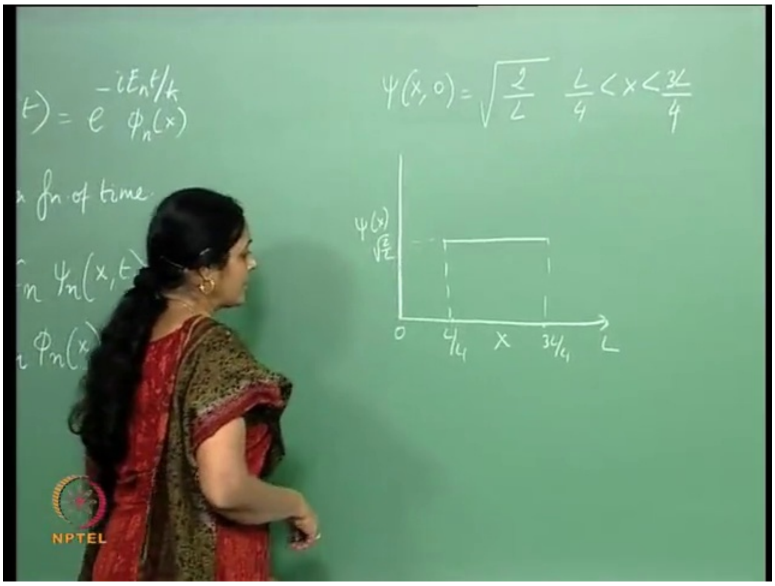

So, the simplest example that, we can think of is this. Psi of x 0 is root of 2 by L, for L by 4 less than x, less than 3 L by 4. Other words, that is x and this is psi of x, takes value root of 2 by L, just from L by 4 to 3 L by 4. So, it is a constant independent of x and this is psi of x.

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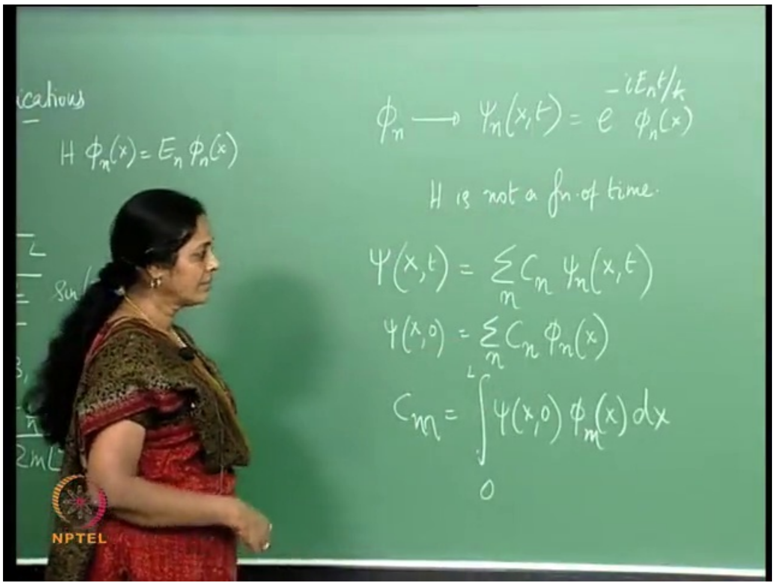

So, if I want to find out the problem is, to find out (Refer Slide Time: 03:04) psi of x, t. The manner, in which I find out C n of course, is to say, that C n and C m in general, is integral psi of x 0, phi m of x d x, from 0 to L. Now, once I know the C m’s I plug it in here and I get psi of x t. Because, I know psi is sub n of x t, from that equation.

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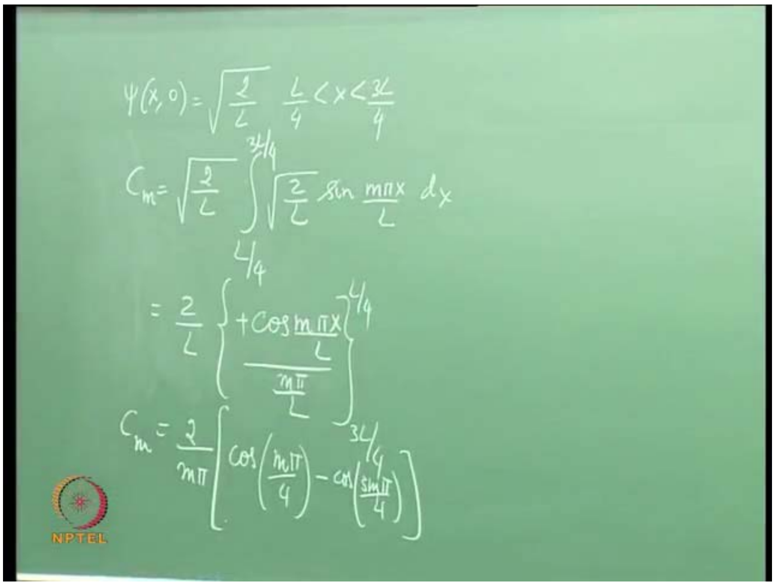

So, let us work that out and see what we get. So, in this example, C m is root of 2 by L, since, the since, (Refer Slide Time: 05:49) psi of x 0 vanishes everywhere, except from L by 4 to 3 L by 4. This integral itself needs to be done, only within those limits. That is a root of 2 by L again, sin m pi x by L d x, which is 2 by L, minus cos m pi x by L, by m pi by L. So, I can always put a plus sin and go from 3 L by 4, to L by 4. So, that is 2 by m pi, cos of m pi by 4 minus cos 3 m pi by 4 and this is what I have for C m. So, certain things are obvious, remember that m takes values 1 upwards. So, there is no problem here. But, for all even m’s this vanishes and it only survives for odd values of m.

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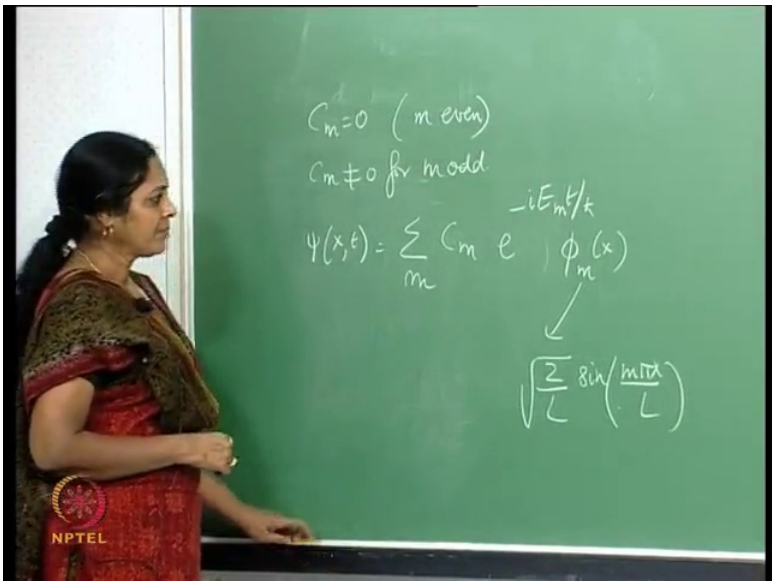

So, I have C m equals 0, for even values of m and C m not equal to 0, for odd values. Of course, m takes only integer values. So, I can substitute now and get psi of x t, as summation over m, 2 by m pi. So, C m is known, you can substitute (Refer Slide Time: 06:26) 2 by m pi, cos m pi by 4 minus cos 3 m pi by 4, e to the minus i E m t by h cross phi m of x, which is known, phi m of x is root of 2 by L, sin m pi x by L and therefore, you find psi of x t.

So, for a given initial state; an arbitrary initial state we now know what the temporal development is like and how it evolves, at a subsequent time. Now of course, at some instant of time later time, this could become one of the Eigen states themselves. It could continue to be a linear superposition of the Eigen states, many things can happen. So, but, this is an example where the initial state (Refer Slide Time: 06:26) was an x independent object.

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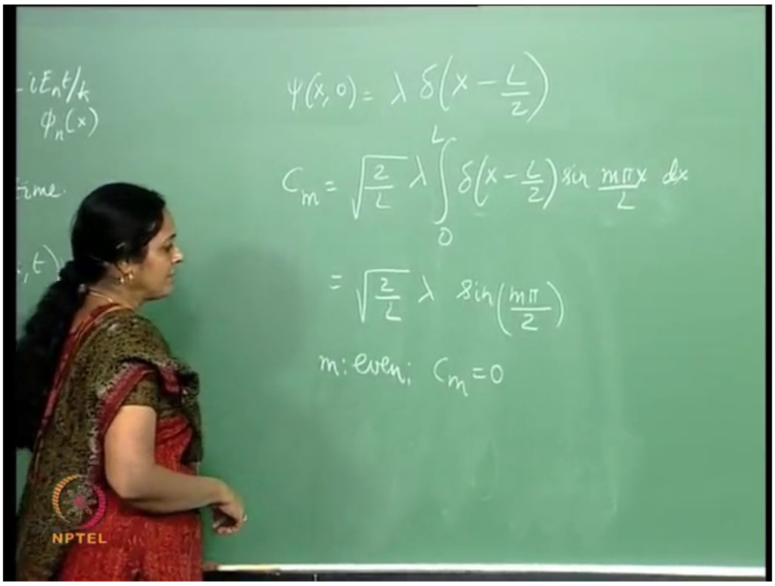

So, let us go to the next level of complication. I am going to use a different psi of x, 0. So, let us say, that this is some positive constant lambda, delta of x minus L by 2. So, that is the delta function there and that tells us therefore, that the state is sharply localized, at x equals L by 2. So, somewhere in the middle of the box and nowhere else there is no spread and there is a weight here, there is a lambda.

So, what happens in this case? So, once more we can find out C m and that is root of 2 by L lambda, integral 0 to L, delta x minus L by 2 sin, m pi x by L d x. To see what this is, this is simply sin m pi by 2, because, I just replace x by L by 2. So, one thing is clear, if m is even, the C m's vanish, as before and if m is odd the C m's survive.

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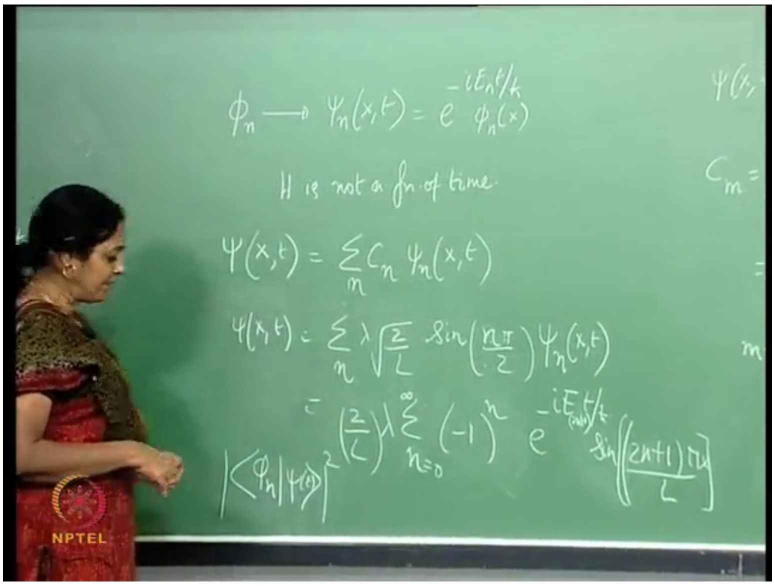

So, what does it mean, for psi of x t? Psi of x t is summation over n, C n. These are constants, sin n pi by 2 and that is as far as C n is concerned, psi n of x t. Where, psi n of x t is simply e to the minus i E n t by h cross phi n. So, this object, is root of 2 by L lambda summation over m, sin n pi by 2, e to the minus i E n t by h cross, phi n of x. But, you can already see what this is going to be. Sin n pi by 2 is minus 1 to the n and this is what I have. And, if I also substitute for phi n of x, I pick up a 2 by L here, times lambda sin m pi x by L, only for L odd.

So, let us say, this is summation over n 2 n plus 1, pi x by L and this n is again really an odd number, 2 n plus 1, where n runs from 0 to infinity. I asked the question, what is the probability that psi of x, t is in some Eigen state phi n? In other words, I ask for this. What is the probability? That this wave function happens to be in an energy Eigen state; Because, you can see there is going to be a mod square, that comes up for the probability and that is essentially going to involve, this object mod square.

In fact, these C n's of course; they are non zero only for odd n's, odd values of n. It is not finite. It is not finite for a very good reason. We did not start, with a normalized state (Refer Slide Time: 09:56) psi of x 0. If you want to normalize this, you do integral psi star of x psi of x d x is 1 and then, the square of the delta function is involved and that is not defined. And therefore, if you take something, like a delta function for the initial state, you see that you cannot have a consistent probabilistic interpretation, because, mod

C n squared, is not even finite and the reason for the problem was here. So, one way of handling this, is to consider an initial state, which is not so sharply peaked, like a delta function, something which is tempered a little bit and then perhaps we can go to the limit of the delta function.

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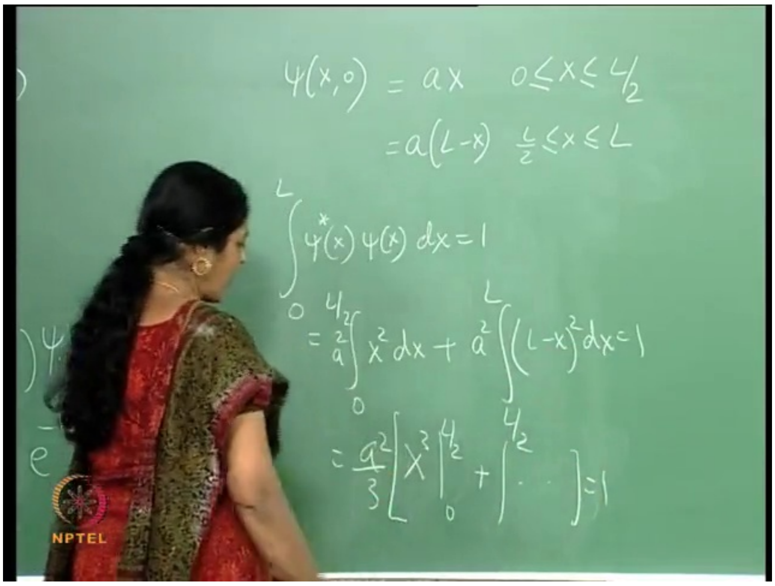

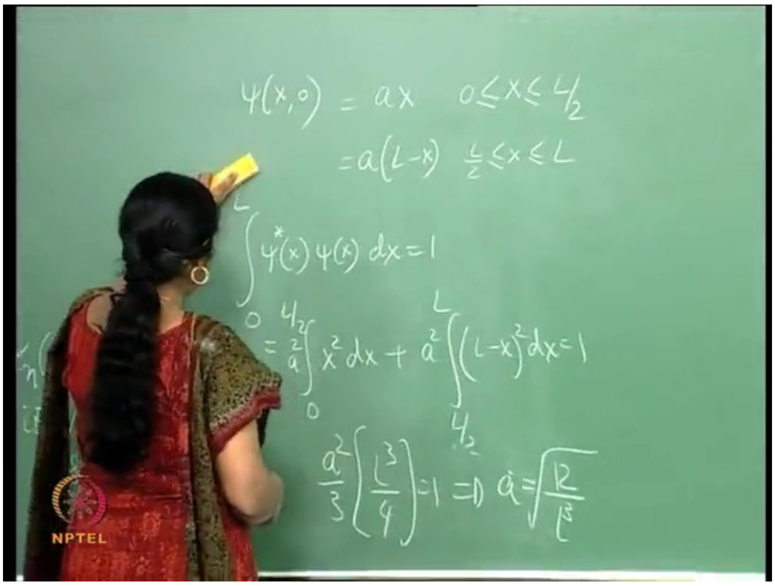

So, let us look at a different kind of initial state psi, which is somewhat tempered. So, I give you this initial state. Some constant a x, for 0 to L by 2 and a times L minus x, for L by 2 to L. So, the whole idea was to take a function, which is normalizable, in other words, which belongs to the class of square integrable functions. So, which means we have to fix a. So, integral psi star of x here it is real, d x from 0 to L should be 1 and that means 0 to L by 2 a squared x squared d x, plus a squared integral L by 2 to L, of L minus x the whole squared d x is 1. That should fix a for us.

So, this tells us that, a squared by 3, times L cubed from 0 to L by 2, x cube from 0 to L by 2. Plus this integral, should be 1 and it is clear what happens, a squared by 3 can be pulled out and I have L cube by plus L cube by 8. So, that is L cube by 4 is 1. In other words, a squared is 12 by L cubed and therefore, a, is root of 12 by L cubed. This is therefore, a good wave function, belongs to the class of square integrable functions and I have fixed a. So, that integral psi star psi d x is 1.

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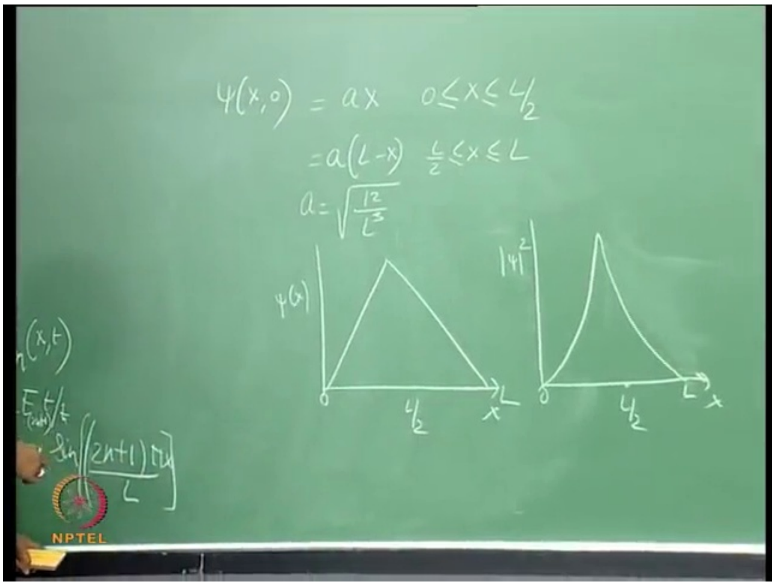

So, with this function, let see how, the initial state looks. So, if I plot psi star psi of x versus x, that is L by 2. So, here there is a linear rise, upto L by 2. So, it looks like a tent and this is psi of x versus x. So, it is not the constant that we considered in our first example. It is not the delta function, which we considered in the 2nd example, something like this. So, how does the probability density look? So, if I plotted, psi star psi, just write this as modulus of psi squared, or psi squared, ((Refer Time: 19:07)) what happens? Of course, it is 0 here and 0 there and the maximum probability is at L by 2, something like that, symmetric of course, like a caspe.

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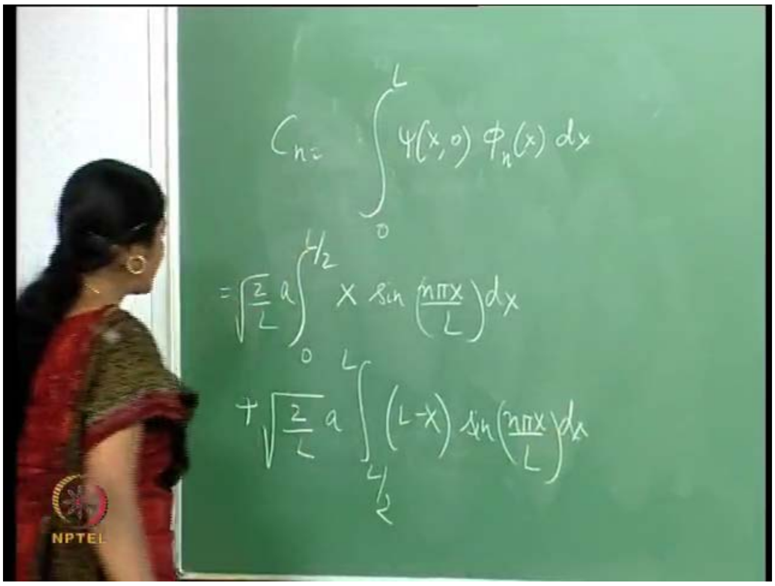

So, this is a good function to consider and now, in this case, let us find out the C n's and therefore, psi of x, t the subsequent evolution. So, C n is integral psi of x, 0 phi n of x, d x from 0 to L. So, from 0 to L by 2, it is a x, root of 2 by L, sin n pi x by L, d x. Plus root of 2 by L a integral L by 2 to L. L minus x sin n pi x by L d x. So, this is the object that I have to find. This gives me C n and I substitute it back and get psi of x t, because everything else is known; (Refer Slide Time: 11:38) psi n of x t is know and C n's are these.

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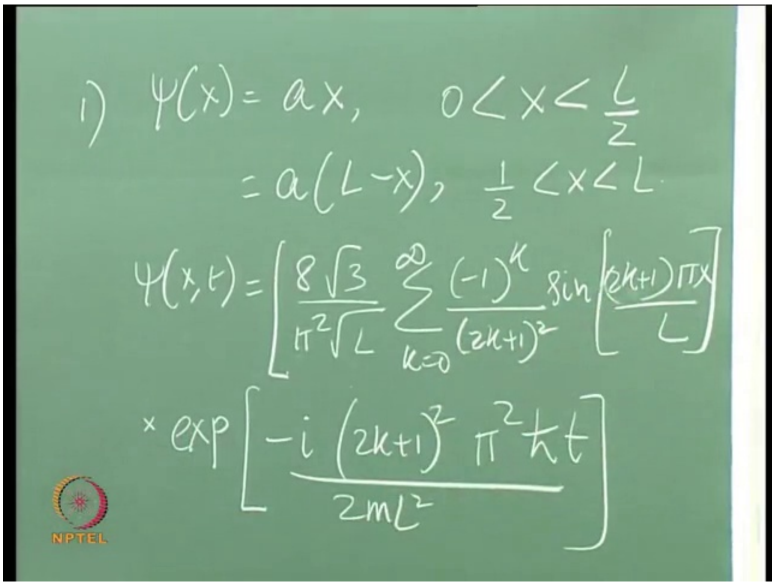

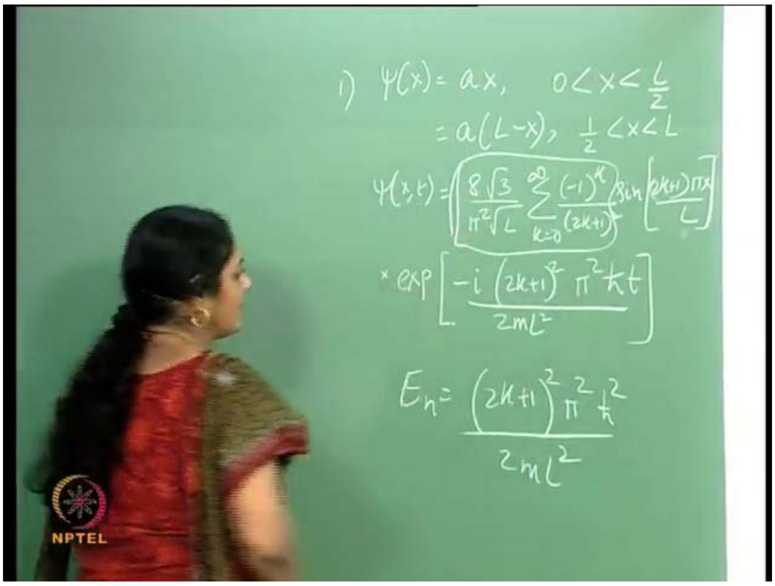

Now, if you did that, you get this to be psi of x t. Apart from a constant, 8 root 3 by pi squared root L. There is a contribution minus 1 to the k by 2 k plus 1 the whole squared, sin n pi x by L. Note that, contribution comes only when n is odd and that is why I have written, instead of n I called it k and therefore, I have written 2 k plus 1 pi x by L.

This, integral out here vanishes the C n's vanish (Refer Slide Time: 19:53) for n even and there is a contribution, only for n odd. And that is why there is a 2 k plus 1 pi x pi L there and then of course, I have to put in an e to the minus i E n t by h cross. So, remember that E n in this problem is n squared, pi squared h cross squared by 2 m L squared and therefore, I have exponential of minus i e t by h cross and that is why, there is just 1 h cross out there. This is what you get for psi of x t. Leave it to you as an exercise, once you do this. You can ask, what is the probability that psi of x t is 1 of the phi n's?

Now, if you looked at that essentially, you will have to look at the C n's. Look at this factor here. The probability of course, this exponential is not going to make a contribution. The C n's it is not C n squared which is going to give us a contribution and that object, is independent of time. And therefore, the probability density for psi of x t, to be one of the energy Eigen states, is independent of time. And, that is what we get, as a result for this problem. So, these are the various cases that, you can consider as examples for initial states, in a square in a particle in a box. And, each of the problems that I have considered, demonstrates different aspects of the problems that you can face, when you work with a particle in a box.

And, in general now, you have a machinery (Refer Slide Time: 11:38) by which you can get hold of psi of x t. Given an initial state psi of x, 0, which is not an energy Eigen state, in many physical problems, 1 is compelled to find the subsequent evolution. For instance, when a laser interacts with an atom, when light interacts with an atom, the light perturbs the atom and therefore, both the initial state of the light and the initial state of the atom change in time. And, what you will have is a subsequent evolved state psi of x t, which could show entanglement. Those will be problems that I will consider, when I do perturbation theory, in the next few lectures.

But, as such now, that we have started doing dynamics, in this case because, we only have an initial state and see how it involves under a Hamiltonian. Now, these are 1 dimensional problems and things are relatively simple. But, for instance when you consider interaction of light with atom, you have 2 different states. You have the state of the atom and the state of the light to consider and initially, they could be factored product states.

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For instance, initially the state of the light could be a coherent state. So, if I refer to the light as subsystem A of the entire system. A could well be the coherent state and it is a factored product, of the atom state. May be the excited state of the atom, may be the ground state of the atom and I call the atomic system B.

Now, this is the factored product but, depending upon the kind of interaction, between the light and the atom. Subsequent evolution may not lead to such a factored product. And while this is psi of x 0, psi of x t could be an entangled state. In other words, you may not be and in general, you will not be able to write psi of x t, is a factored product of the atom state and the laser light state; the light state. So, the radiation field state. So, these are the kind of problems that we will consider subsequently. But, to begin with since we know the particle in a box problem, it is good to look at this and since, we know the harmonic oscillator problem. It is good to look at the coherent state of the oscillator and see how it evolves in time.

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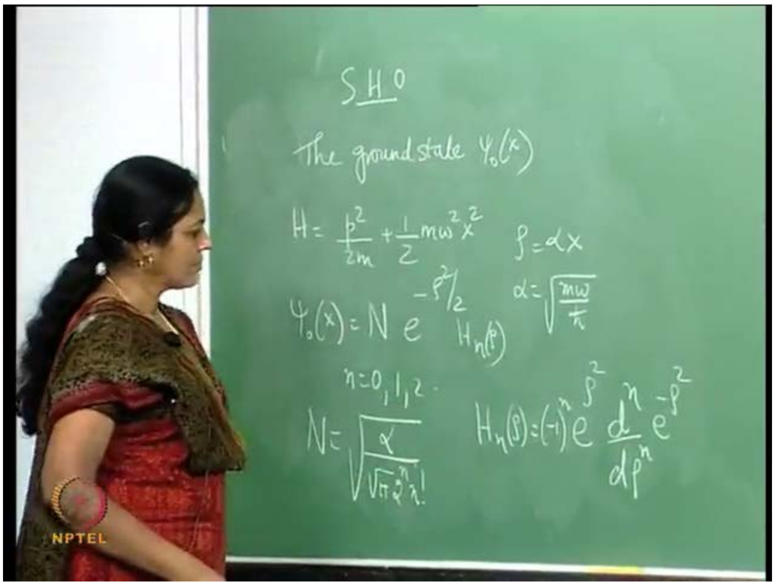

So, therefore, bearing this application in mind, which, I will look at in subsequent lectures, I now wish to do another example; not the particle in the box. But, the simple harmonic oscillator consider, the ground state psi 0 of x. You will recall that the Hamiltonian was p squared by 2 m, plus half m omega squared x squared. You will also recall that the ground state of the oscillator, in the position representation was a Gaussian and in fact psi 0 of x, apart from a normalization n, it had e to the minus rho squared by 2. I define the dimensionless variable rho is alpha x, where alpha had dimensions of inverse length, times H n of rho and n of course, took value 0, 1, 2 and so on.

So, this was a normalization. This came by looking at what happens when psi goes to infinity; the asymptotic x goes to infinity, or rho goes to infinity. The asymptotic form and the H n of rho's are the hermit polynomials. This normalization n itself is root of alpha, by root pi 2 to the n, n factorial and H n of rho, is the hermit polynomial. It is e to the rho squared, minus 1 to the n, d n by d rho to the n, e to the minus rho squared. So, this is what you will substitute here, to get the ground state of the oscillator.

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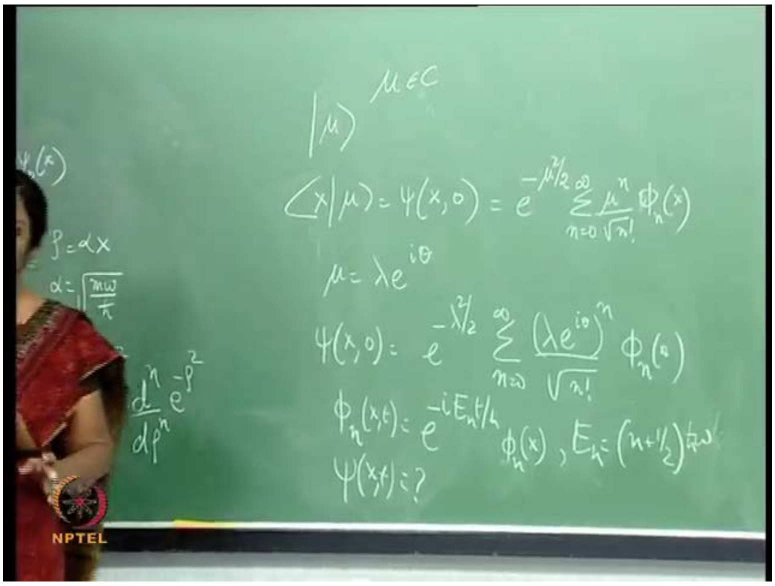

Now, with this ground state of the oscillator, you know that you can construct a coherent state, which is got from the ground state, by using the displacement operator. The coherent state itself is a minimum uncertainty state. It is another Gaussian and I will refer to that as state mu, in abstract notation, mu being a complex number. Of course, I could do this and this gives me psi of x. My initial state in this problem is this object, which is e to the minus mu squared by 2, summation n equals 0 to infinity, mu to the n by root of n factorial, psi n of x. Where, psi n of x is the energy Eigen state, maybe I should call it phi n of x. Because: all the while I have been referring to the energy Eigen state, as phi of x.

So, call this phi 0 of x. This 2 is a minimum uncertainty state, as we have seen and if I write mu as some lambda e to the i theta, because, it is a complex number in general. Psi of x 0, in this problem is e to the minus lambda squared by 2. Summation n, which goes all the way to infinity, lambda e to the i theta to the n by root n factorial, times phi n of x. This is (Refer Slide Time: 26:50) phi n of x. This is n-th Eigen state phi n of x. This is a normalization which I have put down here. So, I can well substitute for all these objects and I will get psi of x 0.

Question is what is psi of x t? It is clear that the phi n of x, are stationary states of the system. And therefore, phi n of x t is e to the minus i E n t by h cross, phi n of x and e sub n itself, is n plus half h cross omega. So, all the information is there and now, we merely need to find out. What exactly is the subsequent evolved state psi of x t, or the initial state, which was the coherent state? How does that look at the end of time t?

Now, in general several things can happen, a coherent state. In other words in this case a minimum uncertainty state, or Gaussian, need not continue to be a Gaussian. In general, therefore, it need not be a minimum uncertainly state for all times and it could do anything.

It could simply be a very complicated looking object as a function of space and time. In fact, at every instant of time, it could put out tendrils; the wave function can be very complicated. But, of course, with the condition that it can be expanded in terms of the basis states of the harmonic oscillator Hilbert space. In other words, it can be expanded in terms of the energy Eigen states, at all times. But, given that it need not have the Gaussian shape and it can change arbitrarily. Whatever, I have said is true for a generic Hamiltonian.

The précised question that we are asking is. How does it look? How does it evolve, under the harmonic oscillator Hamiltonian? So, that is the problem that we would like to look at now.

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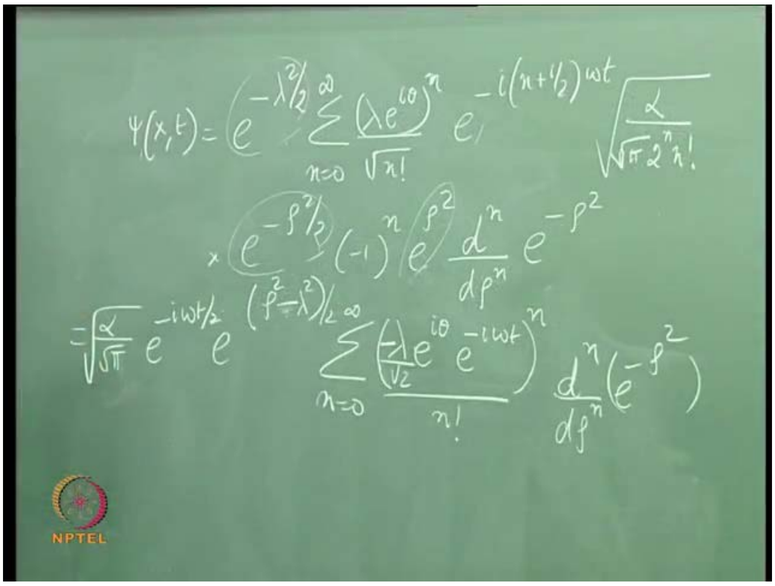

So, let us find out, psi of x t in this problem. Substitute for everything. So, (Refer Slide Time: 29:11) psi of x t, is simply going to be equal to, e to the minus i E n t by h cross. Of course, let us start with the fact that it is a coherent state. So, e to the minus lambda squared by 2, summation n equals 0 to infinity, lambda e to the i theta to the n, by root n factorial. (Refer Slide Time: 29:11) phi n of x t, which is e to the minus i, n plus half omega t.

So, I have taken care of the time evolution, of the stationery state and I substitute for (Refer Slide Time: 26:50) phi n of x. It has a normalization, root of alpha by root pi, 2 to the power of n n factorial, ok and then (Refer Slide Time: 26:50) e to the minus rho squared by 2 h n of rho and h n of rho was minus 1 to the n, e to the rho squared d, by d rho to the n d n d rho to the n, e to the minus rho squared. So, this is the object that I have. So, there are certain things that I can pull out here already, first of all I can pull out a root alpha by root pi and then e to the minus rho squared plus lambda squared by 2.

This together gives me an e to the plus rho squared and therefore, it is rho squared minus lambda squared by 2. So, I have taken care of this object and this and this. I guess I have to put everything else inside the summation. So, I have summation n equals 0 to infinity, lambda e to the i theta and then I have an e to the minus i omega t. So, e to the minus i omega t, the whole to the n. But, then I also have a root 2 here. So, perhaps, I can write a lambda by root 2, the whole to the n.

There is an n factorial because; there is a root of n factorial there and a root of n factorial out there and therefore, that gives me this object. But, I also have an e to the minus i omega t by 2, which, I can pull outside, minus 1 to the n. Therefore, I will put a minus sign here and then, I have d n by d rho to the n, e to the minus rho, d n by d rho n e to the minus rho squared of e to the minus rho squared.

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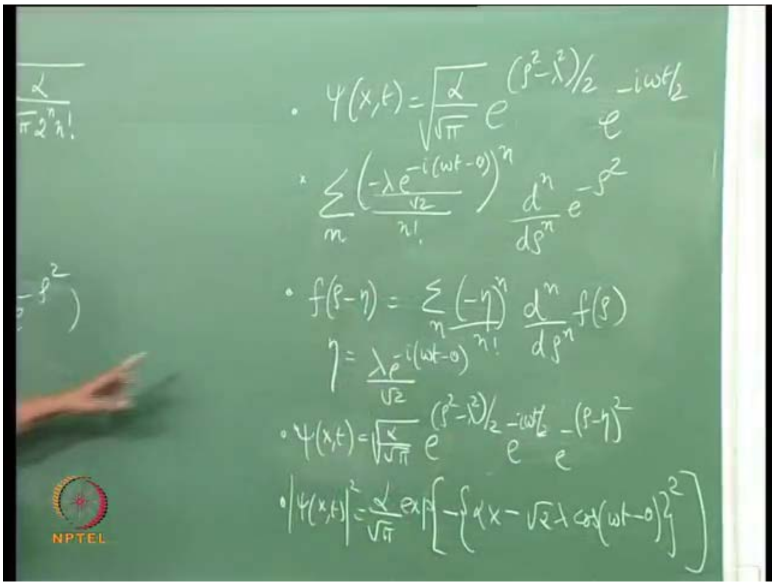

So, this is what I have, which is precisely the expression that I have written out here for you. If you look at it root of alpha by root pi, came from the normalization. For the Eigen state, energy Eigen state, e to the rho squared minus lambda squared by 2, e to the minus i omega t by 2. Is what I have pulled out here (Refer Slide Time: 33:37). Then there is a summation over n, with precisely the factors that I have put down there. But, you compare this psi of x t, this expression for psi of x t, with the Taylor expansion for f of rho minus eta.

You can Taylor expand; write a tailor series for any function of rho minus eta in this fashion. So, if you compare, with whatever you have here. There is a summation over n, minus eta to the n by n factorial, d n by d rho n e to the minus rho squared. So, this f of rho is e to the minus rho squared and the eta is out here. So, I have identified eta, in the Taylor expansion; as lambda e to the minus i omega t minus theta by root 2.

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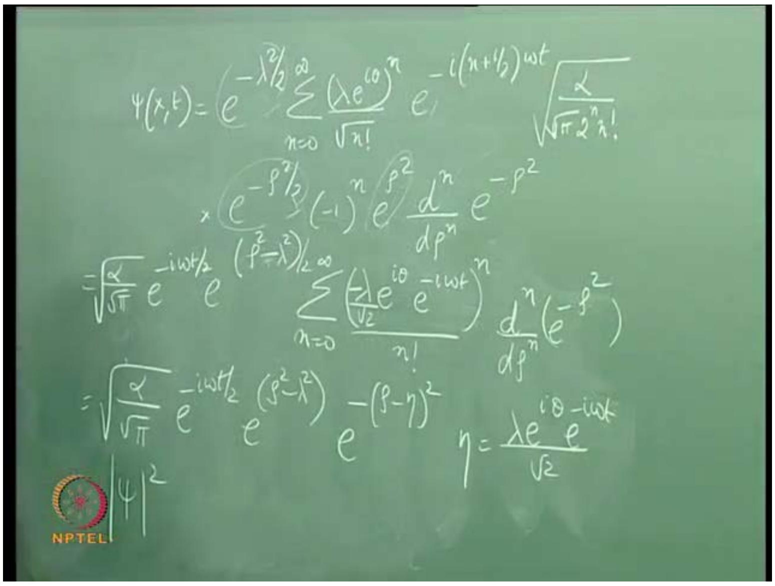

And therefore, what I have, is psi of x t, which can be now written, as root of alpha by root pi, these things do not change, e to the minus i omega t by 2, e to the rho squared minus lambda squared. But, I can see that this (Refer Slide Time: 37:04) compares with my Taylor series and therefore, I can write this as e to the minus. This is my function of rho. Where eta itself is lambda e to the i theta, e to the minus i omega t by root 2. So, that is what I have.

By itself this is not particularly instructed. Suppose, I find psi star psi, suppose, I find the manner in which, the probability density varies. Of course, this would cancel out, I have an alpha by pi, remember that the eta has a time dependence in it and if you put that all in. I leave it to you as an exercise to show this (Refer Slide Time: 37:04) this is the exercise. Modulus of psi the whole squared is alpha by root pi exponential of this object.

The time dependence came not from this. Because: in psi star psi this term would cancel out. But, eta has a time dependence and from there (Refer Slide Time: 37:04) the time dependence turns out to be cosine omega t minus theta. So, it is an exponential of minus of this object, the whole squared remember, that alpha has dimensions 1 by length. It is root of m omega by h cross. So, this is what I have. So, what does it tell me. First of all the initial state is a Gaussian function of x (Refer Slide Time: 29:11) psi of x 0, if you put things in was a Gaussian function. (Refer Slide Time: 26:50) It was e to the minus alpha squared x squared by 2. So, this was a Gaussian function of x.

At any instant of time, it continues to be (Refer Slide Time: 37:04) a Gaussian function of x. The motion itself is simple harmonic. So, what happens to the ground state of the oscillator, or a coherent state that taken a coherent state a Gaussian: As a function of x what happens to it? First of all, at time t equals 0 it starts of as a Gaussian and then it simply executes simple harmonic motion, without changing it is Gaussian form. So, in the harmonic oscillator potential, a coherent state continues to be coherent and in a very popular sense of the term. You can understand why this is called a coherent state.

It does not change its form. It does not change its height. It does not change it is width. It simply continues to oscillate in time, executing simple harmonic motion. In that sense, it is like a classical oscillator and the coherent state, again in that sense is a closest you can get. Among the quantum states and their super positions to something very classical. It is a minimum uncertainty state and in the oscillator potential, it simply moves back and forth, executing simple harmonic oscillation. Of course, the amplitude of the oscillation would depend upon lambda (Refer Slide Time: 37:04). But, unlike the case of other more complicated potentials, which, I would talk about in my subsequent lectures. In the harmonic oscillator potential, the coherent state continues to be a coherent state, merely, executing simple harmonic motion.

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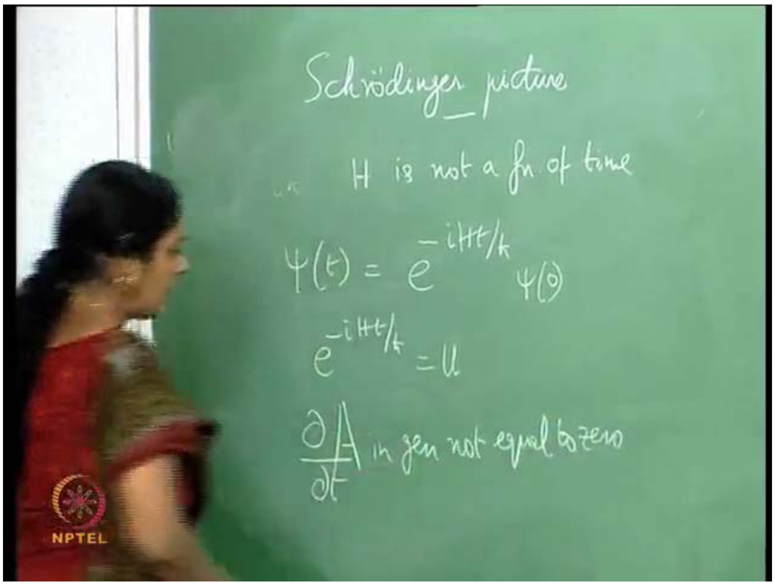

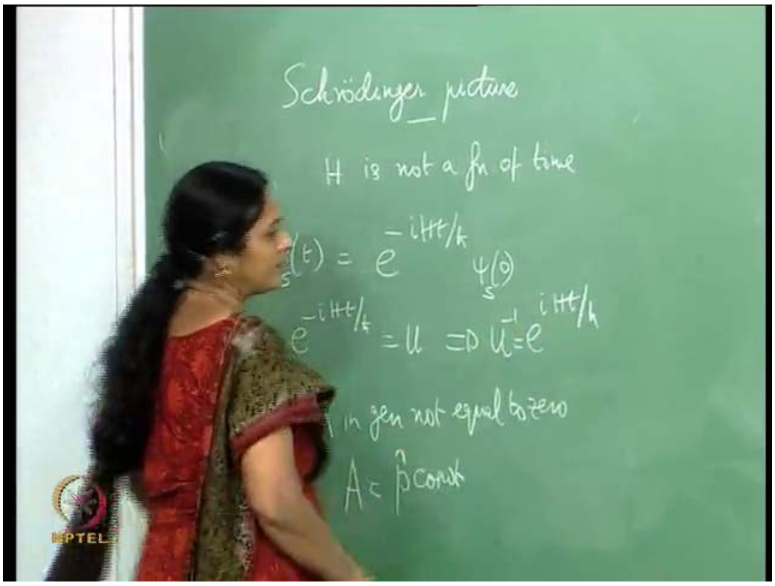

So, having looked at these examples: Let me finally, look at an alternative picture; the Heisenberg picture which, I was mentioning. Where operators evolve in time, given by the Heisenberg equation of motion and the state vectors do not. So, as a prelude, I have already indicated to you what happens in the Schrodinger picture. There is the time dependent Schrodinger equation. Let me start by saying that h is not a function of time; the Hamiltonian is not a function of time. Then psi of t is e to the minus i H t, by h cross psi of 0 and I call e to the minus i H t by h cross as u, since, it is a unitary operator. A itself any operator A can have explicit time dependence. I gave an example could be A cos omega t, or p cos omega t. All this is possible, where p is a linear momentum.

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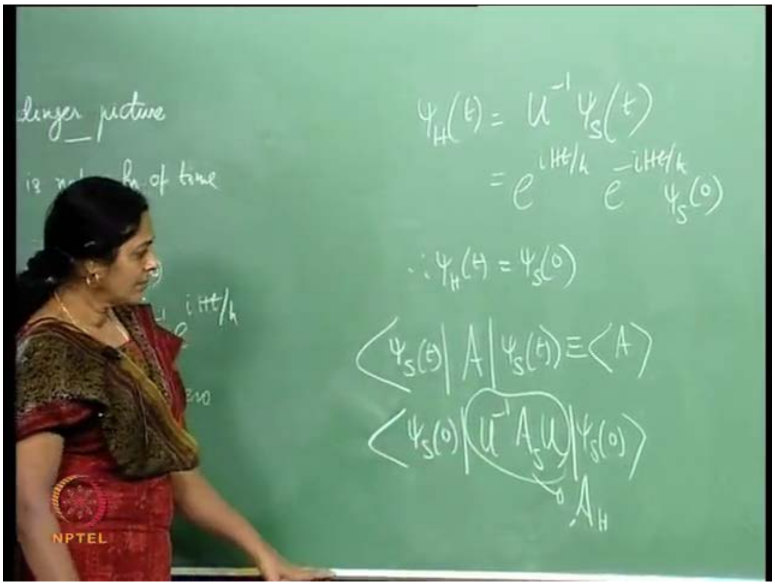

Now, having said that, suppose I consider an alternative picture where, I do the following: I say that in the Heisenberg picture, which is the other picture that, I will consider psi h at any instant of time, is u inverse psi Schrodinger, which is what I was considering here. So, let us put (Refer Slide Time: 42:44) psi s of 0 psi s of t, to show that we are in the Schrodinger picture.

So I go from the Schrodinger picture wave function at an instant of time. To the Heisenberg picture wave function at the same instant of time, by applying u inverse. And, remember u inverse is e to the i H t by h cross. So, I have e to the i H t by h cross, psi s of t is e to the minus i h t by H cross, psi s of 0. And therefore, the state vector in the Heisenberg picture does not evolve in time.

You take the initial state in the Schrodinger picture and that is going to be the Heisenberg picture state vector, for all times. So, how does the dynamics come in? The dynamics comes in through the operator evolution. Expectation values at any time will have to be the same whether, you are in the Schrodinger picture, or in the Heisenberg picture.

Because, expectation values are measurables and the picture you choose to work with, is often just a matter of convenience. And therefore, I could have well done this by starting with psi s of 0, u inverse A u psi s of 0. Remember that, u psi s of 0, is psi s of t and therefore, u inverse psi s of 0 on this side, gives me the bra psi s of t. So, if I now say that u inverse A u is the operator. In the Heisenberg picture and it is evident, that why psi h of t is simply psi s of 0. The operator A h is u inverse A is Schrodinger u and then it is clear, that the expectation values at any time, in both pictures are the same. And, that is what we want, because; the measurables should not depend upon which picture we use for convenience of calculation.

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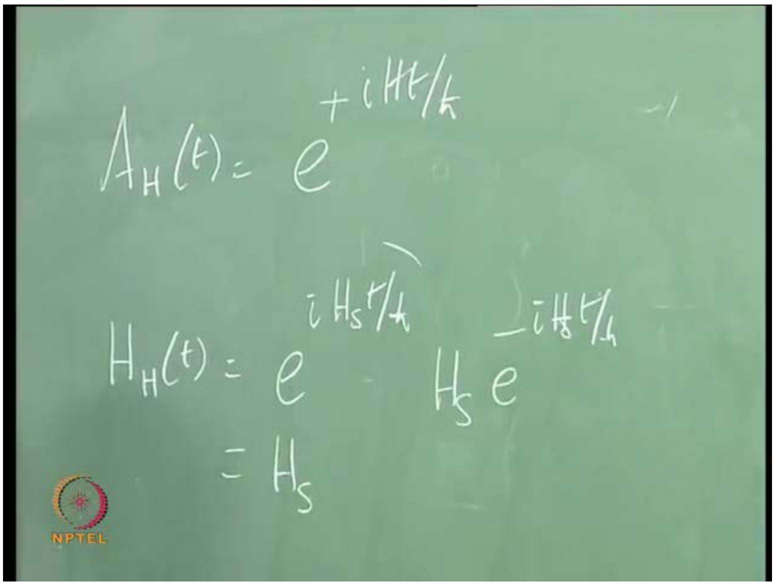

So, what does that mean? By way of an equation of motion for the operator, I have A H of t, is e to the minus u inverse. So, e to the i H t by h cross. Oh! Remember by this token, what is the Hamiltonian in the Heisenberg picture? It is e to the i Hamiltonian in the Schrodinger picture; this is u inverse, H s u and since things commute with each other, that is just H s. So, and the Hamiltonian therefore, is not a function of time and continues to be the same, be it the Schrodinger picture or the Heisenberg picture.

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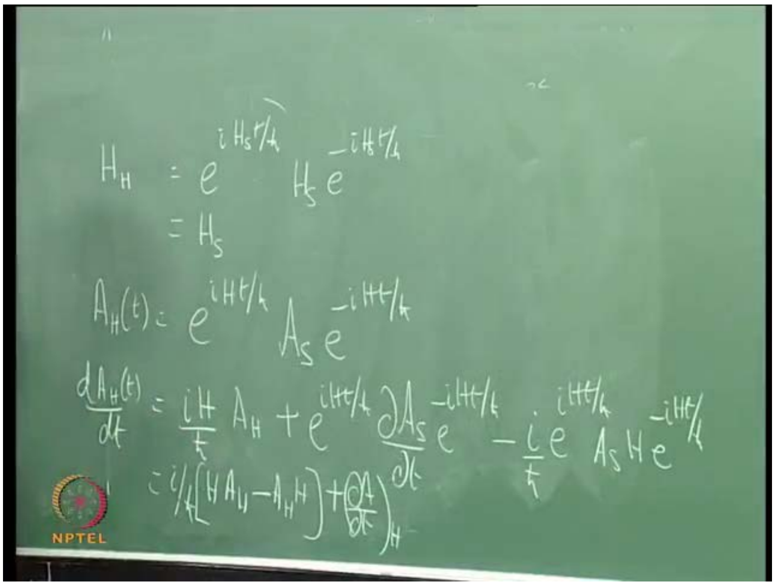

But, now let us look at any operator, A H of t. That is e to the i H t by h cross, we will not put a suffix, s or H here, because, we have said the Hamiltonian's are the same; through this transformation. A s e to the minus i H t by h cross. So, let us get an evolution equation for A H. So, d A h of t by d t, this can only have explicit time dependence, if at all. So, in the differentiation, I have an i H by h cross, e to the i H t by h cross A s e to the minus i H t, which is A H. Plus each of the i H t by h cross delta a s by delta t e to the minus i H t by h cross. This differentiation is non zero; this value is non zero, if A s has explicit time dependence.

Then, there is a 3rd term, which is minus i by h cross, e to the i H t by h cross. A s H e to the minus i H t by h cross. So, Therefore, this is i by h cross H a H, minus notice that this object, commutes with H. The same thing out there, and therefore, that term can be written as minus A H H. So, I have a commutator, plus this quantity, I am going refer to it, I am going to refer to this quantity as delta A by delta t in the Heisenberg picture.

This is a notation that I use; by this I mean differentiate with respect to time, just the explicit time dependence in the Schrodinger picture. Sandwich it, between u inverse and u and you get this object, in the Heisenberg picture, this is my definition.

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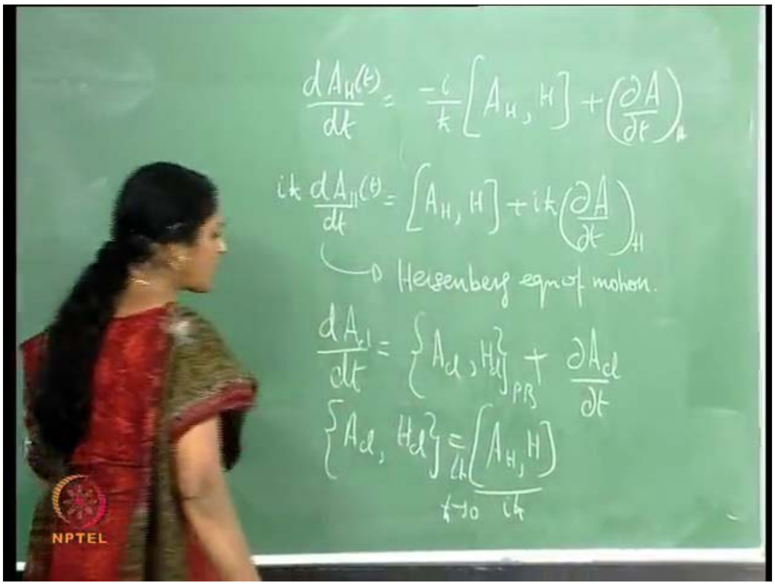

So, what does that imply; for the equation of motion? I have the following, d A H of t by d t, which is the total time dependence. Is minus i by h cross, commutator of A H with H, plus delta a by delta t, in the Heisenberg picture. In tune with the Schrodinger equation, let us work with i h cross, d A H by d t. This object is simply the commutator of A H with H, plus i h cross delta A by delta t, in the Heisenberg picture.

Now, this is called, the Heisenberg equation of motion. Like you have the Schrodinger equation in the Schrodinger picture, which, tells you how the state vector evolves in time, or the wave function evolves in time. In the Heisenberg picture there is an evolution of the operator with time. T he total evolution has two parts: one given by the commutator of the concerned operator with the Hamiltonian and the other comes through a possible explicit time dependence. Compare this with what you have in classical physics. In classical physics any dynamical variable A, if you want to find out it is evolution; temporal evolution, A classical d A classical by d t is the Poisson bracket, with the classical hamiltonian, plus delta A classical by delta t.

Compare, these two equations. The Heisenberg picture seems to be a closer to classical physics, because, the equations are very similar and all certain done is a dynamical variables in classical physics. That are replaced by operators, in quantum physics. How do I make the connection? Well, the Poisson bracket, here is simply the commutator bracket by i h cross. So, let us write that down, the Poisson bracket of A classical with H classical, is just the commutator bracket, by i h cross in the limit h cross going to 0, because, that is quantum physics, going to classical physics.

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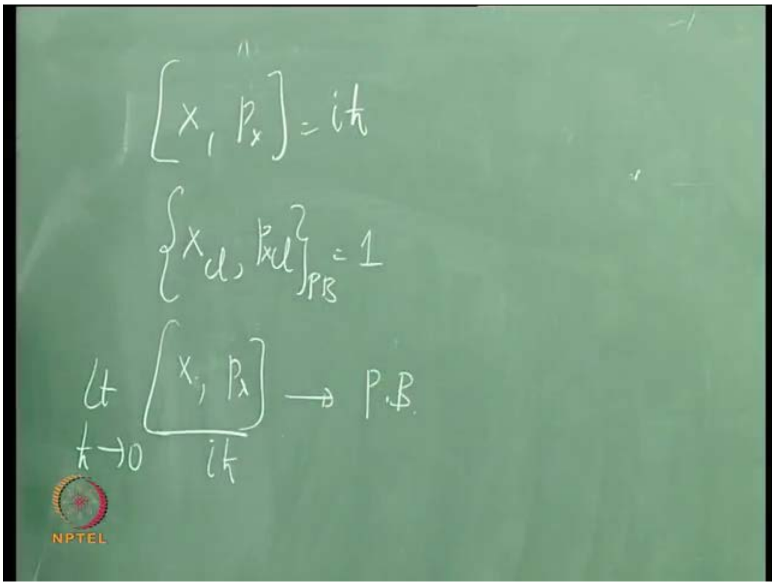

Take a simple example, take for instance, the commutator bracket, between x and p x, i h cross and what is the classical, x classical, with p classical, p x classical Poisson bracket. That is 1, so what is the commutator x p x by i h cross, in the limit of course, h cross going to 0, which is ok in this case, the h cross cancels out the numerator and denominator. This goes to the Poisson bracket. So, that is one of the good things about the Heisenberg picture, it looks very similar the evolution equation (Refer Slide Time: 50:18) is for the operator, which is simply the dynamical variable with a hat on top of it.

And the evolution equation looks very similar, the evolution equation, the Heisenberg equation looks very similar, to the evolution equation; the dynamical variable in classical physics.

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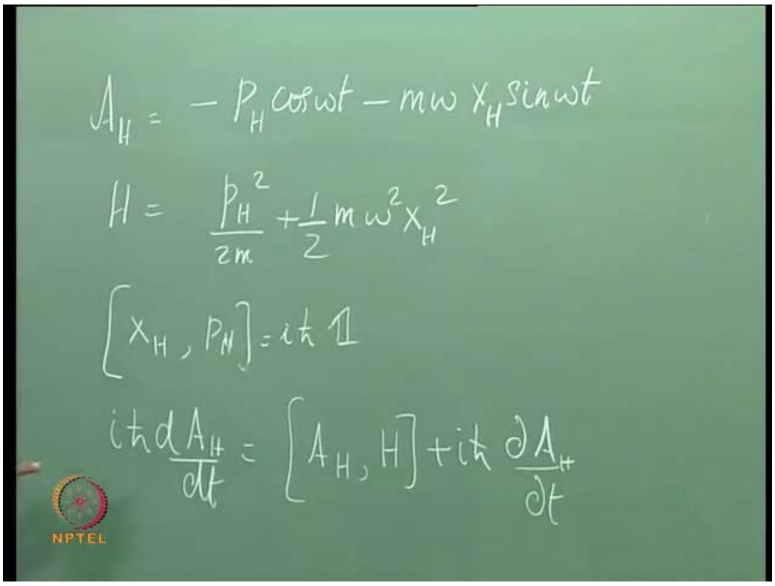

So now, let me do an example, applying the Heisenberg equation of motion. So, I consider an operator in the Heisenberg picture, A H, which is minus P cos omega t, minus m omega X sin omega t. Now, P H is simply the momentum operator, let me put down the Hamiltonian. A Hamiltonian is the Hamiltonian for the simple harmonic oscillator, which is P H squared by 2 m, plus half m omega squared X H squared.

So, you see I have got the momentum operator here and of course, the position operator, both written in the Heisenberg picture, this was an explicit dependence on time. Please remember, that the Heisenberg picture operators are unitarily transformed versions of the Schrodinger picture operators. Therefore, this is certainly true, the lie algebra holds, in whichever picture you are in. I did not write H sub H, because as I have already told you, the Schrodinger picture Hamiltonian is the same as the Heisenberg picture Hamiltonian.

So, now let us look at the equation of motion i h cross d A H by d t, is a commutator of a with H. Plus i h cross delta A by delta t, this is the explicit time dependence. In this problem, let us calculate these two terms.

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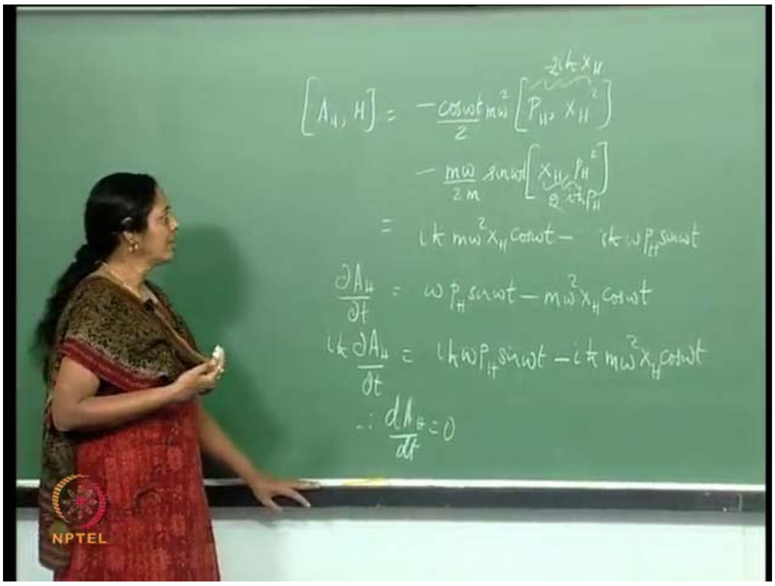

So, what is the commutator of A with H? It is minus cos omega t, by 2 m omega squared commutator of P with X squared. Because, this anyway commutes with itself, there is another term, minus m omega by 2 m, sin omega t. Commutator of X with P squared. But, this quantity is minus i h cross and this is plus i h cross. This is plus 2 i h cross P H and that is minus 2 i h cross X H, I have just used the a b c rule for commutators.

So, this object is simply i h cross m omega squared X H cos omega t. That is the 1st term and the 2nd, is minus i h cross m omega P H sin omega t. So, this is the commutator. Now, let us look at the explicit time dependence. Delta A H by delta t the 1st term gives me an omega P H sin omega t and the 2nd term gives me a minus m omega squared X H, cos omega t. And therefore, what I need is i h cross times that, the m cancelled out here, the m cancelled out here. So, there is no m out there and this is what I have, yeah.

So, now coming back to this step, i h cross delta A H by delta t gives me an i h cross omega P H sin omega t, minus i h cross m omega squared X H cos omega t. So, the commutator of A with the Hamiltonian, plus i h cross delta a by delta t is 0. This cancels out with that and that cancels out with this. What is the point that 1 is making? A does not change with time, however, that does not mean that it commutes with the Hamiltonian.

I have created a problem, a situation, where the commutator is not 0, there is an explicit time dependence and that makes a non zero contribution. That between the two of them the total change (Refer Slide Time: 54:22) d A H by d t is 0. So much for the Heisenberg picture I have given an example and in the subsequent lectures, we will take up dynamics more intensely and look at the various ramifications of time evolution.