Quantum Mechanics - I
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Lecture - 2
Linear Vector Spaces – I

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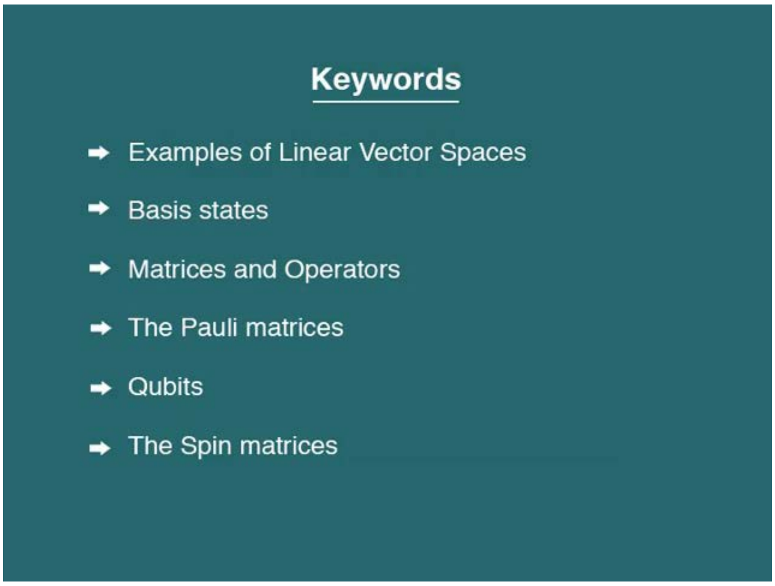

In the last lecture, I mentioned that linear vector spaces are very important in understanding the structure and framework of quantum physics. So, today I will introduce certain salient features of linear vector spaces and also point out to you towards the end of the lecture, how exactly these concepts, mathematical concepts become important in the discussion of physical systems.

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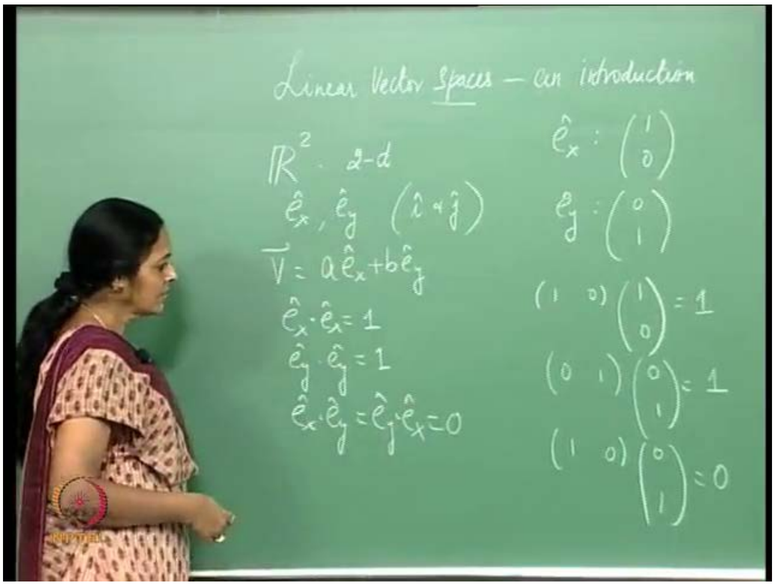

So today, I will talk about linear vector spaces. It is an introduction. Let us start with something we already know, which we are familiar with. And that is the two dimensional linear vector space, which I will call R 2 a 2 for two dimensions. You will have basis vectors e x and e y. You could also call them unit vectors i and j. This would be my notation. So, any vector in this space and this is a two dimensional space could be the plane of this table, the top of the table or the plane of the black board and any vector V in this space is some a times e x plus b times e y. So, a is the component of V along x axis basically because e x is the unit vector along x axis and b is the component of V along the y axis.

Now, e x and e y themselves are orthonormal vectors. So I have the following: e x dot e y equals e y dot e x is 0 and this is the dot product, the scalar product of two vectors. I could give a different notation. I could say the following. Represent, e x by the column 1 0 and e y by the column 0 1. Now if I did that, how do I generate the number 1 from this? Well, certainly if I take the row vector 1 0 and multiply it with the column 1 0 I get 1. So, that is the same as saying that e x dot e x is 1.

Similarly if I did this, that too gives me 1 and that is the same as saying that e y dot e y is 1. And of course, if I did this if I multiplied the row 1 0, with the column 0 1, I get a 0 and that is like saying that e x dot e y is 0 and so on. So I can associate with the unit vector e x in two dimensions the two component column 1 0 and with e y the two component column 0 1. And I have reproduced these relations in this notation. But, this subtle change of notations tells us something that indeed, there are 2 types of vectors that we need to consider, one is the column 1 0 and the other is the row 1 0.

Certainly that is not obvious here, but here I am able to distinguish between two types of objects, the rows and the columns, all these vectors, which can be expanded as a times e x plus b times e y, a and b being real constants, real scalars are states in the two dimensional linear vector space R 2. So, basically what are the properties of a linear vector space? I have started with giving you an example of a linear vector space, the simplest that I can think of.

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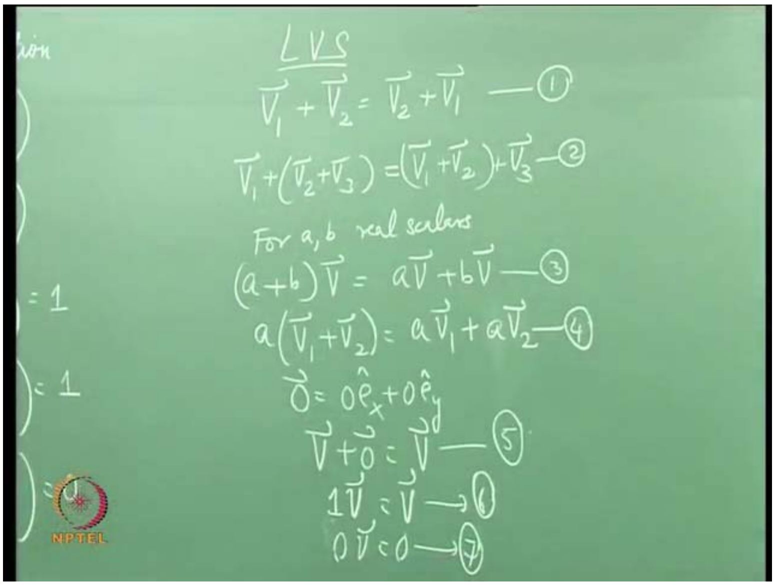

So, what are the properties of a linear vector space? I have to define what happens to states in this linear vector space when I do addition or when I multiply by a scalar. Now, if I have two vectors V 1 and V 2 in this space, there is commutativity under addition. That is certainly true, take any two vectors of this form and addition is commutative. So that is the first property. There is also associativity under addition, by this I mean a vector V over bar or the over arrow, I just used that as a notation for a vector. This is certainly true for all vectors in this space. Take any two vectors and you could add them, any three vectors and you could add them like this.

First add V 2 and V 3 and then add the result to V 1 or add V 1 and V 2 and then add it to V 3, get the same answer. Now, for a and b being real scalars, a plus b times a vector

V is a times the vector plus b times the vector. So, that is the 3rd property and then of course, I have two vectors V 1 and V 2 and I multiply the result of the two vectors, the addition of the two vectors with a, that is the same as a times b 1 plus a times b 2. I can define a null vector. I define it a 0 e x plus 0 e y, in this example of R 2 and any vector added with 0, the null vector gives me the same vector.

Then, I also talk about what happens because of multiplication by the number 1, that leaves the vector unchanged and multiplication by the number 0, that just gives me 0. This is not to be confused with this, that is a vector and this is just the number 0. So, these are the seven properties. I expect of a linear vector space, it is very easy for you to check out, that R 2 the two dimensional Euclidean space. All vectors or states in R 2 satisfy these seven properties.

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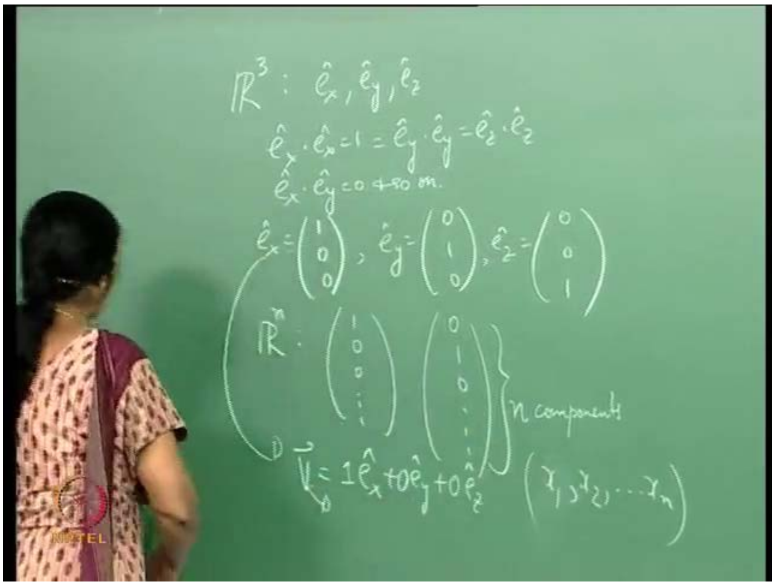

So, this is the simplest example of a linear vector space. I can think of more examples. For instance, a straight forward extension of R 2 is R 3, the three dimensional space, where I have vectors e x, e y and e z. So, there are 3 unit vectors and these are normalized to 1 and they are orthogonal to each other and so on. Now in this case, if I want to give column vector representations to e x, e y and e z. I could think of e x being represented by 1 0 0, e y by 0 1 0 and e z by 0 0 1. It is easy to check out that the fact that the vectors are normalized to 1 and that they are mutually orthogonal follows from this. I can talk of R n, where I have extended 3 to n. So, in R n once more I can define n such vectors, each one being n componented with 1 as one of the entries.

So it would be 1 0 0, 0 1 0 and this whole thing has n components. What is it that I have done here? Take this for instance, if I have a vector V, which is 1 times e x plus 0 times e y plus 0 times e z, that is really this vector 1 0 0. So, if I have a string of real numbers x 1, x 2, x 3 to x n, this n-tuple of numbers would represent a vector in R n, where the x i's are all real. I can now extend the concept further, instead of having x 1 x 2 to x n. I can talk of the entries being complex.

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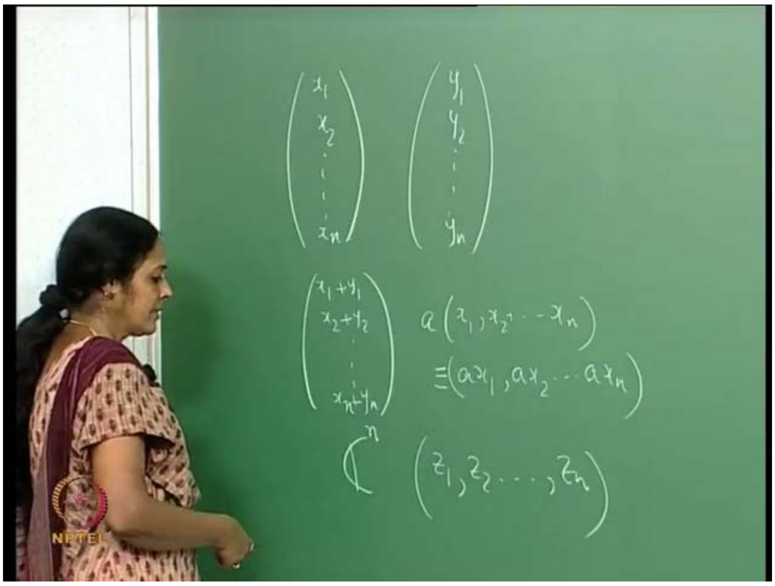

How would I define addition? Here for instance, if I have the column x 1, x 2 to x n and y 1, y 2 to y n. How do I add these vectors? When I add them the resultant would be x 1 plus y 1 x 2 plus y 2. So, there is a component wise addition and if I have to multiply these string of numbers, which I can put down in this fashion: By a, I mean a x 1, a x 2 to a x n, where a is a real scalar. So, what I am trying establish is that a vector in R 2 or R 3 or R n is in general given by a string of numbers, real entries all of them and the number of components here, you will have an n-tuple of numbers, if you are talking of R n.

The whole concept can be extended to C n, this is the complex version of R n. So, the entries would be z 1, z 2 to z n, where the z's are all complex. Once more we can check that C n is an example of a linear vector space. Are these three only examples? That is not true. I can think of other examples of a linear vector space. In fact, I can think of function spaces.

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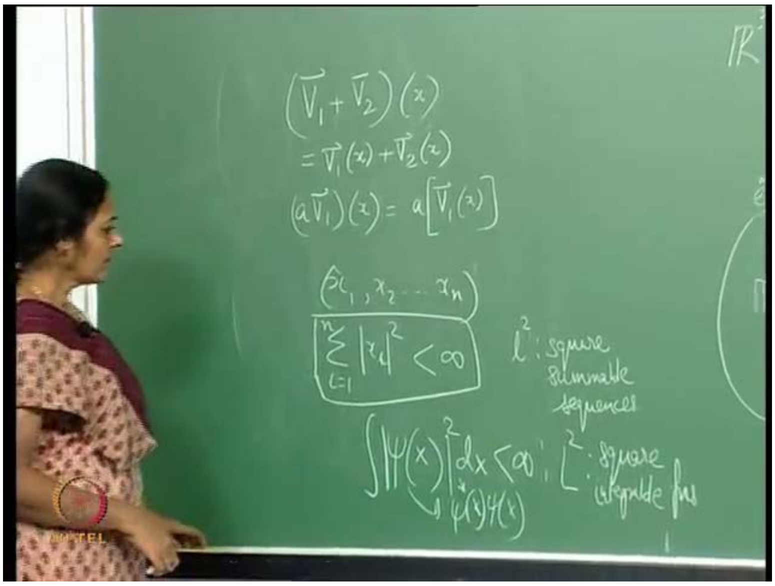

So, I want to define functions of x. Now, this is simply V 1 of x plus V 2 of x, where x is the real variable. And then of course, if a is a scalar and I wish to find this, that is the same as a times the function, V 1 of x. Now given this, I can talk of linear vectors spaces, examples of linear vector spaces being function spaces, function spaces as examples of linear vectors spaces, where I can talk of the manner in which functions combine to produce new functions. So, states in a linear vector space could be columns, could be functions, could be vectors, symbolically denoted like this. They will all be states in an appropriate linear vector space provided; these properties are all satisfied by the states.

There are some interesting linear vectors spaces. Suppose, you consider the string of numbers x 1 to x n, where x i is being real and suppose summation i equals 1 to n mod x i squared is finite, with addition and scalar multiplication defined as I mentioned earlier for an n-tuple x 1 to x n, this space is a linear vectors space denoted by l 2, the space of square summable sequences. As I mentioned earlier, these would be like components along various directions, if you wish e x, e y, e z and so on. And this property becomes important, when we do the probabilistic interpretation of quantum mechanics which is a very a salient aspect, a very crucial aspect of quantum physics.

The fact that summation i equal 1 to n mod x i squared, where these are the various components along the various directions in the linear vector space, if you wish crudely speaking. Then the fact that this is finite is an important input into the probabilistic interpretation of quantum physics itself. Similarly, when it comes to function spaces suppose, these are functions of x that I am considering. Suppose, psi of x is a function of x and in general psi of x is complex then mod psi of x the whole square d x, over the range in which x is defined. This could be minus infinity to infinity or any a to b. This is less than infinity where psi of x is a function of x, a function space where this holds is called l 2, the space of square integrable functions.

And these two become very important in quantum mechanics. In fact, Schrodinger version, wave mechanics, the Schrodinger version of quantum mechanics uses psi of x, the wave function, which represents the physical state of a system. By mod psi of x the whole squared, I mean psi star of x psi of x, where psi is in general a complex function and psi star of x is the complex conjugate of psi of x.

I have therefore, given you several examples of linear vector spaces, ranging from the familiar column and row vectors, column vectors, 1 0 0 1 and so on to function spaces. So, the first thing one understands is this. That a state in a linear vector space need not be a column, it need not be a set of real entries, n-tuples of real entries. It could be one of a variety of things that particular linear vector space that we consider could even be a function space. In particular could be the space of square integrable functions or it could be a space of square summable sequences. So, there are very many examples of linear vector spaces. The point is, one should be able to define addition and multiplication by scalars in such a fashion, that these properties are satisfied. Be the vectors of the kind that we know of an R 2 or some columns or functions, we should be able to define addition of functions and so on.

Now, one thing is obvious. Let us go back to this example, I see rows and I see columns, it is clear that rows and columns cannot be in the same linear vector space, because look at property 1, I cannot add a row to a column. Therefore, there is a linear vector space in this example for instance, the linear vector space is one where there are different columns with entries are states in a linear vector space. Then there is a dual space, the corresponding rows are members of the dual vector space. So, this brings us to the definition of dual spaces, the dual of a linear vector space.

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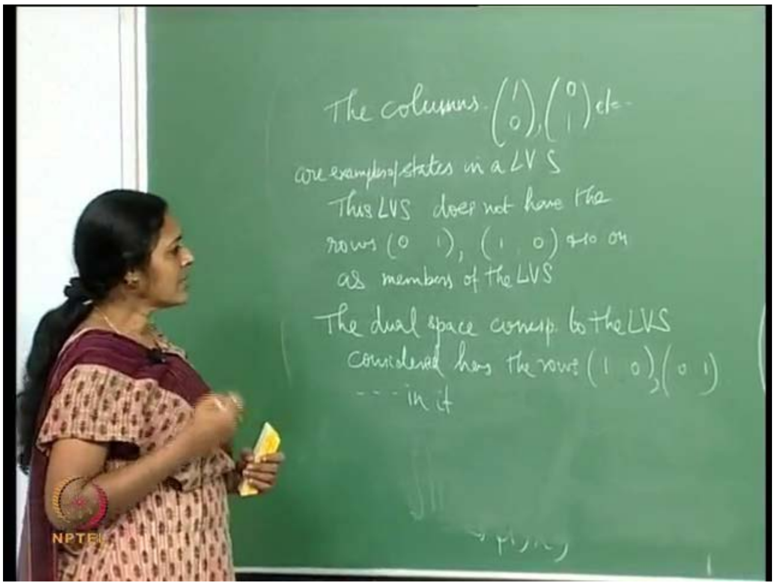

So, in our example the columns 1 0 0 1 etcetera are examples of states. This linear vector space does not have the rows and so on as members of the linear vector space. The dual space corresponding to the L V S has rows 1, 0, 0, 1 etcetera in it. Now, this aspect would not have come out if we had just stuck to e x, e y and e z and in that sense writing e x and e y in terms of columns and also introducing the row 1 0 0 1 and so on, in order to put down these relations in the language of columns and rows has helped us understand, that for the linear vector space there is a dual space.

This is merely an example every linear vector space has a dual space corresponding to it. And the members are in general different. Since, we have a wide variety of linear vector spaces and I wish to study the properties of linear vector spaces in general, this is an appropriate time to introduce a very powerful and compact notation to represent the states of a linear vector space. This notation is given to us by Dirac, one of the founding fathers of physics and quantum physics.

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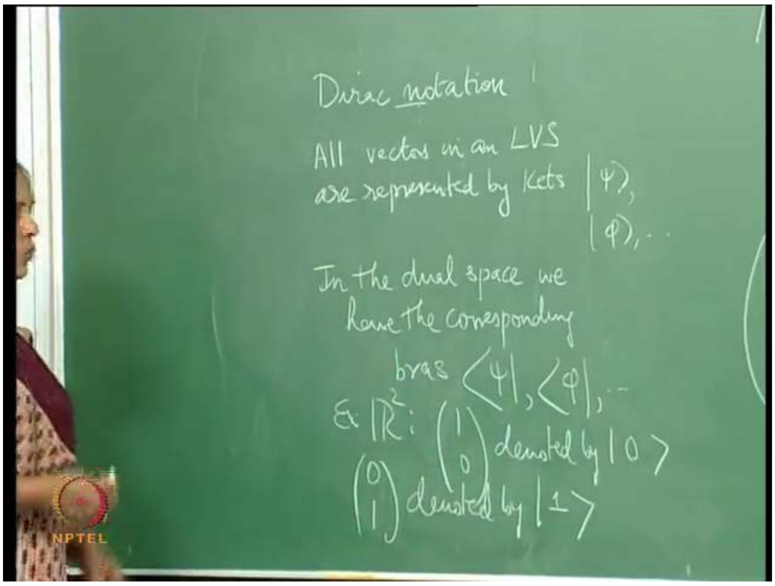

The Dirac notation is as follows: all vectors in an L V S are represented by kets, what are kets? So this is a ket as ket psi, ket phi and so on. A psi is a vector, phi is another vector in the linear vector space, you can call it a vector, and you can call it a state in a linear vector space. So, I put it like this, this is a ket. In the dual space, we have the corresponding bra vectors or I call them the bras, so this is how you represent a bra. So for instance, let us look at R 2, 1 0 is a ket, denoted by a ket but I have to put down something to denote 1 0 let me just call that ket 0. Then I have 0 1, 0 1 is denoted by another ket let me call that ket 1. So, in the Dirac notation the state 1 0 is denoted by ket 0 and the state 0 1 is denoted by ket 1.

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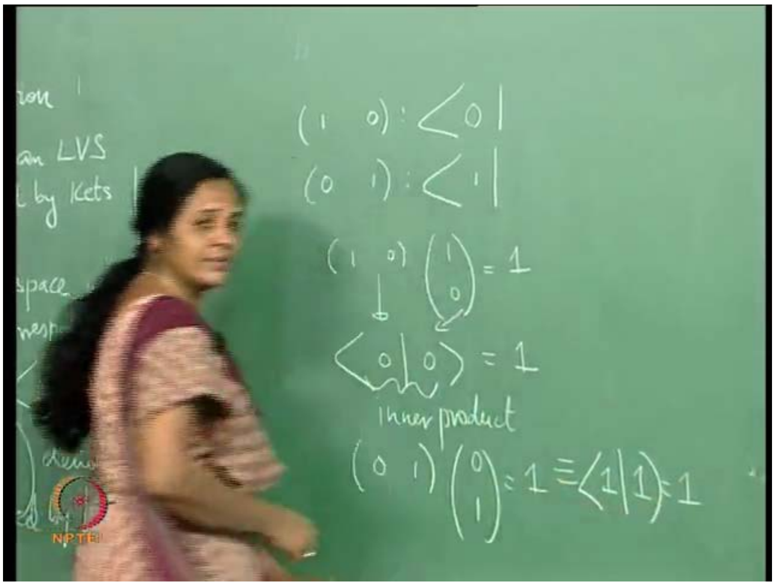

What about the corresponding notation for the rows? The rows 1 0 would be denoted by a bra (Refer Slide Time: 24:15) and since I have used the entry here to be 0 to denote 1 0 so that is the bra. And then 0 1, so it is a bra with entry 1 because the column 0 1 had an entry 1 inside the ket. Then, how do I represent the fact that 1 0 with 1 0 is 1? Well this object is bra 0, this object is ket 0 and this inner product I is 1. Similarly, 0 1 with 0 1 is 1 in the Dirac notation is identical to bra 1 ket 1 is 1.

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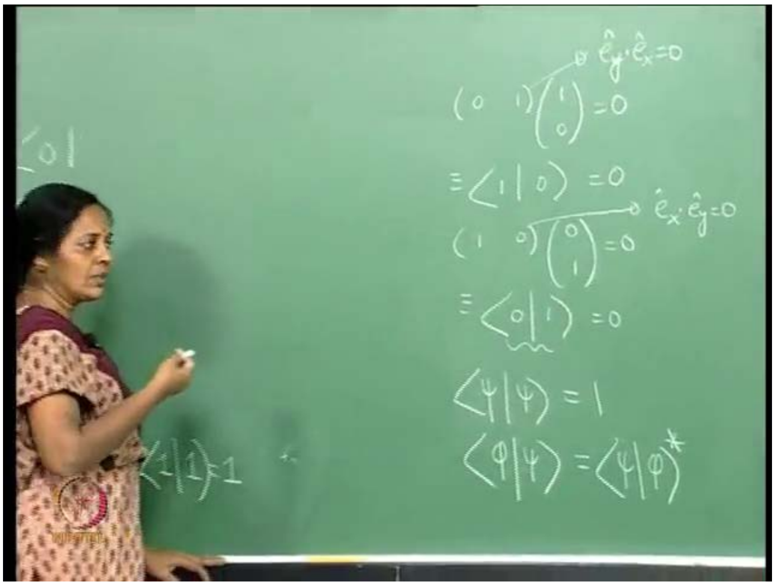

Then of course, I need to worry about the fact that 1 0 and 0 1 are orthogonal to each other. I denote it in the following fashion. So, 0 1 with 1 0 is 0 that is identical that is a bra and since 0 1 is represented by bra 1 and this by ket 0, that is 0. Similarly, 1 0 with 0

1 is 0 is identical to ket 0 bra 1 is 0. This inner product is clearly the generalization of the dot product, because remember that this is the same as e y dot e x is 0 and this is the same as e x dot e y is 0 and so on.

So, in the Dirac notation quite independent of whether, the states concerned are columns or functions quite independent of all that I used this notation, that all the states in the linear vector space are denoted by kets, ket vectors as they are called. And the elements of the dual space are denoted by the bra vectors and this is a straight forward powerful universal notation that I can use for all scalar products or inner products that I form independent of the kind of linear vector space, that I am concerned with. Certain properties of the scalar product emerge and I can generalize this. So, if I have a state psi in a linear vector space, the inner product of psi with itself would be this object. And if psi, when normalized to, 1 look at this example for instance 1 0 with, 1 0 is 1 that means that this is normalized to 1.

Remember this is the same in the other notation to e x dot e x is 1. So if it is normalized to 1 then the fact that it is normalized to 1 is represented in this fashion. In general, I can have the inner product of two different objects like this or like that. But it is clear that this thing is psi phi star, all the entries considered were real in the example. However, in general you know that the entries could be n-tuples of complex numbers, if you are talking about C n. Then, in that case how do go from the column to the row? Well you have to take a transpose instead of the column, you take a row and every entry, its complex conjugate has to be put in and therefore, if the state if the element of the linear vector space as complex entries.

Then when you do the corresponding bra, every entry you have to take its complex conjugate and therefore, the general statement is that an inner product could be a complex number. And it is clear that, if you work with ket psi bra phi this number is going to be the complex conjugate of that number. So, this is a property of the inner product. Till now what we have seen is basically to make numbers, to make scalars out of the kets and the bras. The scalar that I got is the inner product structure.

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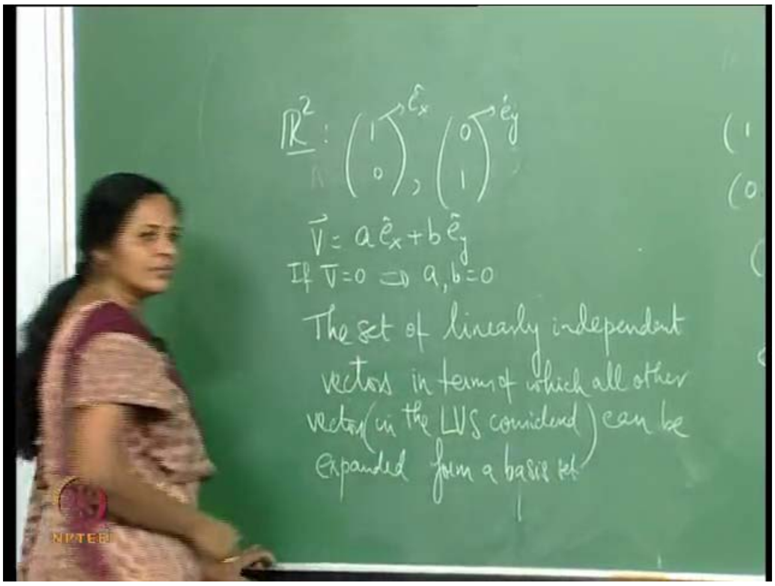

But many other things can be done, apart from producing scalars; I can produce matrices, after all this is true. Consider the same example that we had, 1 0 and 0 1. I can always construct a matrix out of this, given column vectors and given row vectors. But before we construct those matrices let us look at some more properties of these vectors. I am looking at R 2, but that is only a matter of convenience is an illustrative example using something that you already are familiar with. These vectors are linearly independent, because if I construct a vector V which is a times e x plus b e y. And if V is 0 it implies that a and b are 0 so e x and e y are linearly independent. In general I will say there are set of vectors are linearly independent in the linear vector space considered.

If I make a superposition of all those vectors. If that superposition is 0 then if it follows that the coefficients are all 0 then you say that the vectors which are used to form the superposition are linearly independent. So, e x and e y are linearly independent and this can be generalized in a straight forward fashion to higher dimensions. So, that is linear independence of vectors. Now, these set of vectors, the set of linearly independent vectors, in terms of which all other vectors in the L V S considered can be expanded. That means as a superposition of all vectors, linearly independent vectors with coefficients a b and so on.

The set of linearly independent vectors in terms of which all other vectors in the L V S can be expanded form a basis set. In the same sense that e x and e y are basis vectors in R 2, e x, e y and e z are basis vectors in R 3 and so on. Of course, to begin with these vectors need not be orthonormal, they need not be normalized to unity and they need not be orthogonal to each other.

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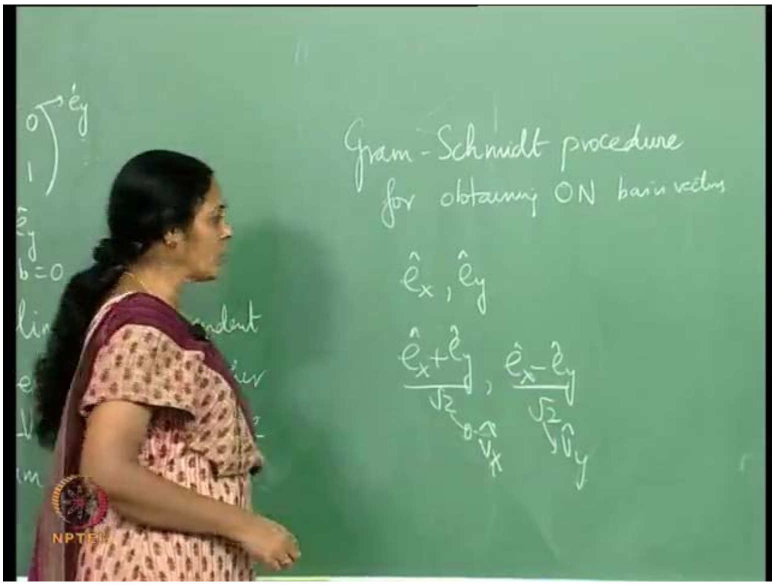

But there is a prescription called the Gram-Schmidt procedure to make orthonormal vectors out of a set of basis vectors and that can be used to produce a set of orthonormal basis vectors. So, there is a Gram-Schmidt procedure for obtaining orthonormal basis vectors. So, when I say basis states, I mean a set of orthonormal states which are linearly independent, I mean that the Gram-Schmidt procedure has already been used, if initially the set considered was not an orthonormal set. And having used the Gram-Schmidt procedure, we obtain a set of orthonormal linearly independent vectors which form the basis set.

So, when I say basis set I assume that orthonormalization has already been done. But one thing is clear that you need not have a unique basis set in any linear vector space. For instance, in R 2 I can expand things in terms of e x and e y or in terms of e x plus e y by root 2 and e x minus e y by root 2. The root 2 has been put in order to make this object, I can call this some vector, v 1, v x and v y. You can check that v x is normalized to 1 and v y is normalized to 1 that they are orthogonal to each other. It follows from the fact that e x is normalized to 1so, is e y and e x dot e y is 0 So, I can have several basis sets and I should be able to go from one basis set to another basis set. So, let us see how exactly this could be happen. This could happen in the following fashion.

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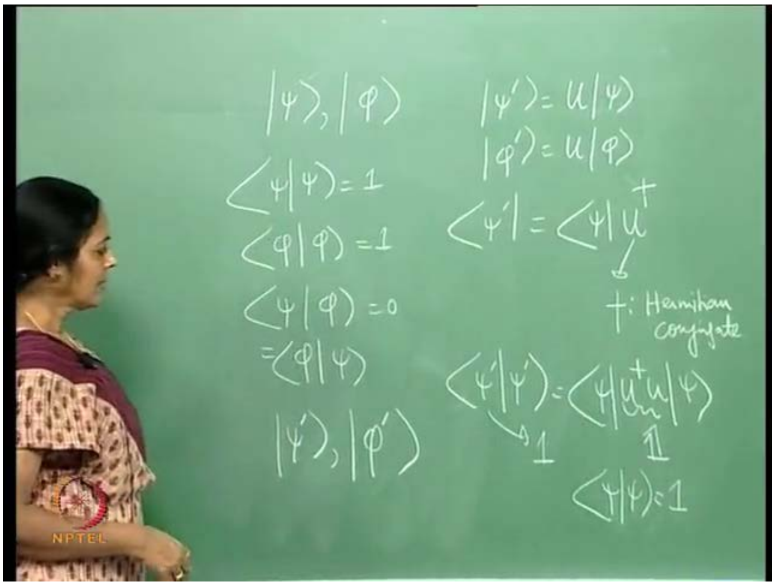

Suppose, I have a situation which is two dimensional, I illustrate it for a two dimensional situation and I have basis states psi and phi. When I say basis states I mean this. Let me use the Dirac notation so that we get used to it. Normally, psi phi inner product would be psi phi complex conjugate that is 0, so I can write it in this fashion. So, I have already assumed that it is an orthonormal basis set. Now, let me imagine that I have another basis set, psi prime and phi prime in that linear vector space. This could be an example psi and phi, psi prime and phi prime.

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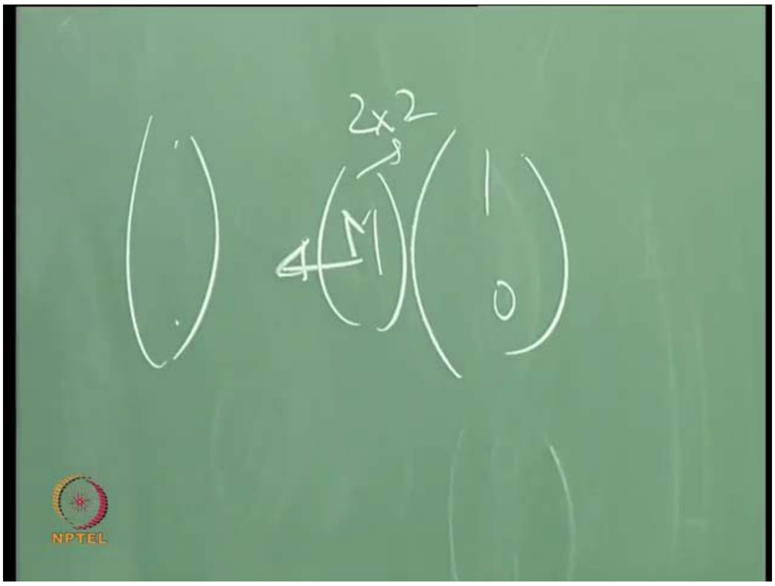

In terms of column vectors it is clear that I would have gone from psi to psi prime through a matrix because if I had a column 1 0 and I need to go to another column of course, with the same number of entries I would have done it with a 2 by 2 matrix. So, let me call that matrix u. So, in general I have psi prime is some u psi correspondingly, phi prime is u phi. So, what is the bra? Means interchange the rows and columns that means, since this was a column it would become a row that means take the transpose and take the complex conjugate of every entry, which amounts to saying take the Hermitian conjugate of this matrix and make that ket into a bra.

So, psi prime bra automatically assumes that the column has become a row in the example that we considered and every entry, its complex conjugate is put in and the Hermitian conjugate of u is taken. This Hermitian conjugate means take the transpose, interchange the rows and the columns and take the complex conjugate of every entry so that is u dagger. So, what is psi prime? This is the same as psi u dagger u psi, but psi prime psi prime is again 1. I went from one orthonormal basis set to another orthonormal basis set. It is clear therefore, that u dagger u is 1, because I know that psi psi is 1.

Similarly, you can show that u u dagger is 1. That means u is a unitary matrix. This is just an example I will establish later that when you change basis and go from one basis set to another basis set in a linear vector space, you would be really using unitary matrices or unitary operators as we call them to move from one basis set to the other basis set. These are not the only matrices that we have. In fact, we have several matrices that we can form using the elements of the linear vector space. After all, as I remarked earlier given column vectors and row vectors, we should be able to form elements of the linear vector space, you should be able to form matrices in the linear vector space.

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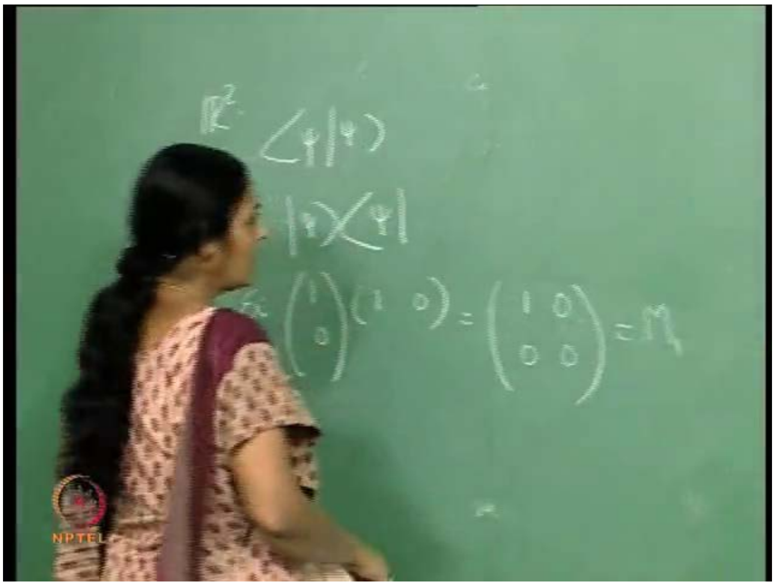

So while this represents a number, what about this? This object is a column, this object is a row and that is the matrix. So, this is a matrix or an operator that operates on states. (Refer Slide Time: 38:58) We will formalize and extend these definitions and explain them better in subsequent lectures. But right now, even in R 2 let us look at the number of matrices that we can form.

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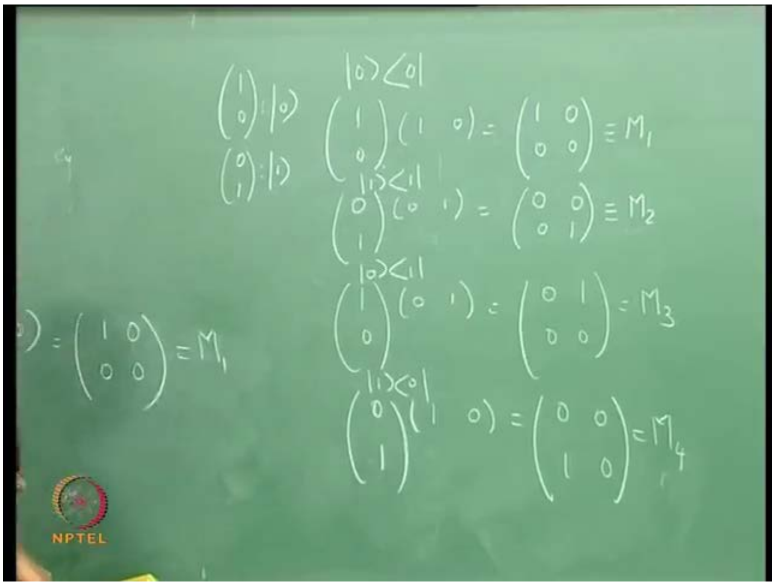

Let me call this M 1 or let me write it here. Similarly, I can have 0 1 with 0 1 and I am going to call that M 2. So what is M 1? This is M 1 that is my notation. This is M 2, then

I can have this. This is M 3 I am just working with the basis states nothing more, but it is clear that I can form several combinations and make many matrices out of these and that is M 4, the Dirac notation. This is a 1 0 in my notation was ket 0 and 0 1 was ket 1. So, this is ket 0 and that is ket 0 so this is the operator over the matrix M 1. I am using the word matrix and operators interchangeably, because I can represent operators by matrices and this is a 0 1 and that is another 0 1. This is a 1 0 with a 0 1 and this is a 0 1 with the 1 0.

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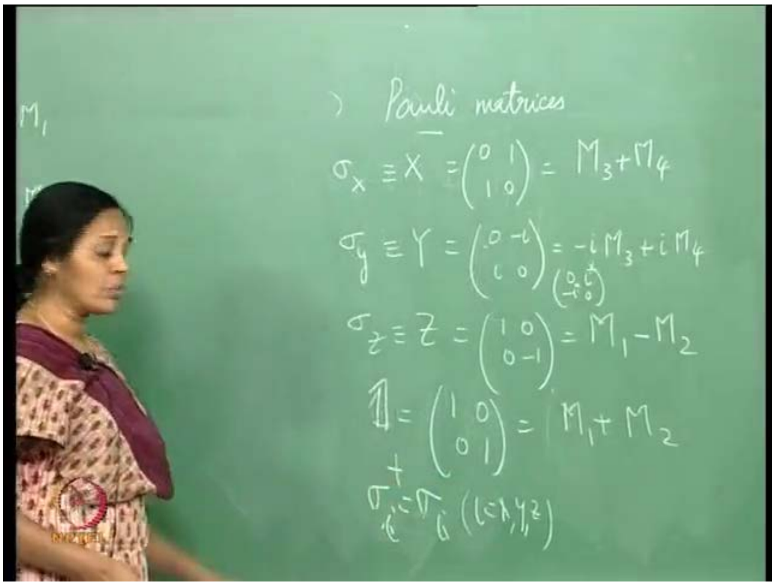

Given these matrices, I can form a very interesting set of matrices, very interesting and very important set of matrices which I will call the Pauli matrices. Sigma x sometimes also called X, particularly in the language of quantum information and computation and sigma X is 0 1 1 0 how do I get it from these? Its M 4 plus M 4 then sigma y also called Y is 0 minus i i 0, where i square is minus 1. So, this is minus i M 3 plus i M 4 then sigma Z which I will call Z, it is 1 0 0 minus 1 and this object is M 1 minus M 2 and of course, the identity matrix which is one along the principle diagonal and that is M 1 plus M 2.

The sigma matrices are very interesting matrices, they are Hermitian matrices. Because any of these matrices if you take sigma x dagger. That means interchange the rows and the columns and also make complex conjugates of every element. So for instance, if you interchange the rows and columns here, it would become 0 I minus I 0, but take the complex conjugate and that will give you the same thing. Similarly, sigma y dagger is sigma y and sigma z dagger is sigma z.

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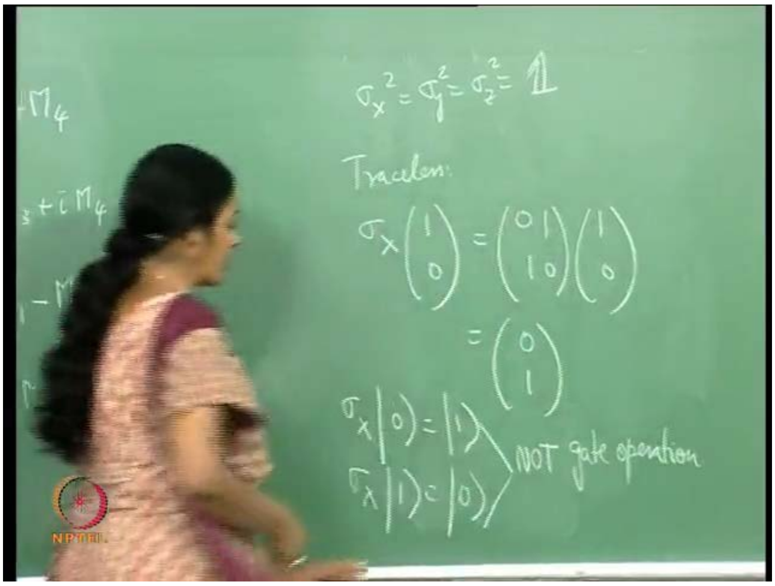

So, I will call that sigma i dagger is sigma i and i, can take values: x, y and z, i denotes: x or y or z. Then, it is also true that the square of each of these sigma matrices is unity. You can easily check that sigma x squared equals sigma y squared equal sigma z squared is the identity matrix, also traceless. That means the entries in the principle diagonal is 0. So, these matrices are very important, keep occurring over and over again particularly, in the context of the angular momentum algebra. Apart from that even at the level of quantum computation which is a subject which is pursued very seriously in the recent past and at present. What is the effect of sigma x on 1 0? Sigma x on 1 0 is 0 1 1 0 on 1 0 and that is the same as 0 1.

So, sigma x on the state ket 0 is ket 1. You can easily check that sigma x on the state ket 1 is ket 0. So, instead of the classical bits 0 and 1, with which you perform classical operations, you could think of quantum bits or qubits. I need to have physical realizations; I need to give examples of how to prepare these qubits. But, suppose, I did that at a later date you could think of ket 0 and ket 1 as quantum bit instead of 0 and 1 and then what happens is that the qubit 0 goes to qubit 1 and the qubit 1 goes to qubit 0. This is essentially the not gate operation that you know from a classical context. In that sense, sigma x becomes very important in quantum computing. It is a different matter that we have to find ways and means of actually carrying out this operation in a physical system. Similarly, if you look at sigma y or sigma z what does sigma z do?

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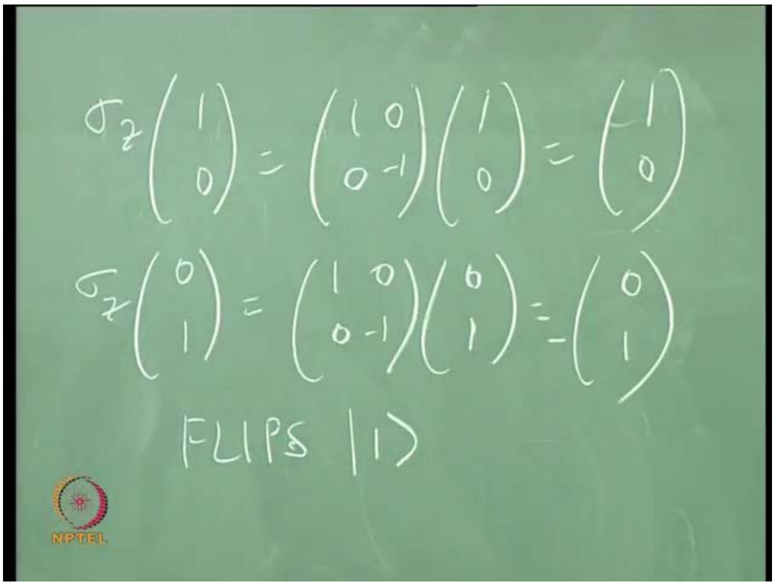

Sigma z on 1 0 is 1 0 0 minus 1 on 1 0 and that just leaves it alone, but sigma z on 0 1 gives me minus of 0 1. In other words, it flips ket 1 and does not do anything to ket 0. So, that is another important quantum logic gate operation which is carried out by the Pauli matrix. To complete the picture there are of course, many important logic gate operations, but right now to complete the picture I will talk about the Hadamard operation, and that just has 1 by root 2, 1, 1, 1, minus 1 what would this do to 1 0. It just takes it to another state which is 1 1.

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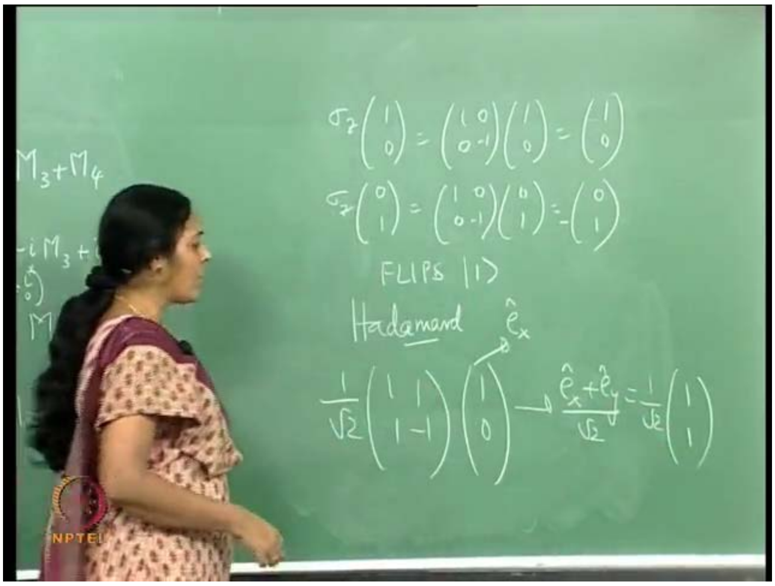

I have the Hadamard operation. The matrix itself is this and what does it do? You can easily check that, when it acts on 1 0 this is e x it takes it to e x plus e y by root 2, because this action is just 1 by root 2 1 1. Similarly, when it acts on 0 1 it takes it to e x minus e y by root 2 so it changes the basis. You can also check that this object is a unitary matrix.

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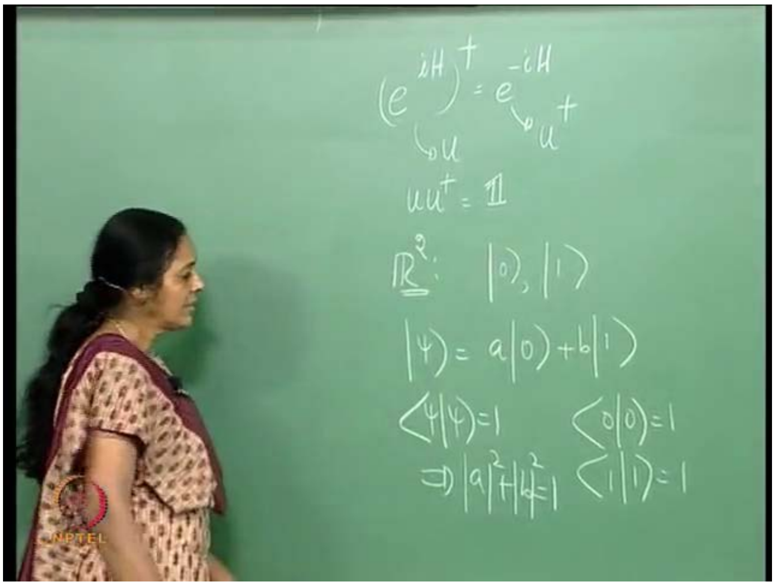

I can also produce unitary matrices by exponentiating the Pauli matrices. Remember the Pauli matrices are Hermitian matrices. So, H is the Hermitian matrix and if I take the

Hermitian conjugate of this, that is simply e to the minus i h. And therefore, if this object is u this is u dagger and it is clear that u u dagger is the identity operator, unitary matrices therefore, find a very natural role in quantum mechanics and all these matrices, Hermitian, unitary and other types of matrices act on states in the linear vector space. How many states are there? In principle an infinite number of states, why? In the following sense, but even if you look at the example that I had R 2 where I had ket 0 and ket 1 as the basis states.

In general, I can form some state psi I will call it ket psi as a times ket 0 plus b times ket 1, and this is the superposition of the basis states. Suppose, psi were normalized to 1 and given that these states are also normalized to 1. It is clear that modulus of a square plus modulus of b squared is 1 and since, a and b could in general be anything. I can form several superpositions so; several qubits are possible, given the basis states ket 0 and ket 1.

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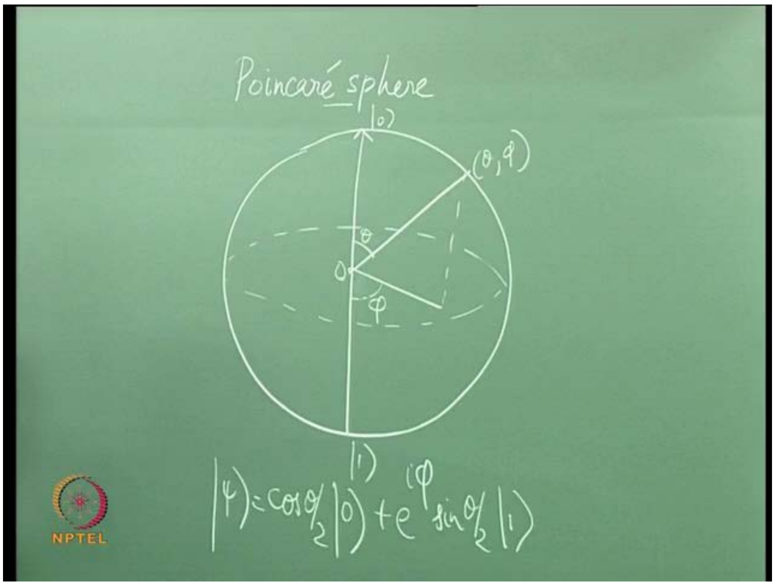

There is a nice geometrical way of showing the qubits and this is in terms of the Poincare sphere. So, I have this sphere of unit radius and this is the equatorial plane. So that is the origin of coordinates, this is ket 0 and this is ket 1. (Refer Slide Time: 53:04) This is a sphere of unit radius and therefore, I know that the kets are normalized to 1. Any point here is represented by the coordinates theta and phi. So, polar and Azimuthal angles and that is the origin. So, I can represent any qubit here as cos theta by 2 ket 0 plus e to the i phi sin theta by 2 ket 1.

And as theta and phi take different values I get various qubits, all sitting on top of this sphere. You can easily check when theta is 0, I get ket 0. You can also check that ket 1 follows from this definition of ket psi. So, this is a very nice geometrical way of picturizing qubits by using the unit sphere. If information is stored in a state, it means in principle an infinite amount of information can be stored. If I combine several qubits or several superpositions, all of quantum computation and information at least a large part of it is about how to harness all the information contained in a qubit?

Of course, once a measurement is made ket psi would collapse either to ket 0 or ket 1 and that is about the information that you will get. Now, the other thing that I want to talk about is the other physical situation. The first one that I spoke about was in harnessing information by storing it in qubits. The other situation is in the context of angular momentum.

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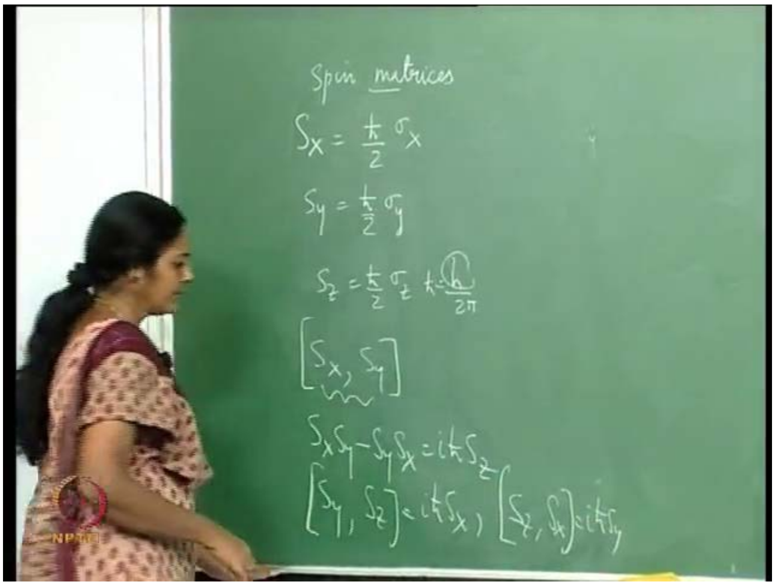

I can use the Pauli matrices to define 3 matrices, which I call the spin matrices: S x, S y and S z. These are of course, 2 by 2 Hermitian matrices and they satisfy what is called the angular momentum algebra. If I find the commutator of S x and S y, the commutator is this by definition the commutator of a with b is a b minus b a. Remember these are matrices so S x S y is in general not equal to S y S x. If I find this commutator, I can show that it is i h cross S z, where h cross is h by 2 pi, the Planck's constant. Similarly, the commutator of S y with S z is i h cross S x and the commutator of S z with S x is i h cross S y. We will come across this algebra in the context of orbital angular momentum, a little later when we discuss orbital angular momentum.

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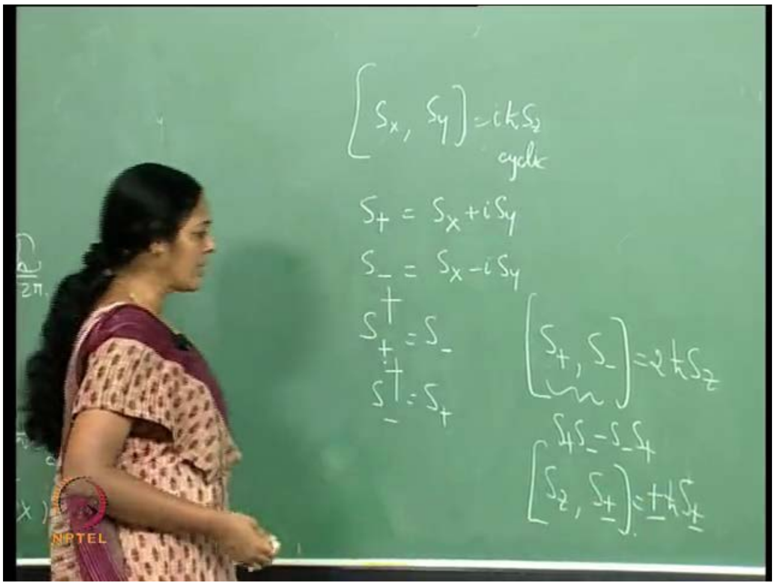

But now, the point that I am trying to make is this. This angular momentum algebra S x, S y is i h cross S z is a cyclic thing. Replace x y by y z, y z by z x. It is given in terms of three Hermitian matrices: S x, S y, S z. Instead I can define an S plus which S x plus i S y and an S minus, which is S x minus i S y.

It is clear that S plus and S minus are not Hermitian matrices, but S plus dagger is s minus and S minus dagger. This is the plus and that is dagger is S plus. Dagger means Hermitian conjugate interchange the rows and columns and make complex conjugates of all the entries. The algebra translates and you can check this out to this. If you find the commutator of S plus with S minus, this will be 2 h cross S z, and if you find the commutator of S z with S plus it is plus h cross S plus. Instead, if you find the commutator of S z with S minus it is minus h cross S minus.

So, commutator of S z with S plus you can check using this as an input, that the commutator of S z with S plus is plus h cross S plus and the commutator of S z with S minus is minus h cross S minus. So, this algebra of the spin matrices can be either written in this way in terms of three Hermitian matrices or in this fashion, where S plus and S minus are not Hermitian matrices, but are Hermitian conjugates of with each other and S z which is a Hermitian matrix. In which physical context does it fit?

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I will discuss the problem of the 2 level atoms in my next lecture, where I will extensively use these matrices and their algebra. As you know in quantum mechanics, energy is quantized for most physical systems. So, not all energy values are allowed, but energy values say E 1, E 2, E 3, E 4 and so on are the allowed energy values for a given quantum system. In the case of the hydrogen atom, these values are unequally spaced and as you go to higher and higher energy levels, they are so close that they look like they are almost continuous. In the case of the linear harmonic oscillator, these energy levels are equally spaced.

A two level atom is one where two of the energy levels are so close to each other that my pumping energy I can take the atom from lowest energy state to the next state. But, the other states are so far away that it is difficult for the atom to go to that state. So, it is really this two level atom, where this would be called ket 0, and that would be called ket 1. I will have these two states, the linear vector space is a two dimensional vector space and I will talk about the matrices or operators which will take you from this level to that level and vice-versa and so on. That is where the spin matrices become very important. So, I will discuss the two level atoms tomorrow, to give you a specific physical situation where the algebra of the spin matrices becomes important.