Classical Field Theory

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Lecture – 4

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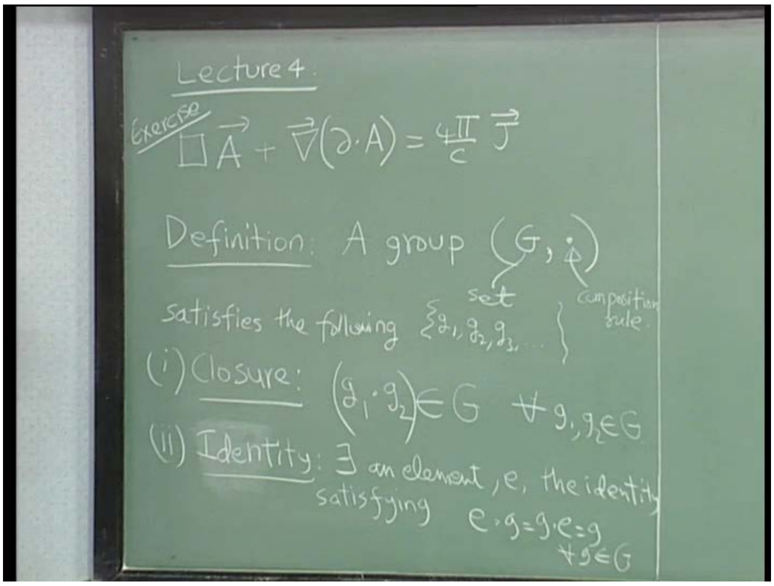

So, I just want to point out that there is a sign error in the exercise, I given you, if you care to do it, we will find that, that has to come with a plus sign. And before we get back to the orthogonal group, I thought I will give you the definition of a group, so that you can correlate with whatever things which you have seen.

So, let us have on the one definition for today, a group consists of 2 things, a set G, and an operation which we will indicate with a dot, which is a composition rule. So, this is a consists of a set, and let us just say that we will write element g 1, g 2, g 3 and this is the composition rule, a group consists of these 2 things, and a group G satisfies the following condition. So, the first important thing is closure, which means if you take any 2 elements g 1 g 2 it belongs the composition rule gives you another element of the set, and for all g 1 and g 2 belongs to G. So, this symbol stands for all. Second is the existence of an identify, let us just call it identify. So, there is a special element, so there exists an element, let us call it e the identity, it is called the identity satisfy for all g, for any g.

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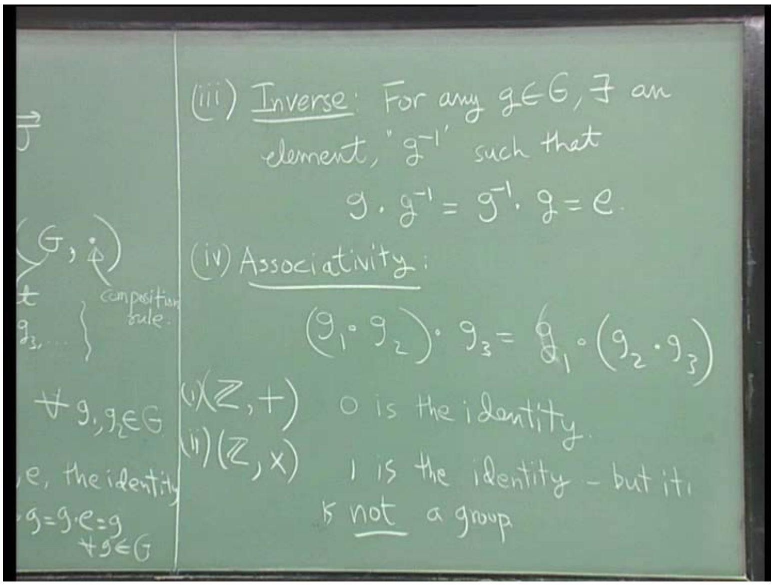

And existence of an inverse, so given for any element, which belongs to this thing, there is there exist an element, which we will call it g inverse it just a symbol for it, such that the composition rule for g, g inverse is equal to g inverse dot g equal to identity. So, I think mathemat, if you want to be mathematically precise, I think it a to give one of these 2 conditions, but we will not get into such a mathematically term.

And the last one is associativity, so the composition rule, so if you are given a (( )) not enough to be given a set, because the operation is important, you need to check both of them. So, just to give you a counter example, you know Z which is the set of integers, with addition at the operation is a group, you can check everything and zero is the, but if you take the same set, and change the operation into multiplication, you can check that a closure works identity also works, but one is the multiplicable identity, it is called, but something is violated the inverse, you have to go outside the set of integers to define the inverse. So, sometimes I guess this is this is called the monoid or something like that well this particular axiom is violated, but this is not a group.

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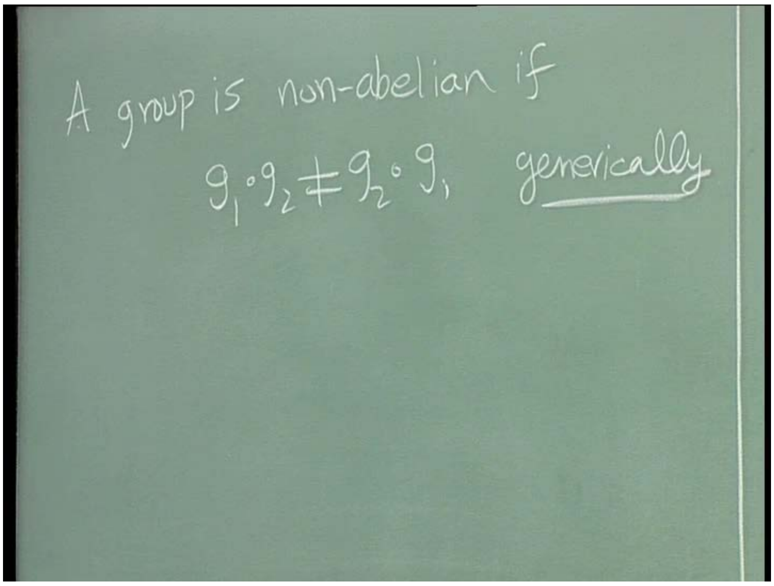

Then there are some more sort of, if statements you say a group is non-abelian, if g 1 dot g 2 is not equal to g 2 dot g 1 generically, there could be pair of element, which commutes with each other, that what it means is that, there exists such elements which do not commute the order matters, g 1 dot 2 is not equal to this.

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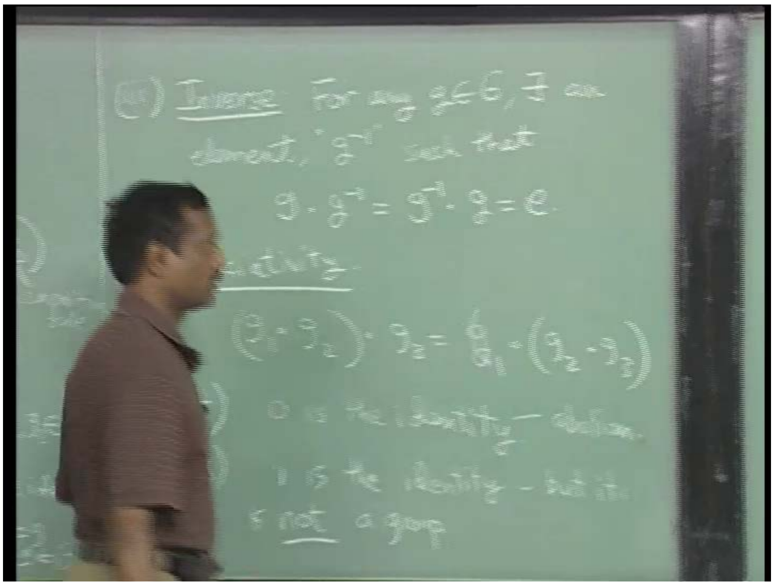

Coming to this group, the order does not matter, 3 plus 2 equal to 2 plus 3, such groups are called abelian groups; non-abelian means not abelian, so this is, this group is abelian.

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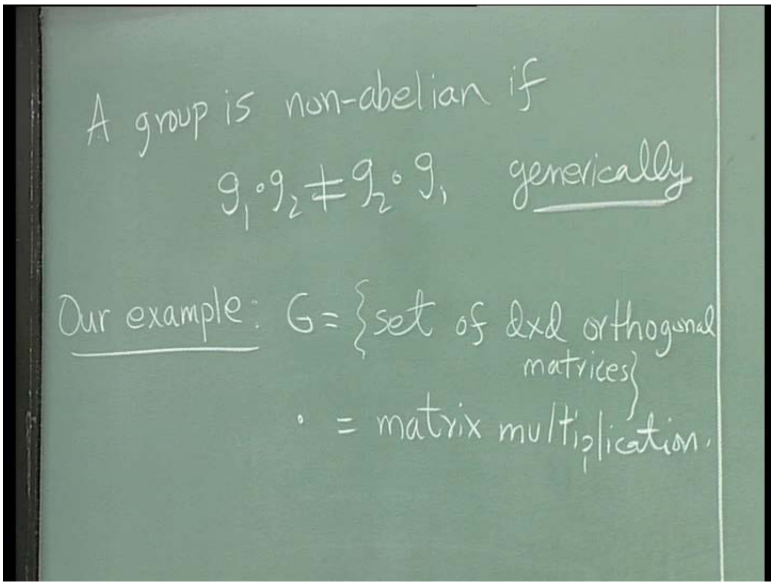

So, coming back to our example, from last lecture, G would be the set of n by n orthogonal matrices or d by d, this is what I call it, and dot is equal to matrix multiplication.

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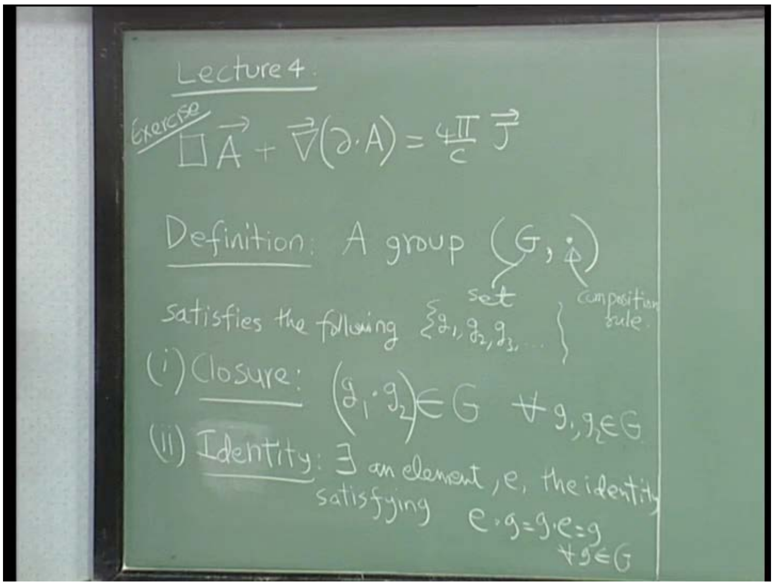

So, one so the first thing which we did yesterday, was to see that the, we or may be, I left it as an exercise, it is to show that the product of 2 orthogonal matrixes is another orthogonal matrix. So, that is the closure part, what is the identity, it is just the identity matrix.

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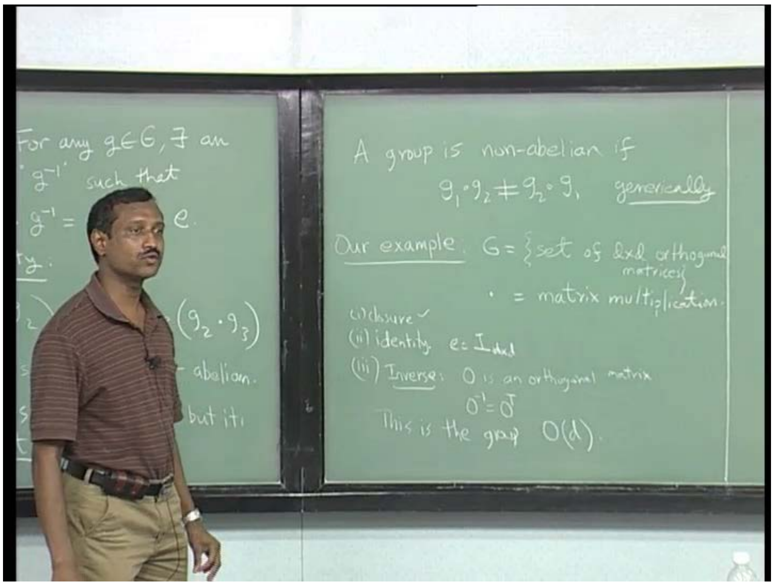

So, closure works what about identity element e is nothing but the identity matrix d by d identity matrix, inverse obviously exists, so let us say that is an orthogonal matrix, then we know, it is easy to show that O inverse is nothing but O transpose which is another orthogonal matrix, and associativity holds for all matrix multiplication, the order or the order does not matter, so I will not check that part.

So, you can see that, this group is it satisfies all the axioms of the group, and this so we call this group O, and right this would this is, the group for which we use the symbol O n or O d like that, and if we put in the extra condition, which is determinant being one that became the special orthogonal matrices, and is there anything else, I want to say about this associativity, I mean it follows from associativity of matrix multiplication good, so is now the question is, Is the group abelian or a non abelian?

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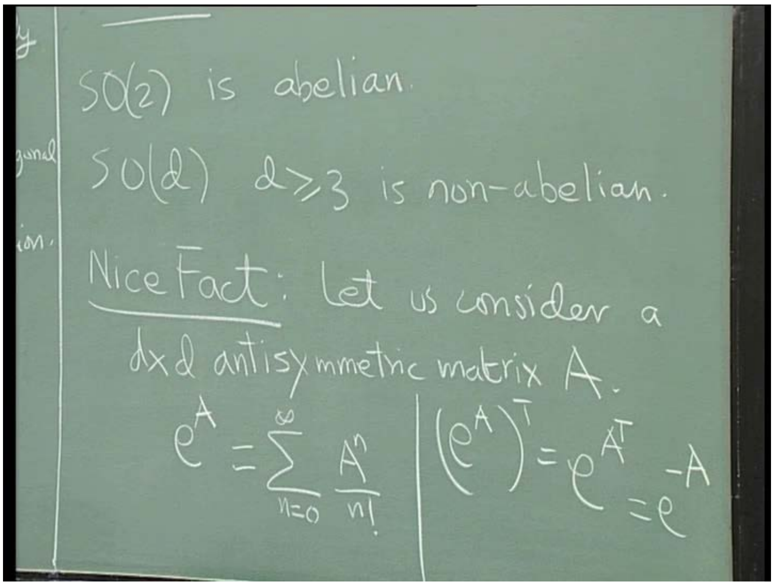

So, let us just choose, SO d, and the question is the group abelian, well typically matrix multiplication does not commute, but there is an exception in this, SO 2 is abelian, but generically, so d, for d greater than or equal to 3 is non abelian, this is something we know for instance, when we define Euler angle every book has its convention, and 3 are not equal to each other, because it matters one will do.

Let us do, there is for instance take first do rotation about the z axis, followed by something about the y axis, followed by again the other z axis; obviously, if you exchange the two, there a second 2 thing, you get, you can combine the two's the rotations about the z axis, to make it into a single rotation about the z axis, and so then they obviously, cannot be the same.

And you know this, and SO 2 being abelian, just tells you that, if your that is just rotation about an axis, if you wish, if you rotate by some angle theta 1, and follow it up with the rotation with the angle theta 2, the it equal into a single rotation with theta 1 plus theta 2. So, the ordering does not matter could have done, theta 2 first, then theta 1, so this is an exception in the whole set, yeah.

There is a very nice, nice nice fact. Consider let let let lets us consider, let let us consider a d by d, a anti- symmetric, and let us consider look at this particular matrix e power A. So, you may wonder some of may not know, what this means? another way of seeing this is just to think out this, as the following, use the definition of exponential.

And you can convince yourself, that convergence etcetera goes through for this, this is the extra convergence is and so you see that this is actually this has proper meaning, but now comes interesting thing, you go ahead and take the transpose of this object, it is an easy thing to show, you should come here, it would looking at the right hand side, that it corresponds to taking the transpose of each of these guys. But A transpose since it is anti-symmetric, A transpose is minus of A, so you can show that this is equal to e power A transpose, which is equal to e power minus A, what this means is that, e power A for any any d by d anti-symmetric matrix is orthogonal.

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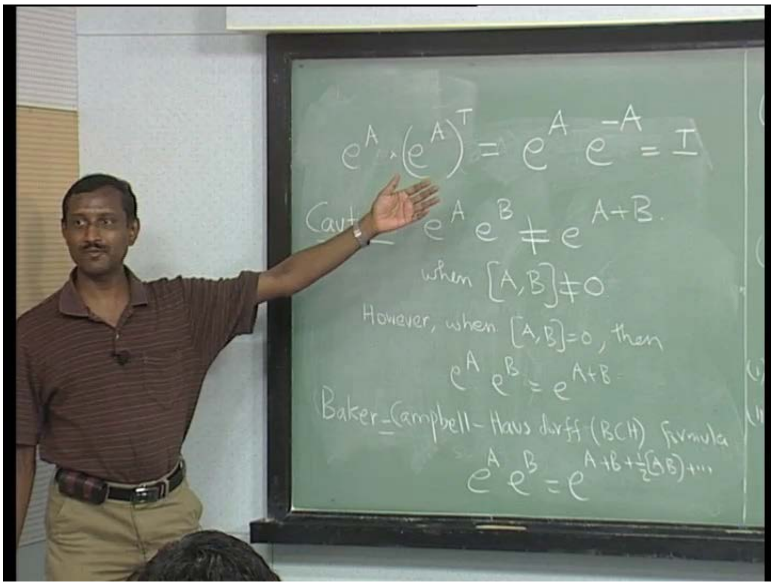

Because, what we get is e power A times e power A transpose is equal e power A is to e power minus A, which is the inverse of these things, but I should caution, you that in general while, this is true for simple exponentials, this is not true when A B do not commute with each other, that is AB minus BA, but how are any matrix commutes with itself.

However, when A and B are equal to 0, then you can show, so in this case in this instance, I can use that formula, do you know there is a formula, which gives what it should be, that as a name does anybody know what, that is it called the BCH formula. So, there is something called the Baker-Campbell-Hausdorff formula, which you can look up look it up, or we will discuss it later in this course. But, I would recommend out that, you look it up and see what it is, it given in the in terms of an infinite series, I thought of remember a bit of it, it is an infinite series. So, but coming back to what we want to prove it, some we do not have to worry about that issue, all we need to do is we have things, which are commuting, so we can use this formula.

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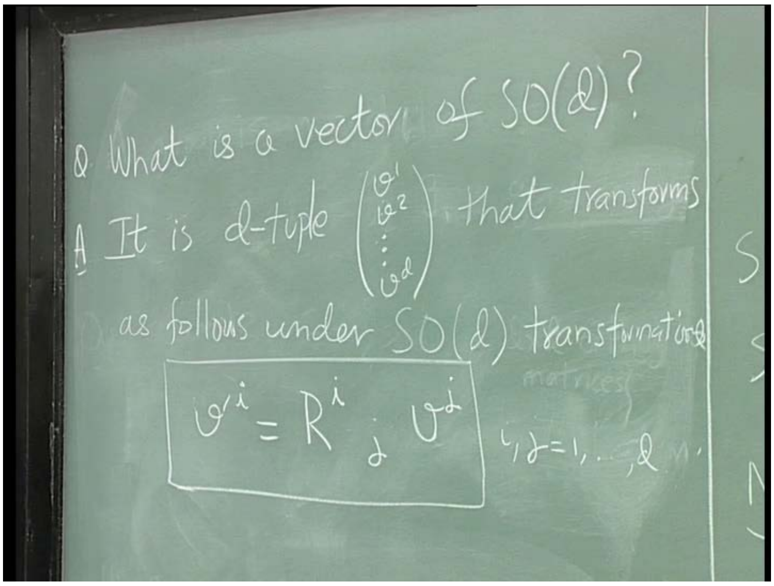

So, what we find is that, a large class of a of a of a of orthogonal matrixes, actually can be realized this way, now the question is how large is this class?

Before we do anything, one more thing we can ask is what is a determinant of an exponential? So, I will not prove it, but I will write it out as a claim is equal to e power trace of A, for any A need not be anti-symmetric, but what is a trace for us A is anti-symmetric, so trace of an anti-symmetric matrix is 0, so for when a is anti-symmetric, so we can say little bit more. We can say that, e power A is an SO d matrix, not O d generic O d matrix, because determinant is plus 1, so it implies that e power A belongs to SO d, so this is the nice fact, which you can actually check.

But, what makes it really neat is, all SO d matrixes can be written in terms of e power A, as in this form,. So, in fact as e power A for some A, and this is related to the theory of lee groups and lee algebras, this is something which we will come back later in the course. But, this just to show you at least once part of the story, you can check that is e power A gives you an SO d matrix, but it is an non trivial statement to say, what a the other part that all SO d matrixes can be written at this way. SO d it is not O d, because

because O d has made matrixes, which has determinant minus 1, but this will definitely not I mean.

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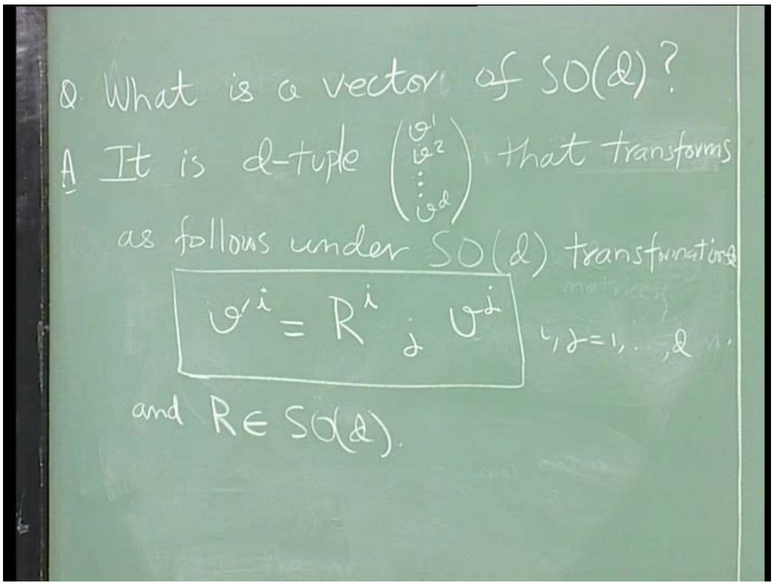

So, now we are kind of ready to define, what we mean by a vector of SO d? it is like chicken and egg with starter on something, and we go back, and we will see that, that will let us generalize in a nice manner. So, we will redefine now, what is a vector of SO d? so answer, it is a d-tuple, it is any d-tuple that transforms as follows under SO d transformation.

So, you just write it out, v prime 1, v 2. No, I will just write it in short, so v prime i equal to lets use this notation, which we used before R i j v j of course, j runs i and j run from 1 to d. So, this is a definition of a SO d vector, and R belongs to, so if you put d equal to 3 we recover, what we would have meant? What we would have written for normal rotation? See the way of looking this, this way of looking at it already, even for SO d lets you write more general objects, which transfer mode in general manner. And now the I want to emphasize the, if you find that there is some equation, which has, which is which manifestly SO d variant.

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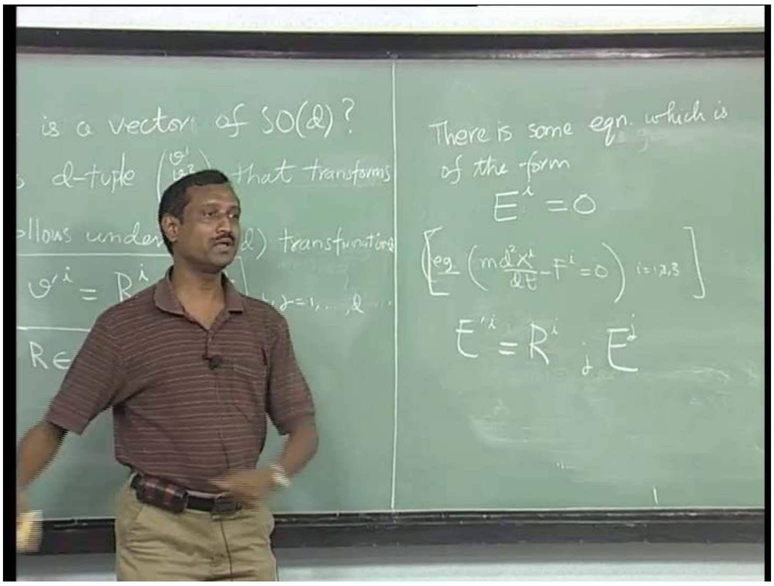

Let us say it is a vector of SO d, so some equation, let us say that there is some equation, which I will write, which is of the form, E some i equal to 0, so it is a bunch of d equations, each of which is a vector. So, just as an example, would be an example, would be something like this, I have written Newton's laws by taking F i to the other side, for i equal to 1, 2, 3, this would be an example of, E 1 would be m d square x 1 by d t square minus F 1 d 2 would be, so on, so forth.

The advantage of saying that, this is a vector equation, and is that under this transformation, that equation itself transforms this way, So, E i would got E prime i would be whatever it is, it would be R i j E j. Now, if we are given that E j equal to 0, and all these E primes are just linear combinations specified by the entries in the rotation matrix. So, if E j is 0, it necessarily follows that, E i is 0, there is nothing to check the key point, here is said this is just a linear transformation.

So, because of that anything you do moving things left, right, and things it does not matter, everything holds, and so so the. So, so that is what when means that, if you write the equations in a form, in a form like this, and say that this is an SO d vector, you are saying a lot. And in some sense, this step should be sort of done in your head, not something which you should do again and again. Because, what we will find later is that, they will replace this SO d with something else, and but we will say, but by that, we will nevertheless write some set of equation like this, and say it transforms like this.

And the most important thing is that, if something is a 0 vector in one thing it will remain, the it will always remain as 0 vector, quite often you will see something in certain examples, you will see where, if you may, if something is 0 in some frame, if you make a change of frame, it is non-zero by any that is sort of first hint, that is not an object which transforms nicely. In genera late g r for instance, there is something called the Crystophal symbol, they vanish in some coordinate systems, they do not vanish in another coordinate system. So, that cannot be a tensor, or a vector, or whatever.

So, we can now are sort of ready to define, what we mean by vectors? by tensors, I mean? Yes, yes that what I will replace the group later, but even within a even if you keep SO d fixed, we will see that, they can replace the word vector with something else. So, but if you want to keep vector yes, you can keep vector, but you cannot replace SO d with. So, four vectors we saw, would correspond to replacing the rotation group with the Lawrence group, but this prob this it is the same story, except the group is the Lawrence group, no no, it is a different group, we will see, I mean it not a I mean, it is not enough to say, it in rotation that followed. It may not have done free writing in the least lecture all we, because we need to prove that it is in variant under Lawrence boze.

What we will see is that, we can repackage Lawrence boze to look little bit more like, like rotations, but it is not quite rotation, yeah, yeah. So, so the thing is that, it depends on how you write it, so you could have some of the symmetric manifest, some may not be manifest going back to Maxwell's equations, the way we started out our last lecture. Even, the manifestly invariant under rotations, because it was either the vector equation, or scalar equation the set of equations, but it was not obvious, how it transformed under Lawrence boze? Or even the fact, that it is non-invariant under under Galilean Boze I mean it does not seem, so obvious.

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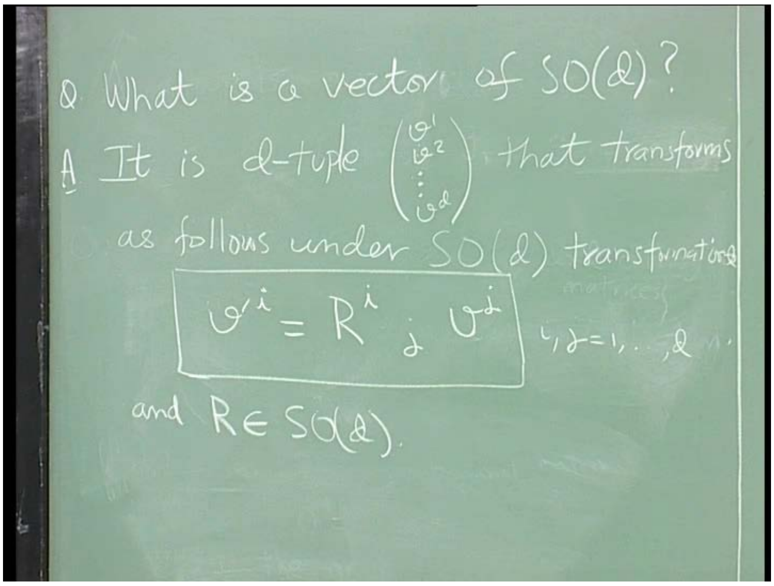

So, in fact given a vector, we can construct other objects, which transform in more complicated ways, but still the most important thing with that, we preserved is this linearity. So, in one frame you have these set of components, and in another frame you get something which is a linear combination of the other guys, so we want to preserve that.

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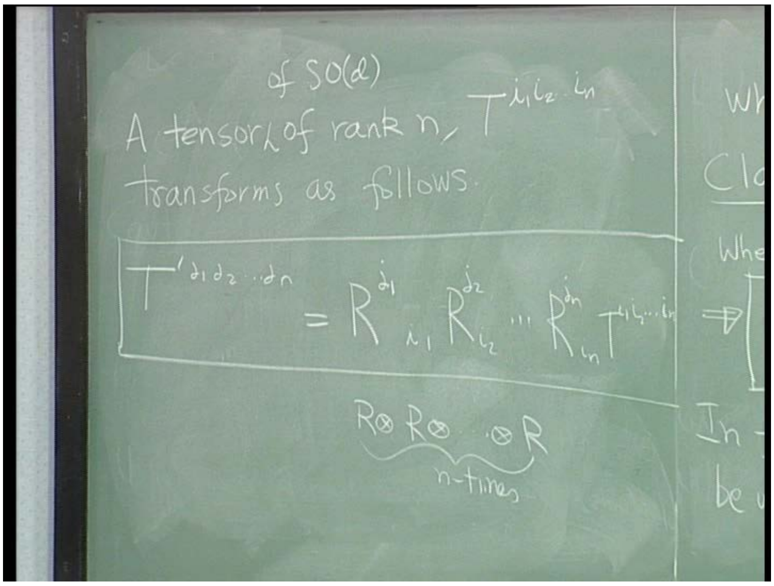

So, we will say that a tensor of rank n, lets call some object, which has n indices transforms. So, tensor now of SO d, so i 1, i 2 and each one of these take d values, transforms as follows,. So, this looks very complicated, but the transformation law is actually very simple, you sort close the thing forget about the fact, that these are other indices j 2 to j n, focus on j 1 and see, where j 1 comes here, it comes with this, and there is i 1,the first index here,. We are follow, we are not at this point, looking at these guys you can see that, this index j 1 behave or i 1 behaves like, it is a vector.

But, similarly I can repeat, forget about j 1, now think about j 2 same thing is happening out here, so it has a bunch of n indices, each of which transforms like a vector. So, this is the defining property, there are you can see here, that there are you know, n copies of these R matrix, but there is only one T out here still, so at the end of the day, it is still a linear combination it is linear n T, it preserves that important property.

So again, you get a nice statement, if which is generalization of that thing, if something is a tensor, and it vanishes in one frame, it will revanish in all frames. So, again if you write an equation, as a tensorial equation, then you are guaranteed, that it if it is 0 in one frame, every equation you can I react, and just said and rewrite it as some tensor equation to 0, which is the statement there it is a 0 in some frame, it will remain 0 in the other frame, does it matter how complicated these guys are. Mathematically, this is a statement that this matrix R can be written as some tensor, called the outer product or whatever R tensor R, so many n times, if you do not know this it is, but this is what you need.

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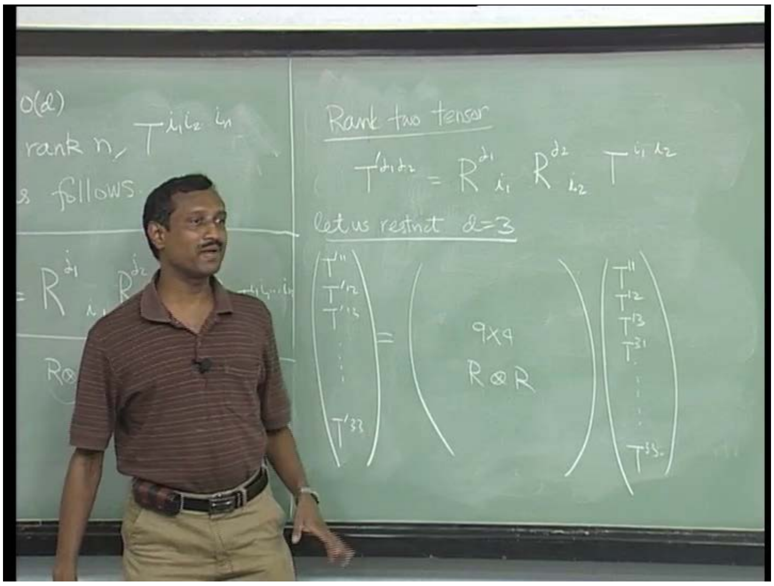

So, let us sort of take the example of a rank 2 tensor, because we can be a little bit more explicit. So, let us consider a rank 2 tensor, I am just rewriting this, and just for simplicity let us let us lets restrict d to d 3. Now, comes the neat stuff, so this will typical tensor, here i 1 can take three values, i 2 can take three values, so that total number of components would be 9. So, we can rewrite this transformation as follows, I could just go ahead and write it like this, T 1 1, T 1 2, T 1 3, T 3 1,so on, so forth, up to T 3 3. I write it as a column vector which has 9 components, and there exists a 3 by 3 metric 9 by 9 matrix, which can be worked out based on this, which will this is some 9 by 9 matrix.

Which I will crudely write, which will indicate as the matrix given by R tensor R and but this is just explicably stating that, it is a linear change of variables, that means, it can be written this thing, in index notation what we have an be rewritten as a 9 by 9 matrix acting on a 9 vector. So, mathematically this matrix is called R tensor n, but at the end of the day you can see this, what is written out here, it is not the most general 9 by 9 matrix, its entries are completely determined by this 3 by 3 matrix, which is started out which so it is not the most arbitrary thing. So, that could be, that could be special components which should not mix with each other, here the way we have written it looks T 1 T prime 1 1 will become linear combination of all these guys.

The answer is that actually that is not true, if you by there exists a nice redefinition, we do not have to write it in this form, there exists a different way of organizing these 9 components such that, these R by R takes a special block diagonal form many 0 come.

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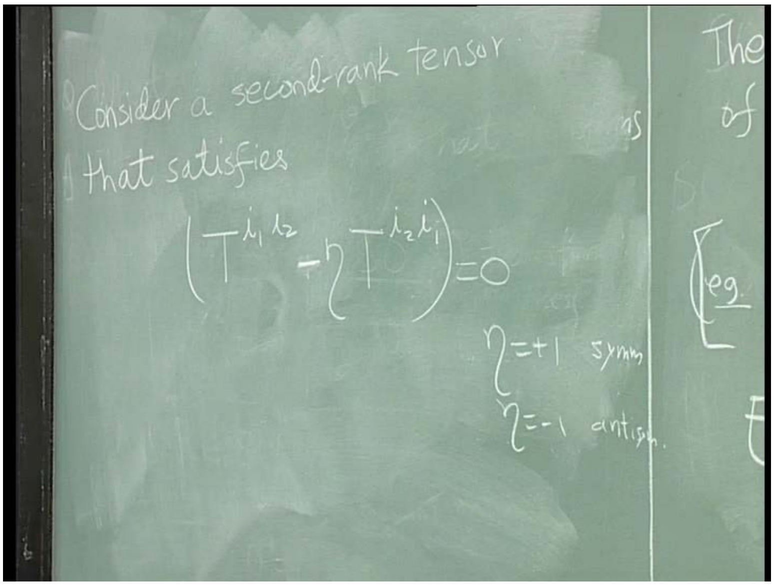

And the way to see that is to look at, consider a situation where yeah consider, where tensor second rank tensor that satisfies this condition, T i 1 i 2 plus eta minus eta i 2 i 1 equal to 0, an eta is either if it is plus 1, it is a statement that its a symmetric tensor, and this is symmetric, and eta equals to minus 1 is anti-symmetric. So, there are 2 ways of looking at it, one way is to say that, find this is, this this is an equation satisfied by this in one frame, it has to hold in all frames, because it is just a linear transformation. In other words if eta is plus 1 that would say, that this is a symmetric matrix, so T is a symmetric matrix, but what about T prime, it will remain symmetric that is what this says.

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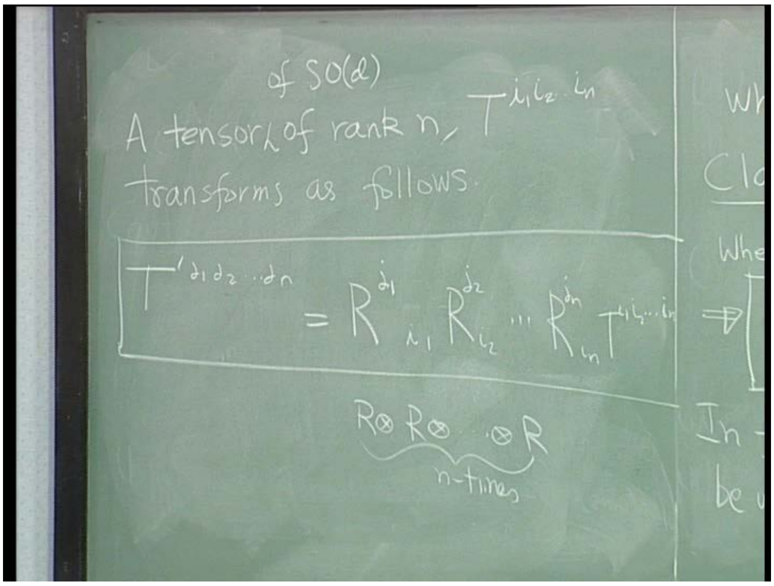

You can go back and prove it, you can go back to this definition, but it is better than this particular way of writing it, you can check it, if T is symmetric into… In this is any 2 actually let us say, it is i 1 and i 7 or something it is symmetric under the exchange, this will also continue to hold that any pair, but since there are only 2 things so. So, now you can see that, so what this implies is that instead of doing something like this, if you give me an arbitrary matrix symmetric matrix, arbitrary second rank tensor, I can break it up into 2 parts.

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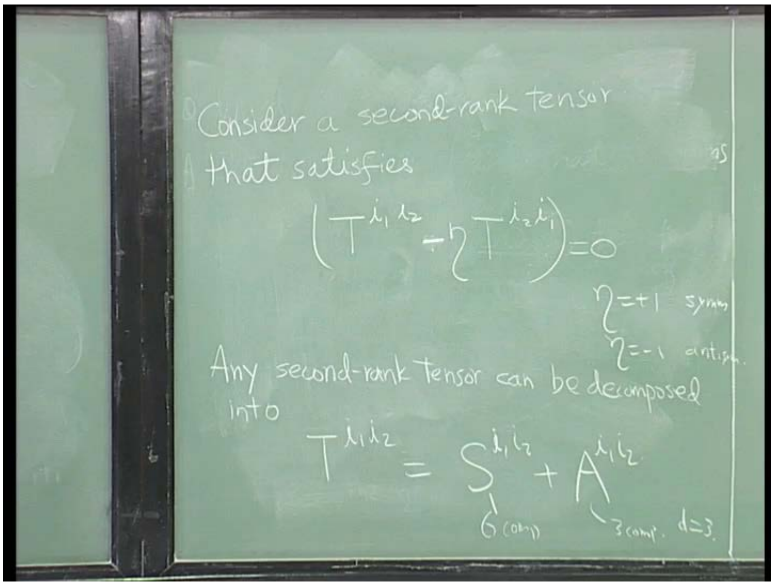

So, any second rank tensor can be decomposed into, so let us write T i 1 i 2 a symmetric part and an anti-symmetric part, if you take this 9 by 9 example, how many symmetric I would have 3 into 4 by 2, 6 components, this will have 6 components, and this will have 3 components, for d equal to 3. And the most important thing is that, this 3 composition does not I mean it continues to hold, if it the symmetric under rotations, the symmetric may part will never mix with the anti-symmetric parts under rotations, or SO d transformations.

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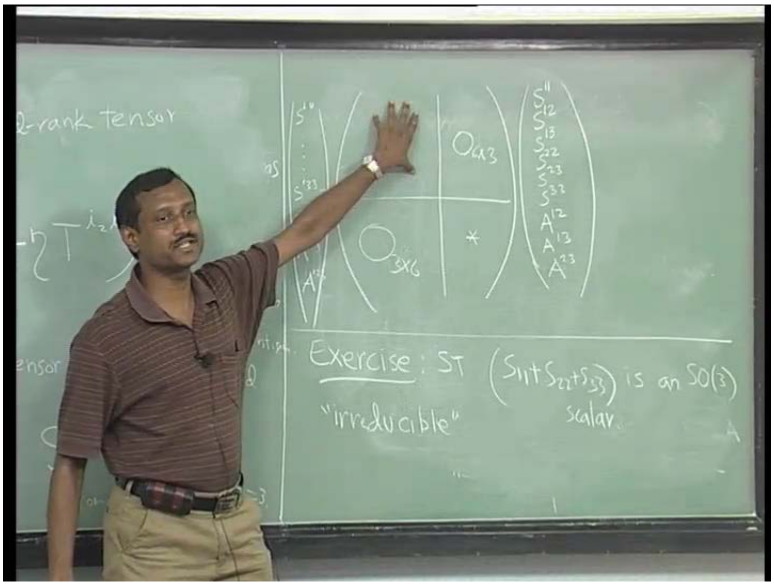

So, in particular if you go back there what, if you break it up like this, so if I now write it as S 1 1, S 1 2, S 1 3, S 2 2, S 2 3, S 3 3, these are the, it is simple to write out the expressions. So, S i 1 i 2 will be T i 1 i 2 plus T i 2 i 1 divided by 2, the other one would be the opposite side, I leave it as an exercise for you to, show that and if I write it out like this, A 1 2, A 1 3 and A 2 3.

So, this is like a change of base, as if you wish you think of this as a vector, just some linear combination acting on there, then if you work out, how it transforms. You will find that, you will get a 0 here, which would be how many rows? 6 by 3, and this would be a 3 by 6 0 and these would be non zero, non trivial entries, S prime 1 1, so on, so forth, till S prime 3 3, A prime 1 2, A prime 1 3, A prime 2 3. So, you can see the fact, that they do not mix with each other is indicated through this zero’s, does not matter what the rotation matrix, and that just follows from the simple into identification, that if something is 0 in one frame, it will remain 0 in all frames, I am just extending it.

In fact, there is something little bit more, you can show this is not the end of the story, you can show that, even this breaks up into a 5 under 1, 5 in the sense, that there is a trace of it behaves like a scalar. So, exercise again, so that SO 3 scalar,. So, you can break this up into a symmetric traceless part, the trace part, and this thing. So, there is something interesting which happens in 3 dimensions, an anti-symmetric second rank tensor as 3 components which has exactly the number of components, as a vector this is special to 3 dimensions. So, usually one converts this into another vector using the (( )).

But the this is the very important point, that the actually even though, I have written it out in this way, there are parts of fed which do not mix. So, what one means is, so the question is can I do better than this, it is already I said first, I have shown you 6 and 3, it seems very easy, but I said there is 5 also breaks into 5 plus 1, you may wonder is there some other kind, can I make anymore changes to make it further smaller blocks.

The answer, and the blocks should hold for R all rotation matrixes, not one particular rotation matrix, for a given rotation matrix, you can always go to some bases, where it is diagonal or whatever, and then it will be very, very simple. But we mean for all rotations, it has to be in form, in this block diagonal form. And the answer is that, this is it for this this example, this breaks up into the trace part, and true for SO d not just for d equal to 3, so the idea is so you turn that say that it is irreducible.

We will see this again, but it is useful to see it, in the simplest example, which we actually I will being seen, you want to break it up into irreducible components, components which do not mix under. So, for instance this 3 by 3 block is irreducible, you cannot break it up any further, and the statement here, that this 6 by 6 block is reducible into 1 block of 5 by 5, and another block of 1 by 1, are there any questions?

So, in the next 5 10 minutes, I will spend some time going back on. So, for as you should notice, as repeated myself to the special orthogonal part, but there is also the the the orthogonal part, I mean the things which have determinant minus 1 such as, the parental transformation,. What that does for you is to add extra transformations, if you wish.

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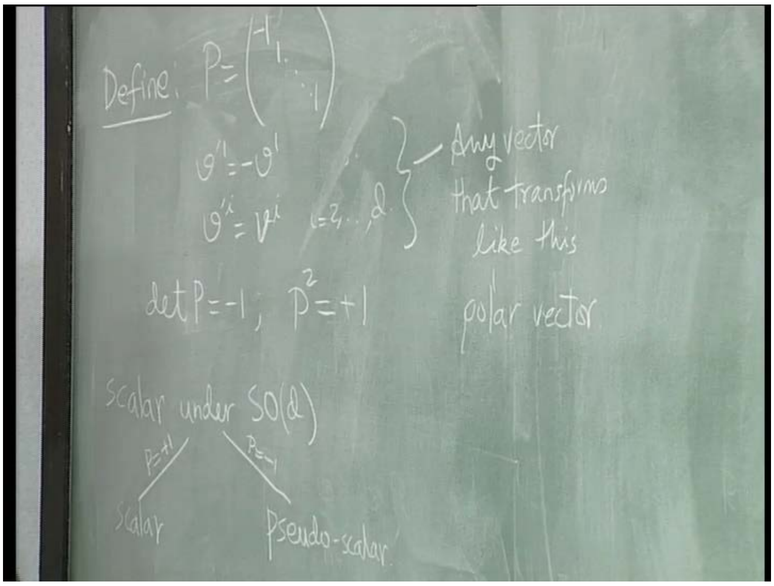

So, so let us consider parity, and I will define parity not as inversion all coordinates, I will define an operation P to be, if you wish the matrix minus 1 1 1 1. So, acting on what what do I mean, by that I mean v prime 1 this is obviously, has two properties. Determinant of P equal to minus 1, and P square equal to plus 1 squares to 1 since, P square P square it it squares to 1, it implies its Eigen values are either plus 1 or minus 1. So, so this lets you distinguish, so if you give me a scalar under SO d, break it up into two parts, and specifying here the Eigen value of P, if it break up into P equal to plus 1, because this a rear scalar, but it could change sign, and we will call that a pseudo scalar and anything which transforms like this, we will call, so any vector this, we will call later a polar vector, because this is changing sign.

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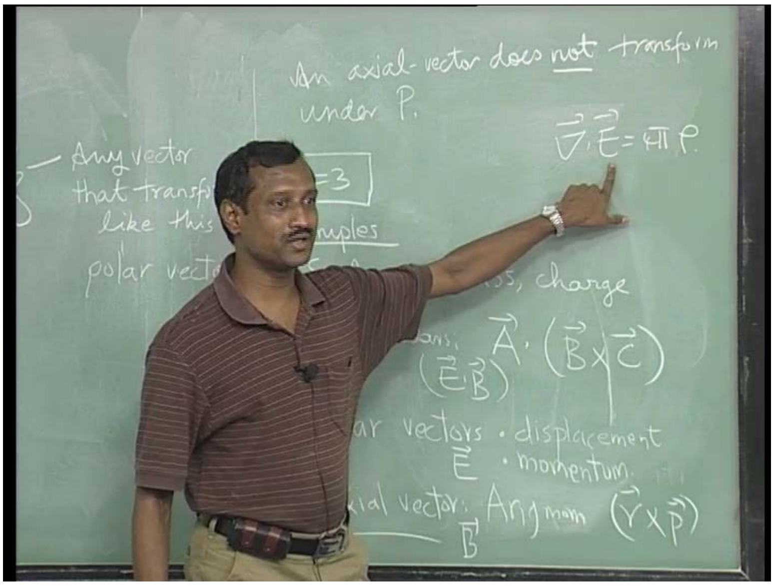

But, an axial vector does not transform under P, so v prime 1 would be 1, but for rotations the rest of ortho, SO d it will transform exactly, they transform the same, but it is how they transform under P they differ. So, examples of this lets now, fix d equal to 3, so I need examples of scalars, can you give me examples? So, that I can write them scalars, mass, charged density, charge, what about pseudo scalars? box product of 3, that is not a scalar, box products got A. You are saying something like this, A dot B cross C where A B C are all polar vectors.

We will add a few more examples, what about normal polar vectors, rotation angle I do not know, displacement would be an momentum, what about electrical magnetic field, do you know what they are? What is axial? Angular momentum is an example of axial vector, fine. Axial vectors, which is just r cross p, so it is a cross product of any two polar vectors is an axial vector.

Exercise for you, go back and check, Maxwell's equations are consistent with E being a polar vector, and B being an axial vector, so I can do something simpler than this, I can write E dot B, right. Are there any other axial vectors? But, most of the examples of axial vectors you start off like this, so but now the thing is how do you go about making this assignments? So, the way it is done is.

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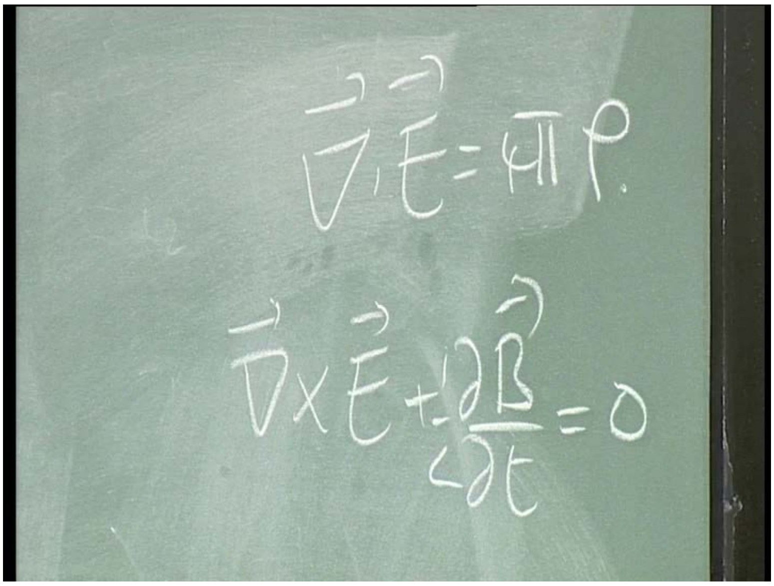

Let us talk of the electric field, that the magnetic field, you take start with Maxwells equations, and find first thing we saw that, charge versus the scalar. So, you look at the equation del dot E equals to 4 pi rho. So, this right hand side is a scalar, not a pseudo scalar, it is a scalar, and this you can convince yourself, del operator is also a an axial vector as a polar vectors. So, this if this now, it forces E to B a polar vector if, if this were axial vector this has to be that, when you go back and look at the next equation, del cross E plus d B by d t equal to 0 I guess 1 by c. So, now, this is, a this thing now you can see cross forces B to B, so the point here is that, you can go ahead and ask, can I make assignments? How they transform under parity? By looking at these equations, and that is how you go about doing things.

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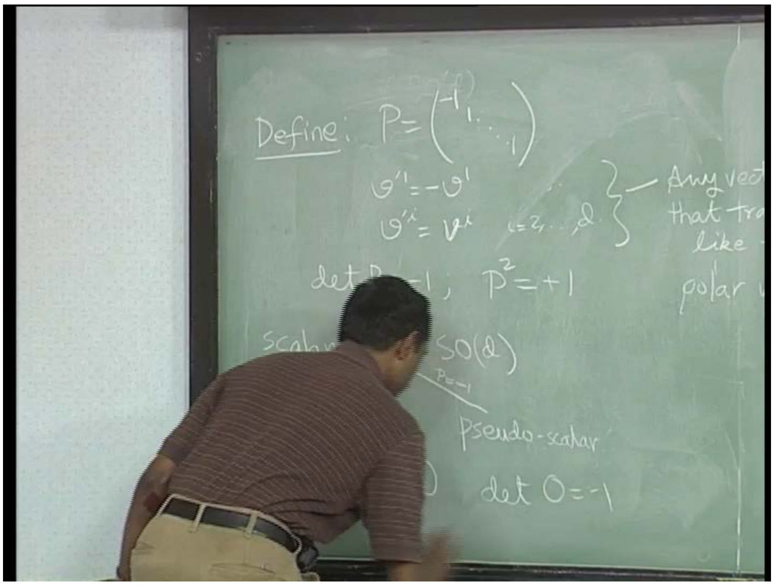

So, force has to be a polar vector. So, you can see that you do worry about the orthogonal, the full orthogonal group, but you separate out the part, we will separate out the part which is the special orthogonal with respect and the part of the orthogonal group, which is not special orthogonal, and its just 1 element. Because, every other element you can convince yourself, that any determinant minus 1 matrix can be written as, P times a special orthogonal matrix, it is not very hard.

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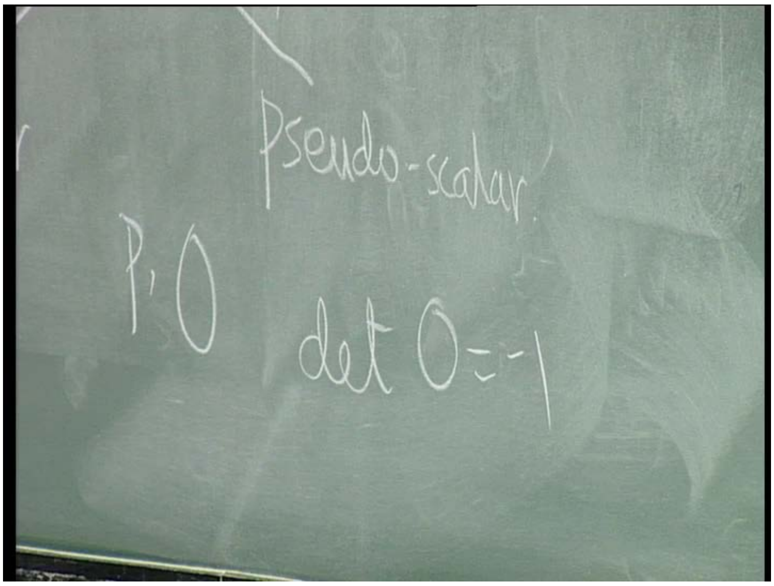

The way to see it is that, if you give me a matrix O such that, determinant O equal to minus 1.Consider the matrix P O yeah, this will be an orthogonal matrix, P square is 1 anyway, so you can see that, it is unique.