Classical Field Theory
Prof. Suresh Govindarajan
Department of Physics
Indian Institute of Technology, Madras

Lecture - 39

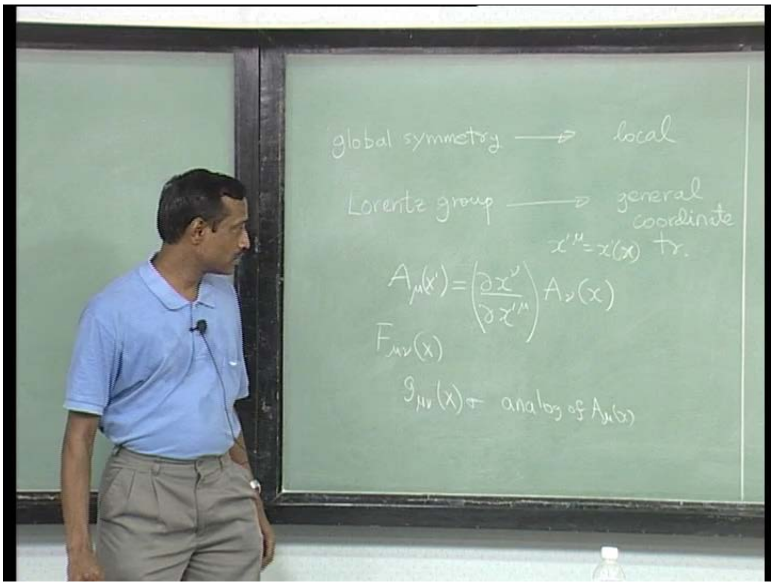
(Refer Slide Time: 00:18)

Way of where kind of things people are looking at and… So, I want a title this lecture as Geometrisation of field theory. So, lot of things we learnt in this course like the (( )) mechanism and enhancement of gauge symmetry things picking up masses, all these things in a very simple field theory. They actually can be given in a geometric meaning and that would not be a big deal, because that would corresponding to choosing a field theory with a particular kind of potential.

But the thing is that field theory is very natural they appear naturally in string theory. And so all the geometric pictures which I am talking about is something one gets by doing string theory and trying to understand field theories which are in those context. Even though string theory is a theory of quantum gravity; that is not the that is not I am looking at I am just thinking of it as something which give you a class of field theories. And the field theories are quite nice, but before I do that there is one important classical field theory which we have not discussed in this course. Does anybody know which one? A very famous general theory of relativity is a very famous classical field theory which we have not discussed in this course. One of the reasons is that there is in there other courses in this place for that topic, but I just want to mention one way of thinking about general theory of relativity which naturally fits into the scheme of things of this course.

And that is actually very anti geometric in some sense I would say anti Einstein, because really what Einstein did was to geometrize gravity. So, even though it is kind of ironic that today's lecture is geometrisation of field theory. I am going to give you one non geometric way of looking at general theory of relativity. And so at least let me not write in this part of this board. I go there and write it out here.

(Refer Slide Time: 02:31)

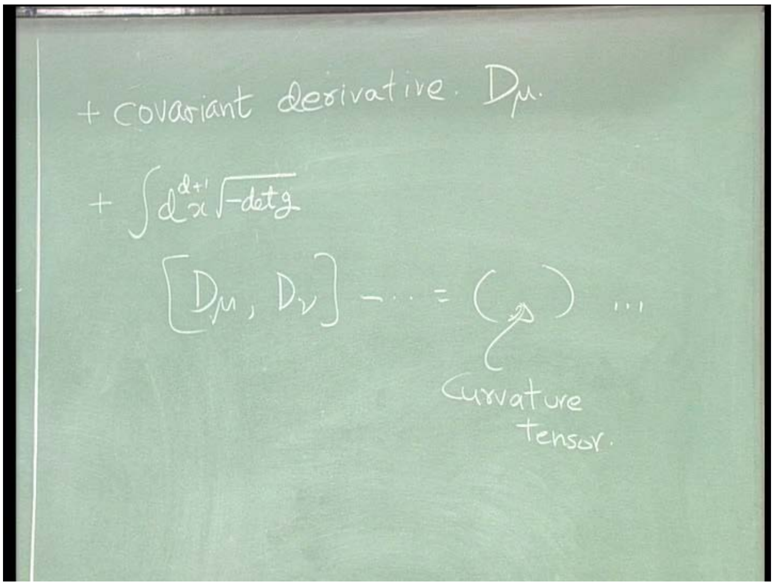


So, if you recall what we did was we took some global symmetries and made it local and all the examples we looked at the internal symmetry which did not act on space time, so but we a. So, we could just take an go ahead and say suppose we consider Lorentz Lorentz symmetries or the Lorentz group. Now this is this is something which acts upon space time and if you ask what is the meaning of making this local it means at every point you make a different rotation or a boast. So, it is a… So, the thing is that. So, you can see that this actually when once you make it local it just corresponds to general coordinate transformation. So, even things like translation fit into that. So, the whole point group becomes sub-zoomed by this because their examples of special this thing. So, the thing is to look is to ash how things will transform. So, for instance if you. So, already we saw if you take a vector field and the general coordinate transformations it should transform. So, a prime let me put even this when you write something like this a mu of x prime equal to. So, this should be the transformation of a vector field.

So, you can define tensors in a. So, this is really the Jacobean of the coordinate transformation change. So, what you doing is say x prime of mu is just some x prime of this coordinates. That is the general coordinate transformation and obviously, you can see things. Like you can ask what happens if I take derivative of this. How does it transform? Thus it transform nicely one exercise is to ask about f mu nu transforms given this transformation you take a derivative.

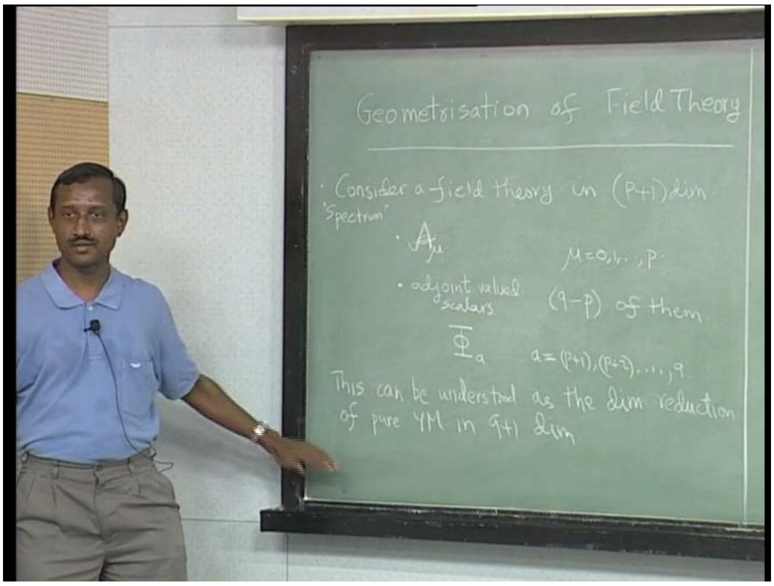
So, you work out x prime nu does it transform nicely. So, what you will find is a in general many thing this will not transform nicely and you no need to introduce a connection and the connection in this case is will turn out to be the metric. And so you can go ahead and you can make. So, then you come up with rules which says that there is covariant derivative which makes things transform nicely. So, you will. So, you... So, you will introduce. So, this is the analog of a mu which we introduced for internal symmetries.

(Refer Slide Time: 06:02)



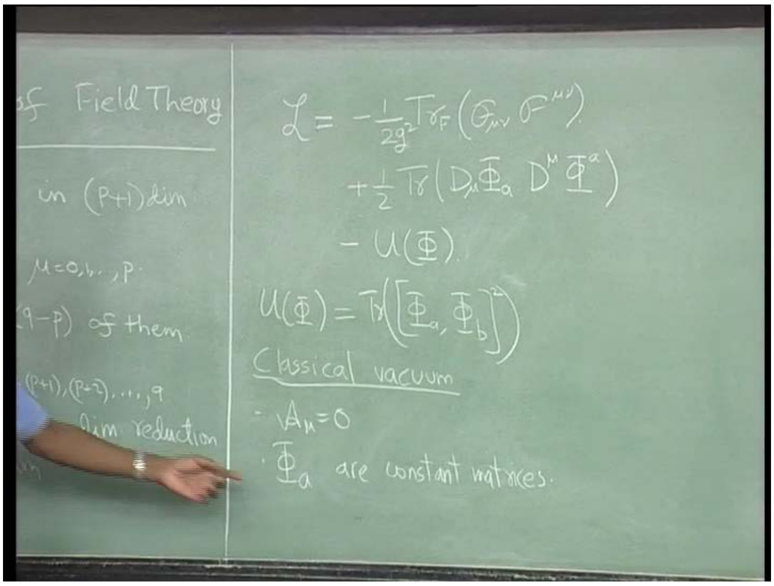
And so we will… So, you will have some notion of a covariant derivative and the measure look something like this minus det g. So, this is a nice measure. So, you can make this and you can also ask. So, let us say we define some covariant derivative like this and you can ask this acting on something will give you the analog of field strength right. So, I am being very schematic out here this object will turn out to be a curvature tensor. So, there is something called the tetra formulation of g r or general relativity where it fits much closely to what I am writing out here much more closely and. So, this is one way of looking at g r, but this is not the way even I would teach a general relativity course. I will start from the geo. I mean the geometrisation of I would say of gravity due to Einstein. That is a different thing, but it still useful to think of it this dual view point. So, coming back to geometrisation of field theory.

(Refer Slide Time: 07:39)



So, the example we will choose is will consider the field theory in P plus one dimensions. And I should give you the spectrum spectrum is a gauge field a mu. So, mu runs from 0 1 to p that is a gauge field. So, for starters will just think of this as a u 1 no will I take it to be mu n or this a non abelian gauge field. So, I was using script a. So, let us leave it that way. And suppose it also has bunch of adjoint valued scalars and there are 9 minus p of them. So, let just call these phi a where a runs from. So, what I will do is since there are 9 minus p of them I will just label them as p plus 1 2 instead of calling them 1 2 to 9 minus p I will just write it as p plus 1 p plus 2. So, so this is a spectrum or the fields and we need to give an action or a Lagrangian for this thing.

(Refer Slide Time: 09:31)



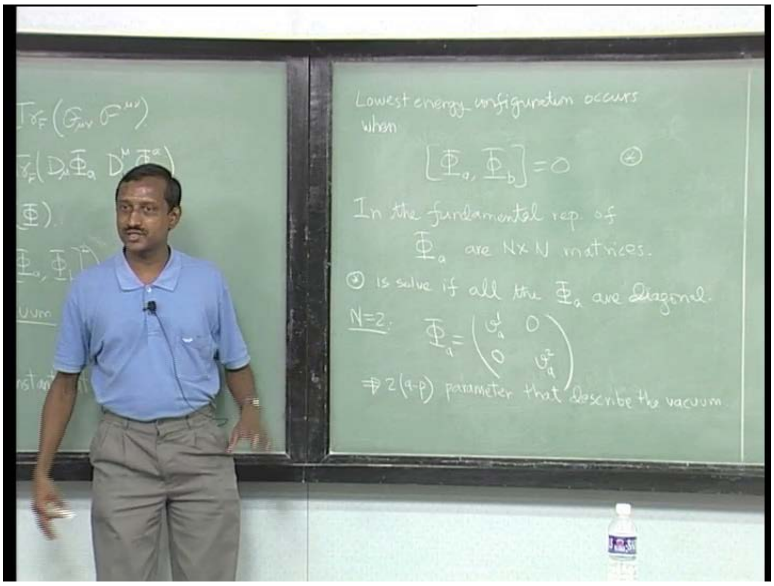
So, the Lagrangian for this gauge field. So, will write the simplest guy we can think two derivative one. So, it should be minus half trace in the fundamental of say f mu nu f mu nu and then. So, here there's no in this trace is the adjoint. Obviously, because it is an adjoint valued things. So, that just d mu covariant derivative of phi a.

So, this is the first things we would have written and we could put let me put a coupling. Let us call it a g square coupling and plus this minus u of phi. This is what you this is what we write. Now, I need to give you u of phi by the way we can also see that the spectrum can be this can be obtained by the dimensional reduction starting from we have seen this before. So, this also this can be understood as a dimensional reduction dimensional reduction of dimensional reduction of ah pure gauge theory; pure young miles in nine plus one dimensions.

And then you will see that these fields are basically what would have been in ten dimensions gauge fields. They give adjoint valued scalar this is clear this spectrum. So, the u of phi is again like we did we had discussed in the just. So, how does I will be something like phi a phi b take its commutator whole square and a trace. So, this is. So, this potentially is somewhat different from what we looked at we were looking at things like lambda phi four and those kind of potential they are very different. And but this is perfectly good potential and this is natural in this theory. So, now we want to lets go ahead and try to analyze the ground states of this theory.

So, the ground state would be; obviously, when things. So, we could we could look at situation, where all the gauge fields are 0 like we did even in the other case and the fields are also time independent. So, they just become matrices, so to analyze the ground states. So, the vacuum… So, will just say a mu is 0 and phi a are constant matrices. So, it comes down to just saying that this thing the square is in the wrong place. So, the phi a is constant matrices and you can convince yourself that this thing the lowest smallest value could get is 0. So, when would that happen when phi a and phi b commute, so the vacuum configuration.

(Refer Slide Time: 14:17)

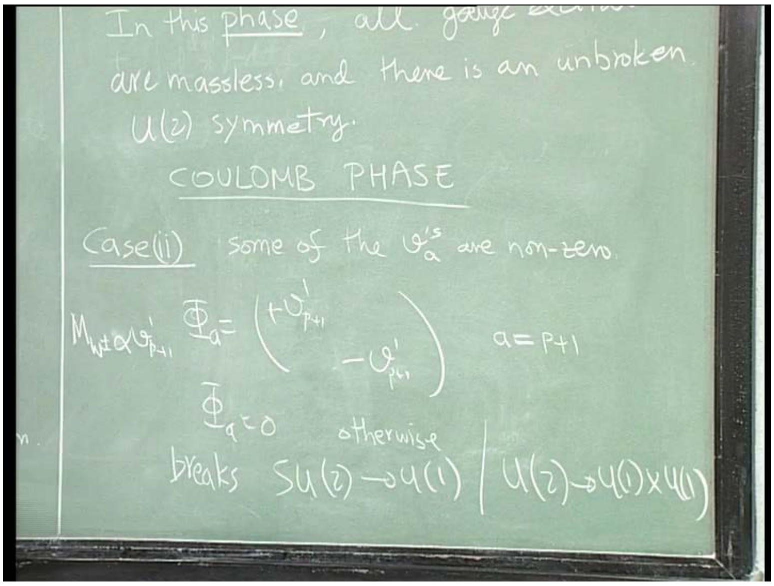


So, the lowest energy or when is this clear. And what we will do is we will think of since this is a adjoint value objects. We can think them as n by m matrices which is as so. So, if you in the fundamental representation, so even here we could make this lets make it fundamental then we haven’t chosen a group have we. So, let us choose the group to be s u n. I will go I will change it to u n pretty soon, so but since we discussed s u n. So, will just the fundamental. So, these phi’s phi a s are n by n matrices and if you want all of them to commute the simplest solution is that they all be diagonal matrices. So, star is solved if all if phi a are diagonal. So, let me just remind lets before we proceed what is the difference between s u n and u n in terms of le algebras is that the basis for u n s u n is traceless a mission matrices. While the basis for u n is you just get rid of the trace condition.

This is related to the fax of the group there is no determinant condition, so out here, so similarly here. So, I think phi a is not a traceless matrices, but I will permit them to have trace. So, let us just take n equal to 2 then you can see that I could write phi a would be v a 1 v a 2 0 0. So, now, what we see here is just that there is a whole hmm whole set of vacquier specified by how many numbers there are a run, so 9 minus b. So, there are two into. So, implies that we have 2 into 9 minus b parameters that describe the vacuum. So, is this clear these are the numbers for all of them the energy is 0.

So, they are equal. So, it is very high degeneracy. In fact, parameterized by this thing you can see very easily that if for s u for u n this just two becomes n. But I will keep using s u 2 as my example the generalization will be straight forward. But we can ask what does this mean. So, we can go ahead and do what we did in our field theory other examples we are asked what are we go ahead and put fluctuations and ask. So, here what was the thing we did a mu is 0. So, you any a mu turn on is a fluctuation, but to this I can add. So, I can add fluctuations to shift this. Obviously, those are mass less guys, but then I can turn on of the diagonal elements. But we have to still preserve the factor the summation. So, really you have all these parameters which you can turn on and the question is what happens.

(Refer Slide Time: 19:15)



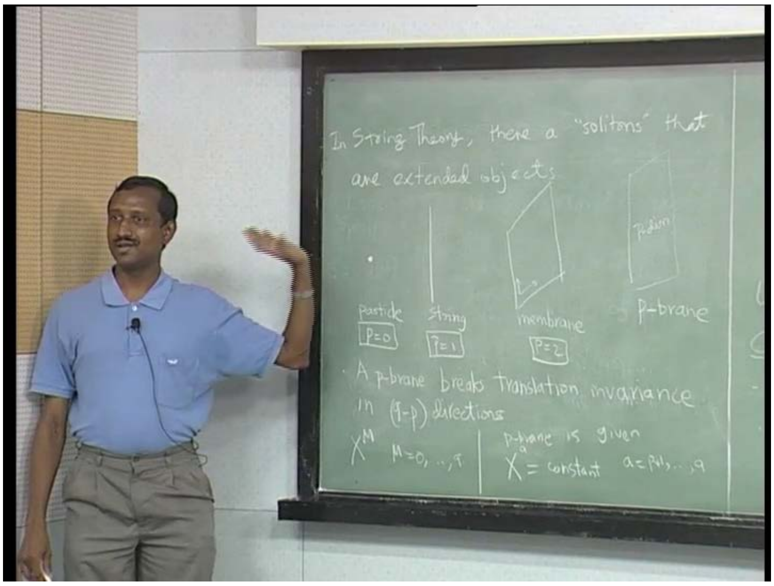
So, there will be a; obviously, when... So, let us first look at the case one which is all the v a 1 equal to v a 2 are all 0. So, this is analogs to the unbroken thing which we would have had where all the fields were 0. So, will. So, in this phase will call it a phase all gauge excitations are mass less. They are mass less and there is a an unbroken u 2 or s u 2 symmetry I will keep going back and forth very easy to see I mean just nothing. So, you work it out quadratic order and you can see that. So, this is... So, this is usually called the coulomb phase the nomenclature is as follows is that in that in coulomb phase is where the gauge fields are mass less and they Propagate if there is a mechanism and something becomes massive those things because they are massive.

They will not they will not propagating degrees of freedom. So, in some sense from physics when you take in the super conductor in the coulomb phase is where you have conductive conduction. If there is a mass then in there would not be. So, when the u 1 becomes massive. So, this called a coulomb phase case two is when v a 1 or v a 2 or everything some of the v a s are non zero. So, in particular lets choose, let us choose phi a lets choose a kind of I mean this is just example of this kind. I choose a same I choose the sane this thing I make it traceless by hand.

I am choosing a subspace in this thing where its traceless they need not be the same for each one of this things and or even better let us just say this is only for one of them say for a equal to p plus 1 and 0 for others. All other and its phi a equal to 0. Otherwise again this is exactly like what happened in case where you said that the only thing which you remember there was the and the potential could have been forgotten. But this is here there is an exactly that is what is happening the potential goes to 0. They do not need to take any limit. Once you put this thing now you look at fluctuations what you will see is that this thing breaks s u 2 to u 1, this is the you looked at. But the u 2 has an extra u 1 that will be unbroken and the mass of the there will be w plus and w minus right. There will be two massive guys and the mass of the w.

So, I will just call them mass of w plus minus will be among that things proportional to v v plus 1 this number there will be g factors all those things, but this is the mass. So, we have the mechanism network and. So, it is kind of felly straight forward, but now, but now s where the geometric picture comes where you interpret these things somewhat differently and have to explain. So, so what I will do for you is explain where there is kind of spectrum comes.

(Refer Slide Time: 24:36)



So, the… So, in string theory there are extended objects there are solitons that are extended objects. So, idea of… So, mo most of the solitons like one is the kings soliton that was like a particle. And if you if you look at time evolution in space time it is like a line, but you could have there could be. So, let us draw the space picture. So, there's always s time.

So, this is what we would call it a particle, but there could be something else which is extended in one direction I am drawing a curve. But let me draw it a straight line and what would you call something like this you will call it a string. If you add time it becomes like a sheet and you could have something which has two such directions. We would call this a membrane and in general there can be a object which carries which is p plus 1 this is p dimensional.

So, I cannot draw that picture anymore. So, you just call this a p brane. So, in that nomenclature particle would be p equal to 0 this would be p equal to 1 p equal to 2. So, we are not looking at the string theory of equation or motion or anything. But I am just saying that there is a soliton, but first thing you can see is that existence of. So, p brane it breaks translation in variance. So, suppose you have a particle in some space.

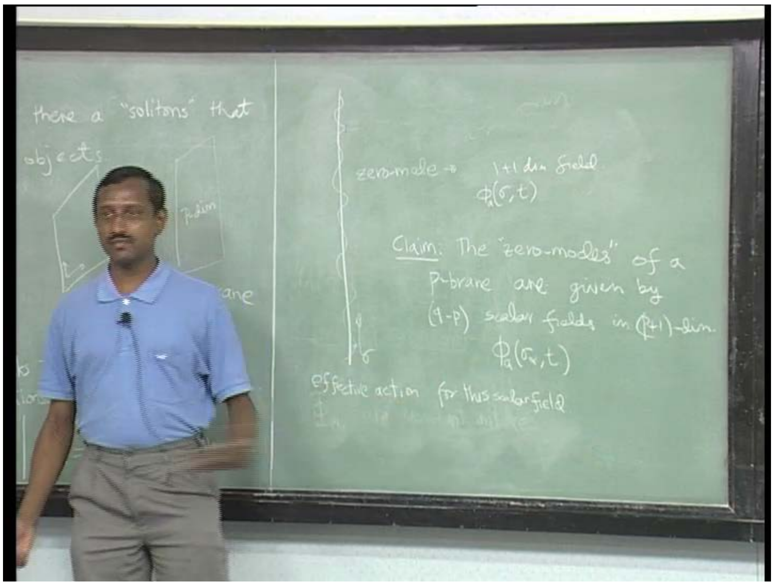
You have to give me its position it breaks translation in all directions. But if you have a string along the direction of the string translation variance is not broken. But translation is broken in the perpendicular direction. So, in a p brane you do not break translation variance in p plus one-dimensions when you break in 9 minus p. So, I should mention that these are objects leaving in 9 plus 1 dimensions.

So, it breaks translation in variance. So, for pictures I will use this and gen generalize because it gives me some this thing translation variance in nine minus p directions in other words. So, let us say we are in ten dimension space x m where m goes from 0 to nine. And let us assume that this brane is along brane p brane is specified. So, we just have simple what I have in mind is something like a flat sheet you know p brane is given by saying that x in a equal to some constants for a equal to p plus 1 to 9.

Notice that the breaking I am doing is exactly like what had happened earlier. So, this is what is going on. So, let us go back to the case of the king soliton king soliton was in 1 plus 1 dimension. It broke translation variance in space. So, then we saw that exactly 1 0 mode. And if you want to look quantum mechanics of the soliton we have to quantize the 0 mode. There will be 1 this thing. So, the quantum soliton would be saying you take that position as a coordinator quantum mechanical problem, but let us take something like a string and ask.

So obviously, there should be this is breaking translation variance in all those directions. So, we should have 0 modes, but the interest so. But the thing is here unlike the king what will happen is that this will become the field on the on the soliton and the way to see that is…

(Refer Slide Time: 29:46)



So, let me draw a string there are may dimensions, but since my board is just two dimensional. So, its preserving translation out here, but its breaking along this direction when now the thing is I can if I am interested in small fluctuations I can replace this by something like this. So, this lets call this direction sigma. So, I can discuss fluctuations if this thing. So, the mode here which corresponds to the you might think that here is one mode that obviously, which means translate everything equally. But why should I do that I can do it at every point I can make it different. So, the 0 mode is actually a field, so the 0 mode. So, in this if it you are a particle it will still be a function of time right I mean I can think of time dependent kind of solution.

Then those in the same sense the 0 mode can be elevated to a field it is a one dimensional field rather one plus one dimensional field. So, let us call this thing I will call that thing phi and it will be a function of sigma and of course, time and how many of them are there because this is p equal to 1 there are. So, many I mean I because I in the board I can draw only one of them, but there are as many of them as this thing. So, I get single scalar field for everything its very clear. Now, for a p brane I will end up getting something which is a p plus 1 dimensional field scalar field.

So, this sort of tells you that a claim is that the 0 modes are the light fluctuations of a p brane are given by 9 minus p scalar fields in p plus one-dimensions and then you can ask. So, the point is that it is only the no it is just transverse direction right directions along the along the thing is not broken. So, in p plus one-dimension is not broken. So, this is just theorem it does not need any glorified stuff. You know it is just theorem.

So, I am just saying that the low energy modes will be that of a field even if you are interested in. So, coming back to the king's soliton even there x 0 which is not a function of d is the exact solution. But if you want to do dynamics of that object you want to quantize that object which you would just make it a function of d. Assume it is a slow varying function of d. And then you will get a effective you can plug it into your action and you will get an effective thing which you would give you the quantum mechanics of that 0 mode. You will see that x 0 plus v t.

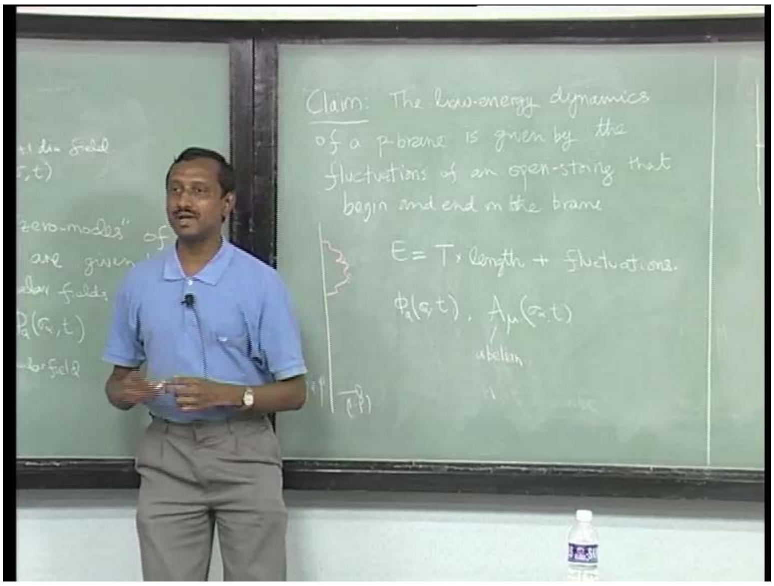
So, that is also go away, but then the next derivative will come. So, it is a nice exercise to go take your king soliton solution. So, and where ever we saw x we put x minus x naught of time or call it q of t that you fell more comfortable and you integrate out. You plug that whatever action you put in there and you will automatically get in kinetic energy term which of course say it should become 0 when it becomes the constant. So, you see things will work beautifully.

You will get some effective action or effective energy function for that thing which will agree with your intuitive thing. So, that is the only thing which everyone is doing here doing something which is similar here to explain to you the lower energy dynamics of this thing is given by this kind of a field, but you could you know you need not do that. You could say let us ask what is the thing lets two derivative etcetera and you naturally write s and there are symmetry in the theory you would write something. So, for instance if you look here rotations grow which would have been 9 comma 1 is broken into s o p plus 1 part and cross s o 9 minus p.

So, you write things which are variant combinations of this phi a s. So, you will end up by getting phi a s or some sigma alphas and t where sigma's are they coordinate on this thing. So, we could write some effective action for this scalar field, but this is not what we saw right. I mean there was also a gauge field in the spectrum. So, that is the surprise the surprise is that string theory gives us a slightly different way of looking at this story. It says that you look it says.

So, I mean these objects p I mean it is quite natural to look at these kind of objects and in the literature for decades. People add to this things and writing effective action for this. You would write lot of things to all these kind of objects and but the miss ingredient was the gauge field in the spectrum which comes from the string theory.

(Refer Slide Time: 35:53)



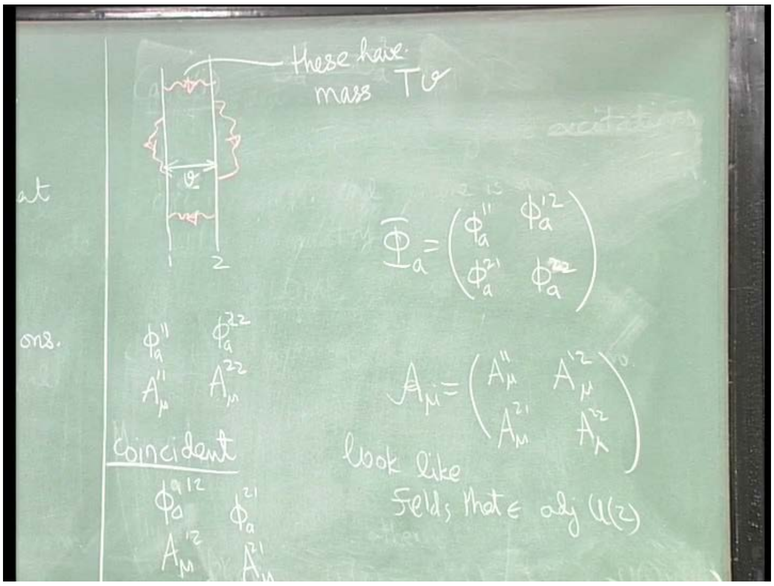
And the way is the claim because I will not be proving and I do not think it is proven in reverse sense. But there is lot of evidence for this the claim is that the low energy dynamics of a p brane of the time. I am talking about is given by the fluctuations of an open string this just word.

They would not make sense for you but it will give you pictures open string that begin and end on a on the brane. So, let me draw the string again. And so now, I will use a different color it is something like this and let us assume it some directed objects something like this, so now…. So, the idea, so the statement implies you should go ahead and quantize a free string is very easy to quantize it is not very hard. And you can ask what it is, it is low energy fluctuations? And if you give me the energy of any string is the following. This is the energy of that string of a classical string will be its tension. This is some dimension its energy per unit length time its length plus quantum fluctuations.

But we are discussing the lowest energy things of all other fluctuations you would. So, you should give me a string which is length l its energy will be lowest possible energy would be tension into its length. So, now, you can see that. So, what would be the lighter string could do you can think of a string or of size going to 0 size. So, in other words that is that is the master which is called low energy. So, you can go ahead and quantize it and what you find is you find that it has the string spectrum is infinite dimension. So, there are massive guys. So, we forget about those guys and you find the lowest configuration has this field, but it also has a gauge field. So, what you find is that there is a phi a, we use the same notation sigma of alpha t. but there is a gauge field a mu this abelian gauge field abelian gauge field.

And this is the very important step it gives you a completely different field theory. Then you would have and if you ask what is the action you can write two derivative its exactly what I wrote. But except the potential energy because it is an Abelian there is everything is commutes. So, there's no potential energy is nothing there is just normal kinetic energy for all these things for the scalar field. That is it that is what you have. So, the question is how do you get non Abelian gauge field, and this where the pictures come this picture. So, this picture right now I do it for a string, but it is best to think of it as all the p plus 1 dimensions going in this way and this direction is 9 minus p, so will call this 9 minus p as direction transverse to the brane. So, now, the question is how do you construct? How do you get non Abelian gauge theory does not look like and this how you do it.

(Refer Slide Time: 39:58)



Consider, two parallel p branes. Remember I told you that these are given by the constant the other one is shifted by some distance let us say they are separated by some distance l. So, what I have in mind, which are parallel and they are separated this way. And let us take this claim the low energy dynamics is given by strings which are separated which sorry which begin and end on the brane. But now there are two branes I can draw many different kinds of strings and let us look at this thing. So, first thing we know is that we believe the two of these things.

So, we will write. So, we will label this fellow one. So, let us just call this fellow this comes from. So, this fellow will give you a one and this will also give you another scalar field one this also give you a gauge field one. So, this is just extending whatever observation I had here to the case of two parallel branes. So, now, let us, but the thing here is if you take this red or pink picture correctly. Of course, this came from this guy this came from something like this. But in principle I can consider strings which begin here and here. So, we are looking at string, which are directed. So, that is the difference between these two. But these strings are massive, because it is separated, so that that is why there are not there in your spectrum.

So, these are massive t times l, actually let me not call it l. Let me call it v t is some tension whatever that is there is some intrinsic function with that thing. But now you can ask what happens when they become coincident the objection that I had is gone when they are coincident you will get you will get extra guys. So, let me just. So, to now let me remind myself by saying that this is something coming from a string, which is starting at one and ending at one.

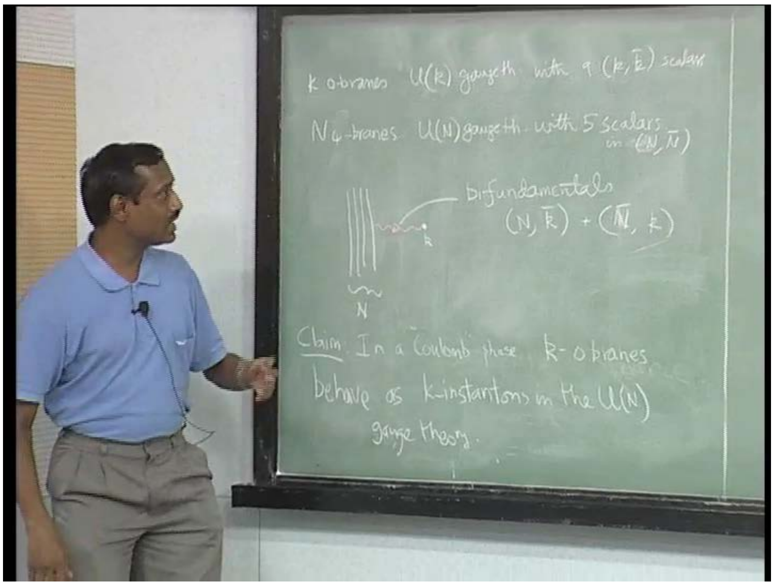
So, let us keep track of both. So, this is 2 to 2 1 to 1 2 to 2, but when they are coincident when they are coincident what you get is you get extra guys. You start getting phi a 1 2 phi a 2 to 1 similarly a mu 1 2 a mu 2 1 they all vector fields. But they have different labels going with them and this happens only when this occur only when they are coincident; when they are not coincident this become massive and not low energy. But now you can see here I can something need happening instead of keeping.

So, when they coincident better it makes sense for me to collect 1 2 3 4 as a matrix and think of this 1 1 2 2 this thing has actually the entries in the matrix. So, I just define for you a field phi a which has all these guys. And similarly here and you can see that these are collecting and becoming an adjoint of u 2. So, these actually now looks like adjoin fields that belong to adjoint of u 2 these, and of course the connection for u 2. Now, let us assume that this is correct.

So, now we can go back and look at this thing this. So, what happening is when you separated them the these two became massive. And what was their mass it was t times v go back and compare with mechanism little while ago the mechanism was precisely that it tells it and that is why it changed l to v. You can identify the separation as with the wave vacuum expectation value at the number putting into in the phase. The other face is called the phase by the way I forgot to mention that. So, you can see that what is going on. So, when you take two of these things and you separate them your getting a mechanism and what is what are the fields which are Becoming massive, it is these guys.

So, it is really this. So, these are the these this to take linear combinations of this became w plus and w minus and there are two unbroken u ones and what are the gauge fields those are these two. So, u 2 broken to u 1 square is precisely visible out here. Now, you can ask how do I get u n gauge theory. It says just take n of them and now you can see that there are many different ways of breaking it. I could, I could, I could take m of them together and separate from the other guys where n minus m. So, you would get a breaking of u n to u m times u n minus m. So, all sorts of things which you could have done in your field theory are realized in this picture. This is clear. So, I am going to conclude with one more example which is which we haven’t discussed buy will give insight into the construction of. And so here we considered branes which are of the same type.

(Refer Slide Time: 46:45)



So, what we will do is we consider a system which is which consists of N 0 branes k 0 branes and N p branes. And I will draw a picture now there are N of these guys and since this is smaller I will indicate this by a dot and there are k of them. So, if you look at it from this view point of this N. You know that what you will get it is some u N gauge theory with not p p equal to 4 with how many. So, there will be 9 minus p would be 5 scalars, if you look at the same thing with p equal to 0. This will tell you have from this once view point you will have a u gauge theory in 1 plus 1 dimension quantum mechanical system. But still will call it gauge theory so but there will be one gauge field a 0 that will behave like Lagrange multiplier u k gauge theory with nine adjoint valued, so with scalars in adjoint of u N. So, I will just write this as N N bar with nine k k bar scalars right. You know that adjoint I can write for s u u N s n times N bar. That is what I am really doing by writing as a matrix. So, these are bi fundamental way of thinking about this. But clearly there are strings which do this start from one of these guys and go like this these will be typically massive. But again if you bring it close enough. There will be excitations you have to consider this thing. But these will be strange things they will have n possibilities for one end and k possibilities for that other end.

So, you end getting, these will give you bi fundamentals of two kinds N k bar plus k rather N bar k better stick to that kind of rotation and these are also certain kinds of multiplex. So, there will be some scalar. So, something’s coming from this thing. Now, comes the interesting part what is this object. So, now, there is one thing which can happen is that when this comes and leaving in this thing.

So, it is dissolving inside that thing and there are these k guys are moving. So, this is now like a claim. So, that will be like a coulomb phase what we discussed when they are on top right in a coulomb phase these k 0 branes behave as k instantons. So, this is viewing from the view point of these 4 branes. So, it dissolves in that it has 4 dimensions which it can go move around these are like particles in that thing and these. In fact, you can show that this carry instant on number and there is certain things from exactly that this will be true. So, this is just our standard construction which we discussed, but we could have asked it from the view point of this u k gauge theory. From the view point of this u k gauge theory you will have a u k gauge theory with these kind of scalars plus.

Additional guys coming from bi fundamental thing. So, look at the details this is precisely the data which enters the construction of course. They knew nothing about 0 branes 4 branes, but it gives you a very, very simple thing and I also mention this gauge theory there's only a 0. But a 0 has no derivative we know that. So, a 0 behaves like a Lagrange multiplier.

So, what you? So, it imposes some constraints and if you just look at the ground state condition you will get some matrix valued condition which is called the hyper hyper-kahler quotient. So, it will be the analog of saying this things commute. So, there is some rules to get something's that have to 0 etc. You get something called the hyper-Kahler quotient and actually you get the low energy condition is indeed the construction its truly amazing.

So, with without the geometric picture you would never have thought of this other possibility. Well that is wrong because human mind is very good and for big mathematicians thought about it before hand. But that was like an abstract setting, but I mean for someone like me it is sort of straight forward to just go ahead and write it spectrum and write what it is pretty fast. So, this is what I called geometrisation of field theory. And the real extreme cases are were the people are actually push things to large classes of examples where you know you know quite a bit theory by asking what is the geometric question. And this is something which you would never have thought of without this kind of a picture.

So, that is this I mean this is really happened in the last fifteen years of. So, where things have become people where geometrized a lot these things and I feel that it makes gives a nice a way to remember things and understand things also. And but you can see that the not all field theories can be realize you can only realize field theories in this manner which are adjoint or bi fundamentals. So, for instance extreme cases people have asked the standard model is s u 3 cross u s u 2 cross u one. So, what you do is you take three of these things branes and some intersecting brane model or whatever it is three of them gives us a two of them which gives u 2.

So, you get a u 3 u 2. You may think even y can come from here. Actually no put something and to intersect in a particular manner to get the spectrum which I haven't discussed again out here. So, there's a whole what I call technology has been developed and even it sometimes call it geometric engineering. There are pep there are geometric engineering of field theories. That is what they call and its sort of very dramatic change in understanding of not just classical field theory, but quantum field theory. So, I am done.