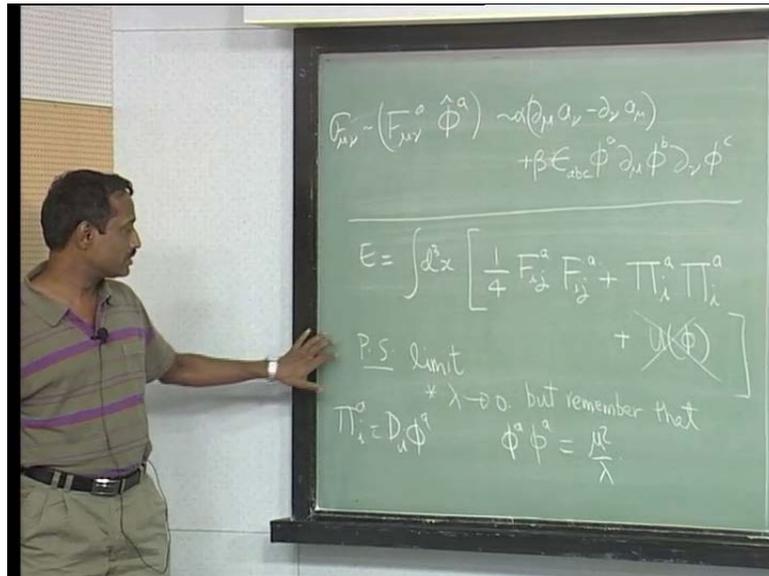


**Classical Field Theory**  
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**Department of Physics**  
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**Lecture – 33**

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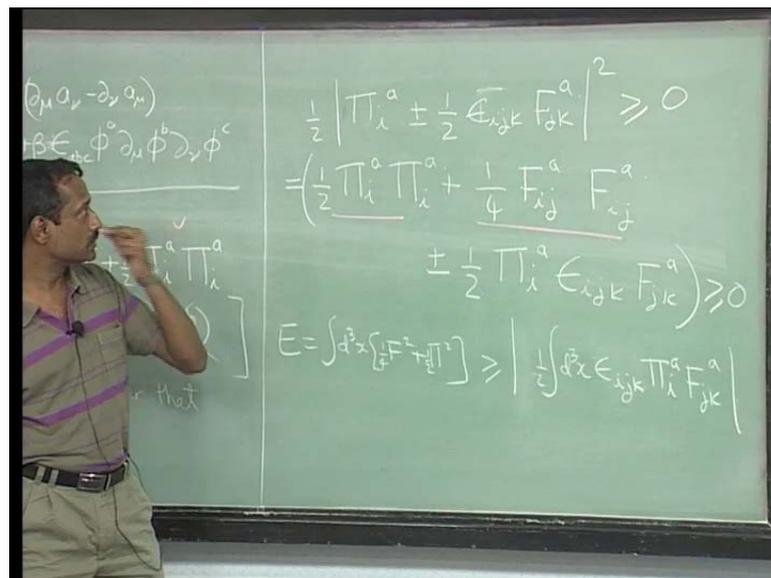
So, last lecture we saw how to go about identifying the u 1 gauge field in some sense and I sought of left with something saying that you look at trace of the non-abelian guys; best to write it in a slightly different way, forget the trace. Lets write as  $F_{\mu\nu}^a$  and  $\hat{\phi}^a$ . So, this where  $\hat{\phi}^a$  I just normalized it to become one  $\hat{\phi}^a$ , this should be the field strength for the abelian gauge field and it had two pieces one was this. So, there are factors which I will not write, but hopefully it is a nontrivial exercise to check that you do get the  $\mu$ ; let me put indices later. So, let me write  $\alpha$  and  $\beta$  times. So, this precisely what we had in this is what we expected for it to be. So, this is what appeared in the topological charge.

So, it kind of shows that you can really replace this guy with this guy because this will not contribute by definition, but it also shows that the fluctuations are abelian. So, let us gets back to the original problem which we were looking at. We had a Hamiltonian density whose energy was, this is what we had if you. So, I threw away; we were looking at time independent configurations and with our boundary conditions the various terms dropped out and we just left with these three terms. And the interesting thing is about the

Prasad-Sommerfield effectively said that the P.S limit said that forget about u of phi or take lambda to zero, but remember that we need to impose phi a phi a should be equal to mu square upon lambda; are there any questions, is this correct? This is what we had, these were the conditions. So, where u of phi was the potential which we define long ago.

So, basically in the P.S limit it corresponds to saying that you forget about this guy. So, what we are going to do now is try to look at as soon that the P.S limit is taken yeah. Sir, we are writing this at lower negative sign. So, that is a good question; here all these things are all there is only i and j which has space indices which I will raise and lower now with delta i j conical. Delta there are no signs, everything is positive definite; that is why even if you noticed I did not write upper i j or lower; I wrote everything as lower i j. So, that say sum of positive terms I mean and this I mean pie i of a is I could have written it this way. It is just the covariant derivative, but this is a notation of Prasad and Sommerfield. So, I told you that I wanted to stick to that and so in notation I am more or less following whatever they did. So, now let us look at something like this and try to understand. So, this is the energy density that we are looking at and we will see why the name b is added that goes with this; we are going to look at energy density which does not have this thing.

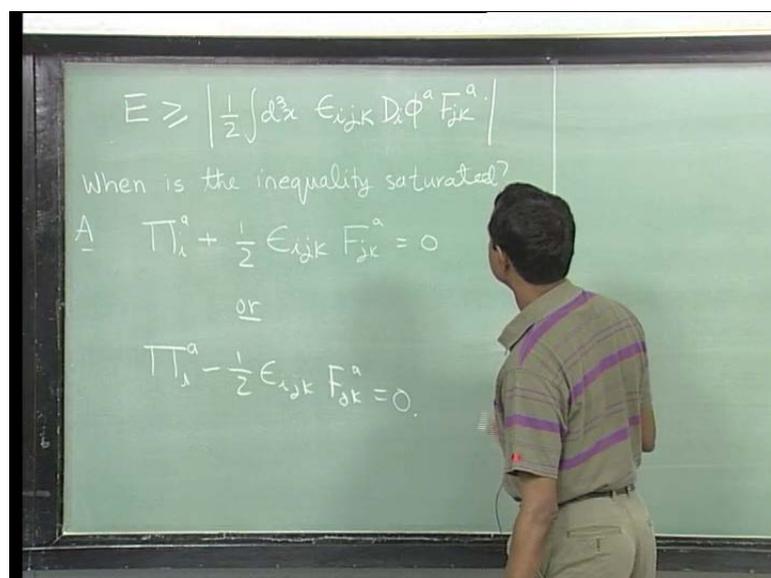
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So, to look at that thing what I will do is consider the following quantity  $\pi_i + \frac{1}{2} \epsilon_{ijk} F_{jk}$  of a mod square. So, by this I mean just taking summing over all  $a$ ; taking the square of it and summing over all  $a$ . This quantity is greater than or equal to 0. This is not yet this, but we will expand it and we will see; we will get two terms which are exactly this. So, of course if this is true its integral over all space will also be greater than or equal to 0. So, let us expand it and what do we get? So, let us look at half of this. So, we get one term which would be half. That is just this term, then the square of this will if you massage it around the epsilons-epsilons will go away giving you 2 delta's that will give you a factor of two, etc and it will give you exactly this term. But now you get two extra terms where the cross term which is plus or minus half  $\text{tr} \, i \, a \, \epsilon_{ijk} F_{jk}$  of  $a$ .

So, this is greater than or equal to 0. So, all I have done here is to expand that thing. So, now we can see that these two terms are precisely what appear here. So, I can rewrite. So, this implies that the energy  $E$  which is defined to be integral  $d^3x$  times because I do not want to rewrite this again I will just call it  $F$  plus  $\pi$  square, may be even at least let me put this much, the factors correct that would be the integral of this should be greater than or equal to minus or plus. So, we see that the energy density is. So, since this thing is minus or plus the strongest one would come when you take the one which gives the positive sign. So, I will take care of that by just saying that it is greater than or equal to modulus of this. So, this is a very important.

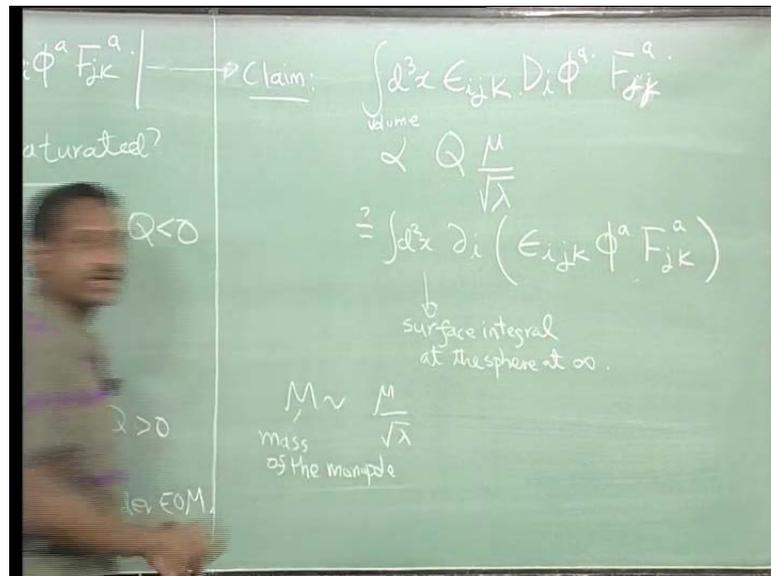
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So, we get that the energy is greater than or equal to. Now I will write instead of writing it as  $\pi \int d^3x \frac{1}{2} F_{jk}^a F_{jk}^a$ , I will write it as  $\int d^3x \frac{1}{2} \epsilon_{ijk} D_i \phi^a F_{jk}^a$ . So, this is what we get, because I forgot to write it, thank you. So, now you can ask two things to ask is when is the equality or when is the inequality saturated by that I mean when is. So, the lowest energy configuration would correspond to  $E$  being equal to 0. So, that now that depends on the sign of what you look at. So, coming back to this, so the question is when will it be true; it will depend on which sign plays the dominant role. So, it will be one of these two things being equal to 0.

So, the answer is  $\pi \int d^3x \frac{1}{2} \epsilon_{ijk} F_{jk}^a F_{jk}^a$ . Let us write it as two terms and determine when  $\epsilon_{ijk} F_{jk}^a F_{jk}^a = 0$ ; in some sense this is similar to the parallels what happened in the case of vortices. Except now the only thing we need to check is this the topological charge and the answer is yes, it is, and the easy way to understand that is to look. So, let us first look at that and then come back to this. So, these two possibilities would have happened. So, let us look at this. So, what would this be? This I can rewrite as this is the covariant derivative of this thing. So, also the Bianchi identity tells you that  $D_i F_{jk}^a = 0$ .

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So, I can rewrite this. So, let me write it as a claim  $\int d^3x \epsilon_{ijk} D_i \phi^a F_{jk}^a$  should be  $\int d^3x \frac{1}{2} \epsilon_{ijk} F_{jk}^a F_{jk}^a$  is proportional to the topological charge  $Q$  and I need something which will give the dimensions of this thing that is the wave of this thing. So,

it will be  $\mu$  by  $\sqrt{\lambda}$ . So, basically the way to see this is that I can pull this derivative out and I can convert this is a volume integral in three dimensions and we can convert this into just ordinary derivative and then this. So, now this will become an integral over this is the total derivative and it is a surface integral now at the sphere at infinity and then we can just follow through this identification which we had and it will show us that what we are getting there indeed is the charge. So, this  $\phi$  I have to just take it the other side. So, basically you need to get those two derivatives and it is just you can work out the details and so you can see that indeed this is what is happening  $Q$ . So, it depends on the topological charge and  $\mu$  over  $\sqrt{\lambda}$ .

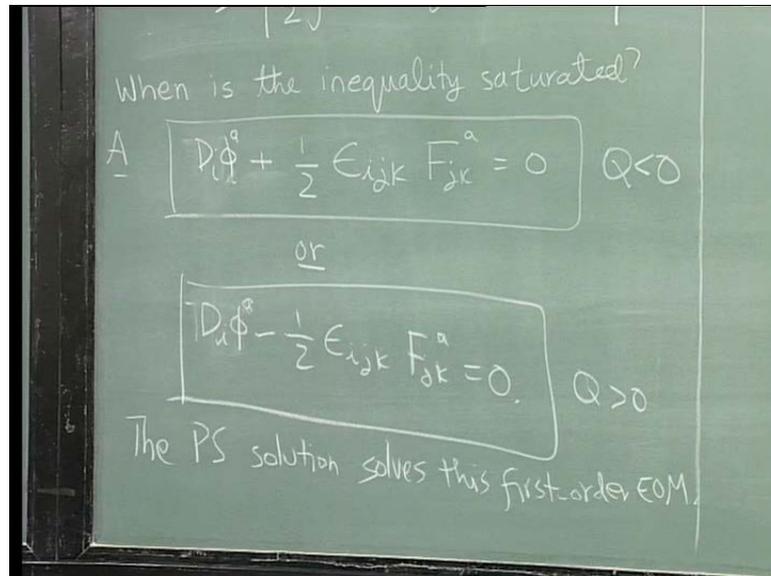
And there are some factors which I am not going to work out that is for you to fix, but what this shows is that this quantity on this side it just depends on the topological charge; this is kind of nice. What it tells you is that if you were in the vacuums configuration, that we know it is a situation with topological charge 0. In that situation, the energy we know is bounded from below by 0 and  $Q$  being 0 that is no contradiction. So, why you applied the total energy and not a diameter density? No in principle I mean the thing here is that when the equality is saturated it happens point by point everywhere, but now the charge I need to integrate over all; what I am interested in is the energy or the mass of this monopole or whatever that is.

Sir, I am taking whatever signless case. So, I can as well do the analysis saying. So, then no that is not quite, how do I say this? That is not true right because the thing is that what you want is the. Why would that why is that not. No no, wait wait; what is your question, repeat? I do not get it. Here you are saturating the total energy, right. So,  $e D$  greater than equal to whatever mod but suppose the lateral is diameter in density. No, yeah in fact I am doing that, right. I mean this is the equation of motion I am writing it not for is out here is the sum of squares. So, if it is something is 0 even an integral is a sum infinite sum. So, it has to hold point by point.

So, I mean in fact that is what I am doing; otherwise I would have written a much weaker constraint or something like that, but here this holds point to point. So, it is really one is working with the Hamiltonian density, but I am looking at the energy when I want to get the relationship between the energy and the charge; this is what you get. So, the point here is that you are solving first order equations. So, let us look at this thing; I still need to fix which goes where. So, I just wanted to write out the charge. So, let us look at what

happened when we took this to this plus sign came to the other side as minus sign. So, the modulus works when  $Q$ . So, this should hold when this becomes positive.

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So, this holds when  $Q$  is less than 0 and this holds when  $Q$  is greater than 0, unless things have been very nasty to me and there is a minus sign in this thing. So, let us assume that this is our definition of  $Q$  then this holds, but now comes the need thing that these are first order equations because this is  $D_i \phi^a$  is nothing but  $D_i \phi^a$  and these are first order equations. So, this trick is due to Bogomolnyi and the Prasad-Sommerfield solution which we looked at the explicit solution in the P.S limit is actually solves; the PS solution solves this first order. This is an exercise for you; it is just a claim but it is very easy to go and check, go back and plug it into these two things and check which one it works for which it works and of course if you look at the Prasad-Sommerfield paper they do give the mass. So, the mass according to them or we already wrote this a long ago it was really  $\mu$  by root lambda went as  $\mu$  by root lambda. There were some  $e$ 's etc, but it is really  $\mu$  by root lambda which we have already seen.

So, this is the mass. So, they are exactly like the king solution which we saw. This energy will be really concentrated in some core region and it will fall off it will go to 0. So, the asymptotic things do not contribute. So, really this again but the point about solving this first order equation is that the mass is determined by the charge. So, in other words if you tell me that it had monopole charge two you can tell me that and it saturates

this equation then you are done. But I do not think Prasad and Sommerfield realized that they have their solutions of course whatever satisfies this equation will satisfy the second order differential equation, but the consumers need to be true because obviously there exist solutions for which  $E$  is greater than this number with this charge and they will not have to at least in principle it exists. So, that is why this these two ideas idea of getting rid of the potential is due to Prasad and Sommerfield and this squaring trick is due to Bogomolnyi and it sort of fits together and.

So, today we call it B P S limits. So, when you hear people talk of B P S black holes, etc, these is exactly situation where the charges of the black holes determine their masses. It is a little bit more complicated setting, but it is really at the end of the day the trick is similar to this. So, what you find is that it is exactly goes. So, if you give me the charge I can give you the mass for these things. It is real fun exercise for you to go back and check that the Prasad-Sommerfield solution which we wrote out a while ago actually satisfies one of these first order equations. So, now the point here is to ask are there more general solutions? There can be dyonic solutions; for instance things which carry electric and magnetic charge. I will show you a beautiful trick which will let you take this solution that you know for the monopole and construct a solution which carries both electric and magnetic charge. So, let us see how that works and that requires some tricks.

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Handwritten notes on a chalkboard:

(i) BPS limit: ignore  $U(\phi)$ .

(ii) Electric charge  $\Rightarrow A_0^a \neq 0$ .

(iii) Things remain time-ind.

$$F_{\mu\nu}^a F^{\mu\nu a} = -F_{ij}^a F_{ij}^a + 2(D_i A_0^a)^2$$

$$F_{0i}^a = \cancel{\partial_0 A_i^a} - \partial_i A_0^a + e \epsilon^{abc} A_0^b A_i^c$$

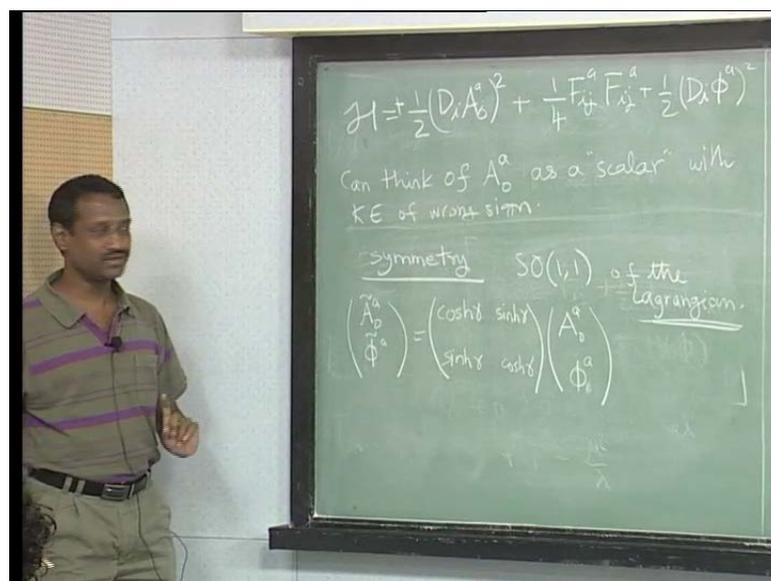
$$= -(D_i A_0^a)$$

EOM

So, we will work in the P.S limit. This we will keep; so that means at the end of the day it means ignore  $u$  of  $\phi$ , but if you want electric charge implies that we cannot say that  $A_0$  is 0. We will still keep everything time-independent, things remain. So, let us look at first thing; in this sort of a situation first let us look at what happened to  $F_{\mu\nu}$  a  $F_{\mu\nu}$  a. So, originally when things were when  $A_0$  was 0, etc, we just said that this is equal to minus  $F_{ij}$  a  $F_{ij}$  a. Now let me pull everything down because I have taken care of the sign. This is what we would have got but now this term this has. Now we cannot do that because  $A_0$  is no longer 0 but now we will get an extra piece. So, to look at that thing let us consider the following thing, let us look at  $F_{0i}$  of a. So, let us write this out. So, there will be one part which would be  $d_0 A_i$  minus  $d_i A_0$  of a.

So, first thing to note is that this term dies down, but what about the other pieces. So, then we will get plus epsilon; is there an  $e_i$  do not know,  $e A_0$ . So, all I have done here is to go ahead and get rid of this time derivative, but the rest remains. But if you look carefully this is nothing but this I can rewrite as minus  $d_i$  of  $A_0$  of a. So, now you can see now I can write out what I get here. It is just there will be a two coming because it could be  $0_i$  this thing. So, I get  $2 D_0$  of  $A_i$  whole square, sorry  $D_i$  of  $A_0$ , thank you. So, now let us just look at what that does. So, now let me just go back and I could look at the Lagrangian or I could look at the Hamiltonian density; let us look at the Lagrangian density.

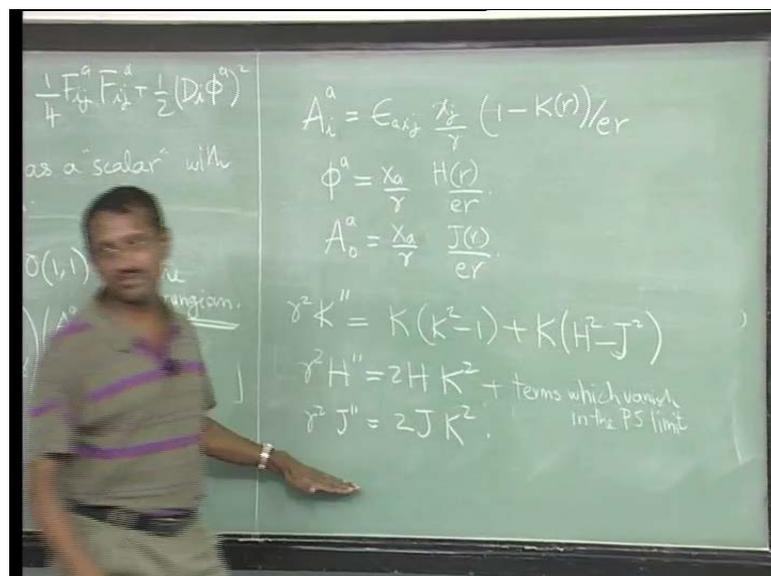
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So, the Lagrangian density for such, no no let me stick to the Hamiltonian density. So, the Hamiltonian density will look like the following half, I keep writing the wrong one right,  $D_i$  of  $A_0$ . I have thrown out this thing, I do not know if this good news or bad news; look at this, there's a minus sign out here. You lose positive definiteness, but there is something interesting which happens. What you see out here is that  $A_0$  is exactly but for this sign is behaving exactly like  $\phi_0$ . So, in other words I can think of  $A_0$  as a scalar with kinetic energy of wrong sign. But in a sense this is expected because this has to do with the signature of the matrix in some sense that is where it came from. So, just looking at this we can see that there is symmetry of the following kind.

So, we are not looking at the most general configurations we have all these conditions put in, but in that limit things look rather interesting. There is way I can think of boost symmetry here which is an  $SO(1,1)$  kind of symmetry which rotates these two guys; not both of them are adjoined in the same representation that is very important. So, I can write something like this. So, it does not act on the index; it sort of rotates these guys. So, what this tells you is that in some sense if you think of this as some kind of boost in some sense you can boost this thing. So, you take the other solution of the monopole solution and just do this boost; at least at the level of solving things it will give you a solution to the equation and in fact if you look at the ansatz which was written out for a dyon that is something which carries both electric and magnetic charge, it was exactly of this kind.

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So, we have already seen. So, this is what we had chosen for as an ansatz for A and phi when we solved for this thing. So, all one does is to add a new variable, no no, H of; have I exchanged J and H, yeah and if you look at the equations of motion what we get is something it reflects. So, you can see that what is the thing which is invariant combination under this boost? But that will translate into H square minus J square as your invariant. So, we will see for instance in the equation of motion for K double prime, we had written something like this K into K square minus 1 plus K into H square is what we had. Now without any calculation we write the answer what this should look like; it should be just minus what is it J square. So, this identification tells you what to write there because this left hand side and these two terms are invariant under this boost.

It does not act on this part; so, it goes through and now we can write the rest of the equation. So r square, so, this is what we would have got in the Prasad-Sommerfield limit plus there would be another term which we had written but that is the potential; in the P.S limit that term is dropped. So, now again it is very easy to write what the equation would be for J; it would be wherever we see H we write J and that is indeed the case. So, the key point here is now we can actually take the monopole solution and do a boost on it. Now that will obviously mix these things and it will create something which has electric charge also.

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Handwritten notes on a chalkboard:

$$J = [cr \coth(cr) - 1] \sinh \gamma$$

$$H = [cr \coth(cr) - 1] \cosh \gamma$$

This solution will have non-zero electric and magnetic charge.

$$Q_{\text{elec}} = \frac{4\pi}{e} \sinh \gamma$$

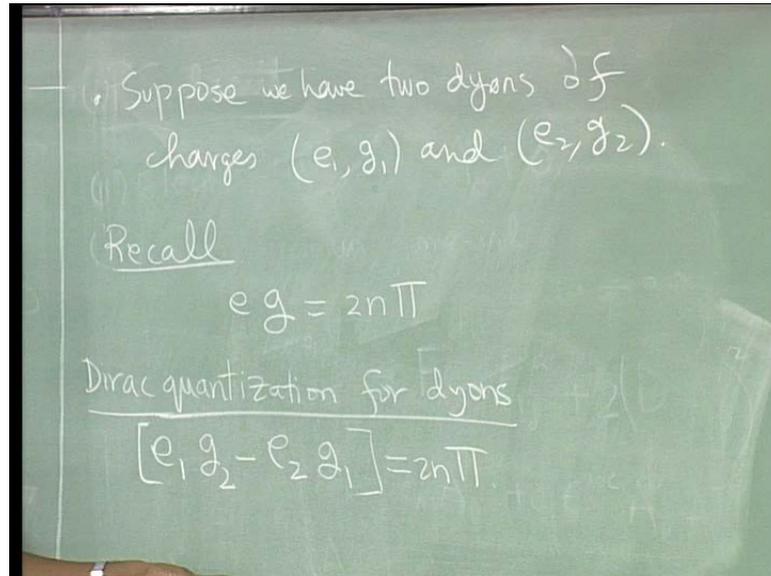
$$M \sim \left( \begin{matrix} \text{earlier} \\ \text{mass} \end{matrix} \right) \cosh \gamma$$

So, boosting what was that solution that was  $K$  equal to  $C r$  by sine hyperbolic  $C r$  and it was obviously  $J$  equal to 0,  $H$  equal to. So, two lectures ago we had written out this thing; at that time we did not have a  $J$ . So, now if you just boost it  $K$  will remain the same and so all we have to do is to multiply this with some cosine hyperbolic  $\gamma$  and this should be the same object  $C r \cot$  hyperbolic of  $C r$  minus 1 and I am just looking at their paper, it agrees with these things. So, the point is that, so you get this part for free by noticing this symmetry. It is somewhat of an accidental symmetry, it is not a real. So, now you can ask what. So, this solution obviously, will have nonzero electric and magnetic charge. So, the electric charge turns out to be  $Q_{\text{electric}} = \frac{4\pi}{e} \sin$  hyperbolic of  $\gamma$ . So, if you switch off  $\gamma$ . So, and the monopole charge will be whatever you had before multiplied by  $\cos$  hyperbolic of  $\gamma$ . So, clearly we can.

So, this tells you and you can see that the finiteness of energy, etc, nothing actually gets messed up but you can work out what the. So, now the thing is. So, clearly it looks like there are solutions with dyonic charge. So, now you can ask what happens to Dirac quantization in the presence of objects which carry dyons. Actually I think this is not correct; the Hamiltonian density is positive definite, it was positive definite when we started. When we wrote it was sum of squares, I think this is positive but the Lagrangian will have the wrong. So, this is still true symmetry of the Lagrangian; I was really bothered by this. In the Lagrangian that will hold because if you remember it just became  $e^2 + b^2$  even if you take the abelian case. So, this is not quite. So, the Hamiltonian density is not an invariant but the Lagrangian is.

So, I withdraw what I said and anyway these equations are determined precisely by that. So, this statement is correct, but my statement comes from I wanted this expression. So, I just say initially wanted to write Lagrangian density and then I wrote the Hamiltonian that was my error. So, this is positive definite. So, the mass actually turns out to be the earlier mass times  $\cos$  hyperbolic square of  $\gamma$ . So, when I came to write the mass I realized that I was off and because this will tell you that the mass should also be invariant, but it is not and that trick also will not I mean the Bogomolnyi trick is not quite you have to do some variant of that.

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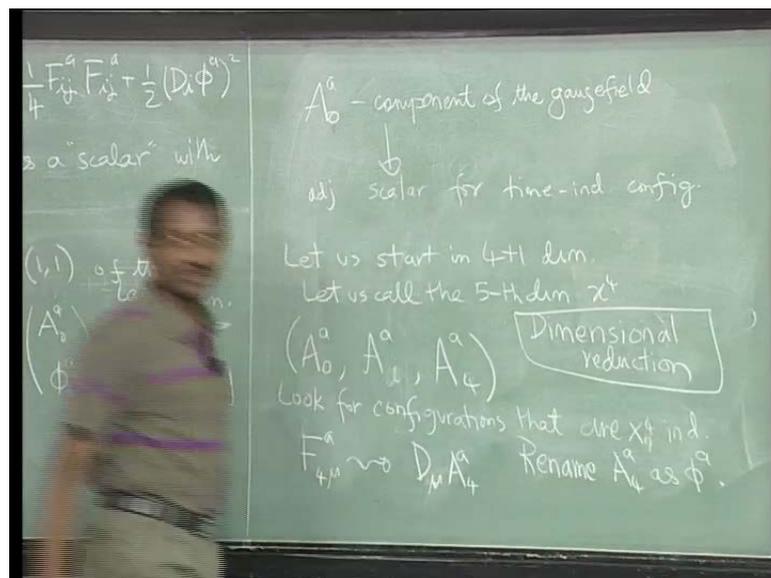
So, now the thing is suppose we have two dyons of charges let us just say I write it  $e_1 g_1$  and  $e_2 g_2$ . So, we are trying to write, recall that what we had was  $eg$ . If you had pure electrically charged object and this thing  $g$  this was equal to  $2n\pi$ , was it in rational units; it was quantized. So, now we just need to find a replacement; I will not prove what it should be but we should find a replacement for this which should reduce to this in this case. So, I will just write the formula. So, the Dirac quantization for dyons takes something like  $e_1 g_2$  minus  $e_2 g_1$  equal to  $2n\pi$ . So, now you can see what happens. So, let us take a case where  $g_1$  is 0 and  $e_2$  is 0. Then this just becomes  $e_1 g_2$  is equal to  $2n\pi$  this term drops out. But if you go the other way round again this works out, but there's a minus sign, but all that matters is really it is an integer. So, again it goes through, but more general would be when you have objects which have both dyonic.

So, this is the formula I will not derive this thing at all, but you can convince yourself that at least in this sort of things that suppose we started out with something where we had one solution with  $e$  one with electric charge and this thing and you write a boost; do carry out this boost. Can you write something which will work out and you will see that that will work out provided you. So, using the simple identity this minus sign is important because  $\cosh^2 \gamma - \sinh^2 \gamma$  should be equal to 1. So, this is just a consistency check of what I am saying. So, in some sense what I have in mind is I take something which is a pure monopole and make into a dyon

with a boost; obviously, the same thing if you give me a thing which is pure electric under the boost even that will pick up magnetic fields.

So, now, that is an example of a dyon but we know. So, this better reduce to that and that happens. So, is this clear? So, we will come back to this picture of let us look at the same set up where one thing we saw is that there was a parallel between this a  $A_0$  out here and this guy  $\phi^a$ ; they were almost on par with each other because we just said we are looking at these kind of solutions. Now that also gives us an idea that may be there was a funny signature had to with  $A_0$  coming from the time like thing, but this came with the correct signature. Suppose I think I took my thing I had a theory in 3 plus 1 dimensions, but instead of that suppose I looked at an example in 4 plus 1 dimensions. So, let us call the four.

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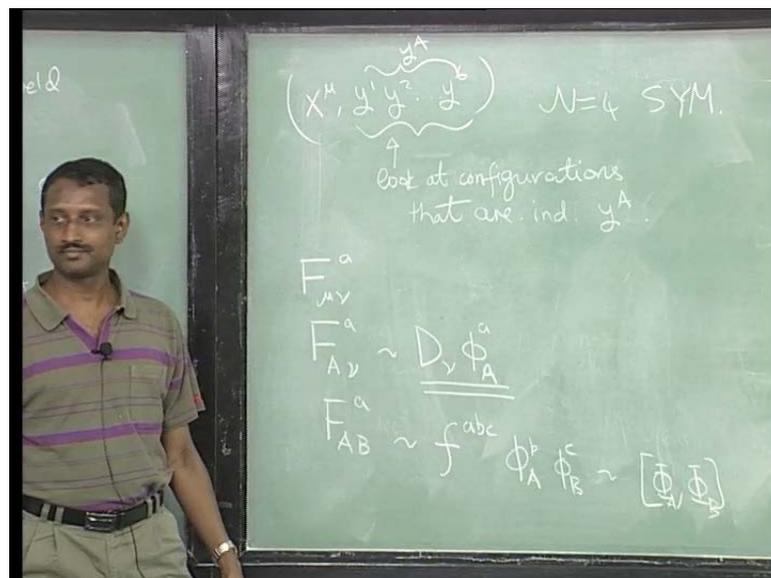
So, what I am saying here is we started out with  $A_0$  which was a component of the gauge field and since we are looking at time-independent configurations, this was behaving like a scalar adjoined scalar for time-independent configurations. I could do the same thing; this could have come from some another fourth dimension where I required the things be independent of the fourth dimension, then I could get something like this. So, that is what I am saying; let us start in 4 plus 1 dimensions and let us let us call the fifth dimension  $A$ , I should call it  $A$ ,  $x^4$  rather, and the gauge field. So, I would start out

with gauge field which should have been  $A_0^a$ , then 3 dimensional guys plus there would be  $A_4^a$ .

So, now again as we did here I will say that look for configurations; what would be the analog of time-independent in this case,  $x_4$  independent. So, again we will do what we did there; we can look at something like  $F_{\mu\nu}^a$  and this will start looking like  $D_\mu^a$  of  $A_4^a$ . Exactly like we did before and if I just went ahead and rename  $A_4^a$  as  $\phi^a$ , then what would you see? You would see no difference between what we wrote; once you took the PS limit that had the potential energy dropped out. So, that is all that remains. So, this idea of starting with some 4 plus 1 dimensional object and then requiring things to be independent of some dimension, we have done two examples today; one is time and the other one is  $x_4$  and this will give you nothing give you a normal scalar, it would not give you funny sine scalars; things will be perfectly fine.

So, it is called dimensional reduction and can you guess what one would call the process of going from this thing in the opposite direction what would you call; you have not done chemistry, oxidation it is called dimensional oxidation. So, the process of adding a dimension in some sense is called dimensional oxidation. So, what we are going to do in the next few minutes is to ask suppose life was so, bad that we had you know 9 dimensions, space dimensions and one this thing. We can go ahead and do this same thing, I can do dimensional reduction I will assume that.

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So, we will take coordinates  $x^\mu$  and let's call the rest  $y^1, y^2$  up to  $y^6$ . So, this is a normal space time coordinates and we will choose six such guys and again look at configurations that are independent of  $y$ . Let us call this combined things we will call it  $y^A$ . The reason to do that is because we will see that there is more structure in this thing. So, out here we just saw that  $F_{4\mu}$  became this way but now we have to look at several possibilities. So, we start out we will have  $F_{\mu\nu}$  which is the usual thing but then I can have one of them to be  $\mu$  the other to be  $a$  and so this will look like  $D_\mu$  of some scalars. There will be so many of those things; I will just relabel them and call  $\phi^a$ 's and they will all carry this gauge index. It is important to note that if this scalar was in some other presentation; I could not have done oxidation because gauge fields in the deduction always give you adjointed valued objects because gauge field cannot come in some other representation. So, it is important that things be in the adjoint.

So, you can see that I get six scalars, but this we are not done; there is some more things which is  $F_{AB}$  of  $a$  and this term will look like  $f_{abc}$ . So, you can see I mean I am not even specifying the gauge group; I am just saying just take some field strength for some gauge field some non-abelian gauge field. So, I have  $f_{abc}$  times. So, they come with the potential; it is better to think in matrix terms, if you think of this as a matrix this is nothing but  $\phi, \phi$ . So, it is a commutator. So, if you look at  $F_{\mu\nu}^2$  this will become the usual  $F_{\mu\nu}^2$ . This will be the kinetic energy, this will be the usual kinetic energy for the 4 dimensional gauge field; you get a nice kinetic energy for the 6 scalar fields, you also get a potential for this scalar fields. Now the potential will be square of this quantity of course. So, you get a nice quadric potential naturally in this there.

So, you get a system which has all these things and you can ask, what is the analog of the Prasad-Sommerfield limit? The analog of the Prasad-Sommerfield limit will be that  $\phi^A$  and  $\phi^B$  will commute with each other; you look for field configurations where  $\phi^A$  and  $\phi^B$  commute with each other. In fact even out here you can see that even here we could have looked at something like  $F_{0a}$  of  $\phi^a$ . We have looked at time-independent configuration, but if you just for a moment kept that together you would have seen that there will be one term which will look like  $A_0$  commutator  $\phi^a$ ; that should have been that term. But they will have to commute with each other and in our solution they did commute. So, this actually does happen.

There is a theory like this and it is called this theory; if you add super symmetry it is a theory which is called  $N$  equal to 4 supersymmetric Yang-Mills and I am giving you the Bosonic spectrum of it. It has one non-abelian gauge field and six scalars, very easy to see. In fact, these theories if you want to construct them as supersymmetric theories directly in 4 dimensions it is very, very hard. So, Sherkin Watts actually constructed them directly in ten dimensions where things were easier and then did a dimensional reduction. So, this idea of dimensional reduction is very powerful and very, very useful and we will come back to this again. We will see how the Bogomolnyi equations are actually a special case of something called self-dual Yang-Mills and that is what we are going to do next.